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On Ramadan Group Transform of Fractional Derivatives: Definitions, Properties, and its Applications to Fractional Differential Equations

Heba M. Arafa[1,*], *Mohamed A. Ramadan*[2] *and Majeed A. A. S. Amini*[3]

[1] Department of Mathematics, Faculty of Education, Ain Shams University, Roxy 11341, Cairo, Egypt
[2] Department of Mathematics and Computer Science, Faculty of Science, Menoufia University, Menoufia 32511, Shebein El Kom, Egypt
[3] Department of Mathematics, Arts and Science College, Ahlia University, Kingdom of Bahrain

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Abstract: In the current study, we introduce for the first time the transformation formulae corresponding to the Ramadan group integral transform (RGT) of Riemann-Liouville and Caputo fractional derivatives. Noting that RGT is a generalization for Laplace as well as Sumudu transforms. The obtained results make significant improvement and complement some known ones in the literature. illustrative examples are explained to demonstrate that RGT is a potent and effective method for finding a fractional differential equation's analytical solution.

Keywords: Ramadan Group transform, fractional Ramadan group transform, fractional differential equations.

1 Introduction

The fractional calculus is a theory that deals with integrals and derivatives of any real number, even complex ones. (see [1]). The fact that it is a generalization of the classical calculus means that many of its fundamental characteristics are retained. The past few decades have demonstrated how useful fractional calculus is for explaining the characteristics of actual materials, such as polymers (see [2]). Applications of the fractional calculus can be found in a variety of scientific disciplines, such as the theory of fractals, physics, engineering, economics, and finance (see Gorenflo and Mainardi [3]). M. Caputo uses his own definition of fractional differentiation to formulate and address some viscoelasticity difficulties (see [2]). In the theory of control of dynamical systems, where differential equations of fractional type are utilised to describe the controlled system and the controller, fractional integrals and derivatives also emerge. Fractional differential equations have attracted a lot of research attention, we recommend the reader to check these papers [[4], [5], [6], [7], [8], [9]]and references cited therein.

In this study, we provide RGT equations for the two most common definitions Riemann-Liouville and Caputo fractional operators which are essential in fractional calculus. The theory of fractional derivatives and integrals, as well as applications of this theory in pure mathematics, strongly rely on the Riemann-Liouville formulation (see [2]). Additionally, the Caputo fractional operator is crucial in practical issues when integer order derivatives with conventional initial conditions are involved.

Integral transforms like Laplace, Sumudu, and Fourier are utilised in differential equations of fractional type solutions, which are crucial in helping to solve issues in applied science, mathematical physics, and engineering. Mohammed A. Ramadan et al. recently introduced a new integral transform that combines Laplace and Sumudu transforms; for more information, please see the publications [[10], [11], [12], [13]] and references cited therein. Here, using the Ramadan group transform, we derive the Riemann-Liouville and Caputo fractional operators. We next use these findings to solve

* Corresponding author e-mail: hebaallahmohammed@edu.asu.edu.eg

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homogeneous and non-homogeneous differential equations of fractional type with non-zero initial conditions.
We provide several lemmas, definitions, and theorems that are crucial to the proofs of our conclusions in the sections that follow.

Definition 1(See [10]). *Suppose a set A described by*

$$A = \{f(t)|\exists, t_1, t_2 > 0, |f(t)| \leq M \exp^{\frac{|t|}{t_i}}, if\, t \in (-1)^i \times [0,\infty)\},$$

then definition of the RGT is

$$K(s,u) = RG(f(t)) = \begin{cases} \int_0^\infty e^{-st} f(ut)dt, & 0 \leq u < t_2 \\ \\ \int_0^\infty e^{-st} f(ut)dt, & -t_1 \leq u < 0 \end{cases}$$

Definition 2(See [10]). *If $F(s)$ and $G(u)$ are the Laplace and Sumudu integrals transforms respectively of $f(t)$, then, we have the following relationships*

$$F(s) = K(s,1), G(u) = K(1,u) \text{ and } K(s,u) = \frac{1}{u} F(\frac{s}{u})$$

Theorem 1(See [13, Theorem 3.1]). *If $K_1(s,u)$ and $_2(s,u)$ are the respective RGTs for the functions $f(t)$ and $g(t)$, then*

$$RG[(f * g)(t),(s,u)] = uK_1(s,u)K_2(s,u) \tag{1}$$

where $*$ represents the convolution of f(t) and g(t).

Theorem 2(See [10, Theorem 2]). *If $n \geq 1$ and $K(s,u)$ is the RGT of $f(t)$, then the following is how to determine the RGT of the n^{th} derivative of $f(t)$:*

$$RG[(f^n(t),(s,u)] = \frac{s^n K(s,u)}{u^n} - \sum_{k=0}^{n-1} \frac{s^{n-k-1} f^k(0)}{u^{n-k}}.$$

Definition 3.*As shown by Podlubny [1], The following are the definitions fractional integral operator of the Riemann-Liouville of order ρ on the Lebesgue space $L_1[0,1]$:*

$$I^\rho h(x) = \begin{cases} \frac{1}{\Gamma(\rho)} \int_0^x (x-t)^{\rho-1} h(t)dt, & \rho > 0, \\ h(x), & \rho = 0. \end{cases}$$

Definition 4.*As shown in [1], the Riemann-Liouville fractional derivative of order $\rho > 0$ is defined by*

$$(D^\rho h)(x) = (\frac{d}{dt})^m (I^{m-\rho} h)(x),\ m-1 \leq \rho < m,\ m \in \mathbb{N}. \tag{2}$$

Definition 5.*The fractional-order derivative with regard to Caputo sense defined by ([1]):*

$$D_*^\rho h(x) = \frac{1}{\Gamma(m-\rho)} \int_0^x (x-t)^{m-\rho-1} h^{(m)}(t)dt,\ \rho > 0,\ x > 0, \tag{3}$$

given that

$$m-1 \leqslant \rho < m,\ m \in \mathbb{N}.$$

So, it is possible to write

$$D_*^\rho x^k = \begin{cases} 0, & \text{if } k \in \mathbb{N}_0 \text{ and } k < \lceil \rho \rceil, \\ \frac{\Gamma(k+1)}{\Gamma(k+1-\rho)} x^{k-\rho}, & \text{if } k \in \mathbb{N}_0 \text{ and } k \geq \lceil \rho \rceil, \end{cases} \tag{4}$$

where $\mathbb{N} = \{1,2,\ldots\}$ and $\mathbb{N}_0 = \{0,1,2,\ldots\}$.

Definition 6.*(See [2]) In terms of two parameters, the Mittag-Leffler function is given by*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+\beta)}],$$

such that $\alpha > 0$, $\beta > 0$ and z belongs to the complex plane $\mathbb{C}$.

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2 Main Results

Theorem 3. *If $K(s,u)$ is the RGT of the function $f(t)$, then FRGT of Riemann-Liouville fractional integral of order α for the function $f(t)$, as follows*

$$RG[I_t^\alpha f(t);(s,u)] = \frac{u^\alpha}{s^\alpha} K(s,u) \tag{5}$$

Proof. Using the Riemann-Liouville integral's definition, we obtain

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau = \frac{1}{\Gamma(\alpha)} [t^{\alpha-1} * f(t)], \tag{6}$$

where $*$ denotes convolution of two functions. Now taking RGT to both sides of the previous equation, then by using RGT-convolution theorem (See(1)) and table of RGT-transformations [10], we obtain

$$RG[I_t^\alpha f(t);(s,u)] = \frac{1}{\Gamma(\alpha)} u \frac{u^{\alpha-1}}{s^\alpha} \Gamma(\alpha) K(s,u) = \frac{u^\alpha}{s^\alpha} K(s,u) \tag{7}$$

Theorem 4. *Let $n \in \mathbb{N}$ and $\alpha > 0$ be such that $n-1 \leq \alpha < n$ and $K(s,u)$ be the RGT of $f(t)$, then the RGT of the Riemann-Liouville fractional derivative of order α for $f(t)$, is of the form*

$$RG[D_t^\alpha f(t);(s,u)] = \frac{s^\alpha}{u^\alpha} K(s,u) - \sum_{k=0}^{n-1} \frac{s^k}{u^{k+1}} [D_t^{\alpha-k-1} f(t)]_{t=0} \tag{8}$$

Proof. Let $D_t^\alpha f(t) = h^{(n)}(t) = \frac{d^n}{dt^n} h(t)$, we can write

$$\begin{aligned} h(t) &= \frac{d^{-n}}{dt^{-n}} \frac{d^n}{dt^n} h(t) \\ &= \frac{d^{-n}}{dt^{-n}} D_t^\alpha f(t) \\ &= I_t^{n-\alpha} f(t), \end{aligned} \tag{9}$$

now by taking RGT to equation (9), next using (5), we reach to

$$RG[h(t);(s,u)] = RG[I_t^{n-\alpha} f(t);(s,u)] = \frac{u^{n-\alpha}}{s^{n-\alpha}} K(s,u). \tag{10}$$

Also, from Theorem (2), we have

$$\begin{aligned} RG[D_t^\alpha f(t);(s,u)] &= RG[h^{(n)}(t);(s,u)] \\ &= \frac{s^n}{u^n} RG[h(t);(s,u)] - \sum_{k=0}^{n-1} \frac{s^{n-k-1} h^{(k)}(t)|_{t=0}}{u^{n-k}} \\ &= \frac{s^n}{u^n} \frac{u^{n-\alpha}}{s^{n-\alpha}} K(s,u) - \sum_{k=0}^{n-1} \frac{s^k h^{(n-k-1)}(t)|_{t=0}}{u^{k+1}} \\ &= \frac{s^\alpha}{u^\alpha} K(s,u) - \sum_{k=0}^{n-1} \frac{s^k h^{(n-k-1)}(t)|_{t=0}}{u^{k+1}}, \end{aligned} \tag{11}$$

the fractional derivative of Riemann-Liouville (See (2)) allows us to write

$$\begin{aligned} h^{(n-k-1)}(t)|_{t=0} &= \frac{d^{n-k-1}}{dt^{n-k-1}} h(t)|_{t=0} \\ &= \frac{d^{n-k-1}}{dt^{n-k-1}} D_t^{-(n-\alpha)} f(t)|_{t=0} \\ &= D_t^{(\alpha-k-1)} f(t)|_{t=0}, \end{aligned} \tag{12}$$

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hence, substituting from (12) into (11), we get

$$RG[D_t^{\alpha}f(t);(s,u)] = \frac{s^{\alpha}}{u^{\alpha}}K(s,u) - \sum_{k=0}^{n-1}\frac{s^k}{u^{k+1}}[D_t^{\alpha-k-1}f(t)]_{t=0}.$$

The proof is finished with this.

Theorem 5. *If $n \in \mathbb{N}$ and $\alpha > 0$ be such that $n-1 \leq \alpha < n$ and $K(s,u)$ be the RGT of $f(t)$, then the RGT formula of fractional derivative in the Caputo sense of order α for $f(t)$, takes the form*

$$RG[D_*^{\alpha}f(t);(s,u)] = \frac{s^{\alpha}}{u^{\alpha}}K(s,u) - \sum_{k=0}^{n-1}\frac{u^{k-\alpha}f^{(k)}(0)}{s^{k-\alpha+1}} \tag{13}$$

Proof. Utilizing the meaning of the Caputo fractional derivative, we are able to write

$$\begin{aligned} D_*^{\alpha}f(t) &= \frac{1}{\Gamma(n-\alpha)}\int_0^t f^{(n)}(u)(t-u)^{n-\alpha-1}du \\ &= \frac{1}{\Gamma(n-\alpha)}\int_0^t h(u)(t-u)^{n-\alpha-1}du \\ &= I_t^{n-\alpha}h(t) \end{aligned} \tag{14}$$

where $f^{(n)}(t) = h(t)$, now by taking RGT to both sides of equation (14), then using (5), we have

$$\begin{aligned} RG[D_*^{\alpha}f(t);(s,u)] = RG[I_t^{n-\alpha}h(t);(s,u)] &= \frac{u^{n-\alpha}}{s^{n-\alpha}}RG[h(t);(s,u)] \\ &= \frac{u^{n-\alpha}}{s^{n-\alpha}}RG[f^{(n)}(t);(s,u)] \\ &= \frac{u^{n-\alpha}}{s^{n-\alpha}}[\frac{s^n}{u^n}K(s,u) - \sum_{k=0}^{n-1}\frac{s^{n-k-1}f^{(k)}(0)}{u^{n-k}}] \\ &= \frac{s^{\alpha}}{u^{\alpha}}K(s,u) - \sum_{k=0}^{n-1}\frac{u^{k-\alpha}f^{(k)}(0)}{s^{k-\alpha+1}}. \end{aligned} \tag{15}$$

The proof is finished with this.

Theorem 6. *If $f(t)$ is an exponentially ordered piecewise continuous function on the range $[0,\infty)$, also suppose that $\alpha > 0$ be such that $0 \leq \alpha < 1$ and $K(s,u)$ be the RGT of $f(t)$, then the RGT formula of fractional derivative in the Caputo sense of order α for $f(t)$, takes the form*

$$RG[D_*^{n\alpha}f(t);(s,u)] = \frac{s^{n\alpha}}{u^{n\alpha}}K(s,u) - \sum_{k=0}^{n-1}\frac{u^{(k-n)\alpha}(D_*^{\alpha k}f)(0)}{s^{(k-n)\alpha+1}} \tag{16}$$

Proof. Using mathematical induction technique and RGT for Caputo fractional derivatives(See (13)), we obtain
For $n=1$, formula (16) becomes

$$RG[D_*^{\alpha}f(t);(s,u)] = \frac{s^{\alpha}}{u^{\alpha}}K(s,u) - \frac{u^{-\alpha}}{s^{-\alpha+1}}f(0),$$

which is true with RGT for Caputo fractional derivative (See (13)).
For $n=2$

$$RG[D_*^{2\alpha}f(t);(s,u)] = RG[D_*^{\alpha}(D_*^{\alpha}f(t));(s,u)] = RG[D_*^{\alpha}h(t);(s,u)], \tag{17}$$

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where $h(t) = D_*^\alpha f(t)$. Using RGT for Caputo fractional derivative (See (13)) twice, hence (17) becomes

$$\begin{aligned} RG[D_*^{2\alpha} f(t);(s,u)] &= \frac{s^\alpha}{u^\alpha} RG[h(t);(s,u)] - \frac{u^{-\alpha}}{s^{-\alpha+1}} h(0) \\ &= \frac{s^\alpha}{u^\alpha} RG[D_*^\alpha f(t);(s,u)] - \frac{u^{-\alpha}}{s^{-\alpha+1}} D_*^\alpha f(0) \\ &= \frac{s^\alpha}{u^\alpha}\left[\frac{s^\alpha}{u^\alpha} K(s,u) - \frac{u^{-\alpha}}{s^{-\alpha+1}} f(0)\right] - \frac{u^{-\alpha}}{s^{-\alpha+1}} D_*^\alpha f(0) \\ &= \frac{s^{2\alpha}}{u^2\alpha} K(s,u) - \frac{u^{-2\alpha}}{s^{-2\alpha+1}} f(0) - \frac{u^{-\alpha}}{s^{-\alpha+1}} D_*^\alpha f(0), \end{aligned} \tag{18}$$

which is compatible with the formula (16). Finally, assume that the formula is true for $n = r$, hence

$$RG[D_*^{r\alpha} f(t);(s,u)] = \frac{s^{r\alpha}}{u^{r\alpha}} K(s,u) - \sum_{k=0}^{r-1} \frac{u^{(k-r)\alpha}(D_*^{\alpha k} f)(0)}{s^{(k-r)\alpha+1}}, \tag{19}$$

we want to prove that it is true for $n = r+1$, now

$$RG[D_*^{(r+1)\alpha} f(t);(s,u)] = RG[D_*^\alpha (D_*^{r\alpha} f(t));(s,u)] = RG[D_*^\alpha z(t);(s,u)], \tag{20}$$

where $z(t) = D_*^{r\alpha} f(t)$. Using RGT for Caputo fractional derivative (See (13)) and (19), hence (17) becomes

$$\begin{aligned} RG[D_*^{(r+1)\alpha} f(t);(s,u)] &= \frac{s^\alpha}{u^\alpha} RG[z(t);(s,u)] - \frac{u^{-\alpha}}{s^{-\alpha+1}} z(0) \\ &= \frac{s^\alpha}{u^\alpha} RG[D_*^{r\alpha} f(t);(s,u)] - \frac{u^{-\alpha}}{s^{-\alpha+1}} D_*^{r\alpha} f(0) \\ &= \frac{s^\alpha}{u^\alpha}\left[\frac{s^{r\alpha}}{u^{r\alpha}} K(s,u) - \sum_{k=0}^{r-1} \frac{u^{(k-r)\alpha}(D_*^{\alpha k} f)(0)}{s^{(k-r)\alpha+1}}\right] - \frac{u^{-\alpha}}{s^{-\alpha+1}} D_*^{r\alpha} f(0) \\ &= \frac{s^{(r+1)\alpha}}{u^{(r+1)\alpha}} K(s,u) - \sum_{k=0}^{r-1} \frac{u^{(k-(r+1))\alpha}(D_*^{\alpha k} f)(0)}{s^{(k-(r+1))\alpha+1}} - \frac{u^{-\alpha}}{s^{-\alpha+1}} D_*^{r\alpha} f(0) \\ &= \frac{s^{(r+1)\alpha}}{u^{(r+1)\alpha}} K(s,u) - \sum_{k=0}^{r} \frac{u^{(k-(r+1))\alpha}(D_*^{\alpha k} f)(0)}{s^{(k-(r+1))\alpha+1}}. \end{aligned} \tag{21}$$

The proof is finished with this.

Theorem 7. *The RGT of $t^{\beta-1}E_{\alpha,\beta}(At^\alpha)$, is given by*

$$RG[t^{\beta-1}E_{\alpha,\beta}(At^\alpha);(s,u)] = u^{\beta-1}s^{\alpha-\beta}(s^\alpha - Au^\alpha)^{-1}, \tag{22}$$

given that
$\alpha > 0$, $\beta > 0$ and $A \in \mathbb{C}^{n\times n}$ with $\mathbb{C}$ a complex plane, such that the real part of the complex number $\frac{s}{u}$ is greater than $\|A\|^{\frac{1}{\alpha}}$.

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*Proof.*Using Mittag-Leffler function definition and RGT table of transformations (see [10]) , we obtain

$$\begin{aligned} RG[t^{\beta-1}E_{\alpha,\beta}(At^{\alpha});(s,u)] &= RG[t^{\beta-1}\sum_{k=0}^{\infty}\frac{(At^{\alpha})^k}{\Gamma(\alpha k+\beta)}] \\ &= \sum_{k=0}^{\infty}\frac{A^k RG[t^{\alpha k+\beta-1};(s,u)]}{\Gamma(\alpha k+\beta)} \\ &= \sum_{k=0}^{\infty}\frac{A^k u^{\alpha k+\beta-1}}{s^{\alpha k+\beta}} \\ &= \frac{1}{u}\sum_{k=0}^{\infty}A^k(\frac{s}{u})^{-\alpha k-\beta} \\ &= \frac{s^{\alpha-\beta}}{u^{\alpha-\beta+1}}\sum_{k=0}^{\infty}A^k(\frac{s}{u})^{-\alpha(k+1)} \\ &= \frac{s^{\alpha-\beta}}{u^{\alpha-\beta+1}}((\frac{s}{u})^{\alpha}-A)^{-1} \\ &= u^{\beta-1}s^{\alpha-\beta}(s^{\alpha}-Au^{\alpha})^{-1}. \end{aligned} \tag{23}$$

This completes the proof.

3 Examples

In this part, we look at a few examples of fractional differential equations that include both homogeneous and non-homogeneous, which were previously solved by Laplace or Sumudu transforms, and here we solve them by RGT method to illustrate the relevance of our work.

Example 1(See [8, Example 3.2]). Take into consideration the following non-homogenous differential equation of fractional type involving fractional derivative of Riemann-Liouville

$$D_t^{\alpha}y(t)-ay(t)=h(t),\ \ n-1<\alpha<n \tag{24}$$

with non-zero starting conditions

$$D_t^{(\alpha-k)}y(t)=b_k,\ \ k=1,2,.....,n$$

where c and b_k are constants and the fractional derivative of Riemann-Liouville of order α represented by D_t^{α}.
Taking RGT to both sides of equation (24) and using (8), we obtain

$$\frac{s^{\alpha}}{u^{\alpha}}K_1(s,u)-\sum_{k=1}^{n}\frac{s^{k-1}}{u^k}D_t^{(\alpha-k)}y(t)\mid_{t=0}-aK_1(s,u)=K_2(s,u), \tag{25}$$

where $K_1(s,u)$, and $K_2(s,u)$ are RGT functions of $y(t)$ and $h(t)$ respectively, now (25) takes the form

$$\begin{aligned} K_1(s,u) &= \frac{u^{\alpha}K_2(s,u)}{(s^{\alpha}-au^{\alpha})}+\frac{\sum_{k=1}^{n}b_k u^{\alpha-k}s^{k-1}}{(s^{\alpha}-au^{\alpha})} \\ &= uu^{\alpha-1}(s^{\alpha}-au^{\alpha})^{-1}K_2(s,u)+\sum_{k=1}^{n}b_k u^{\alpha-k}s^{k-1}(s^{\alpha}-au^{\alpha})^{-1} \\ &= uRG[t^{\alpha-1}E_{\alpha,\alpha}(at^{\alpha});(s,u)]RG[h(t);(s,u)]+\sum_{k=1}^{n}b_k RG[t^{\alpha-k}E_{\alpha,\alpha-k+1}(at^{\alpha});(s,u)]. \end{aligned} \tag{26}$$

Taking inverse RGT to both sides of equation (26)and using the convolution theorem for Ramadan group (See (1)), we get

$$y(t)=\int_0^t h(\tau)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(a(t-\tau)^{\alpha})+t^{\alpha-k}E_{\alpha,\alpha-k+1}(at^{\alpha}).$$

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Example 2(See [8, Example 3.4]). Think about the following homogenous differential equation of fractional type involving Caputo fractional derivative

$$D_*^\alpha y(t) + ay(t) = 0, \ \ 0 < \alpha \leq 1, \tag{27}$$

with non-zero initiating condition

$$y(0) = c$$

where c and a are constants and D_*^α denote Caputo fractional derivative of order α.
Taking RGT to both sides of equation (27) and using (13), we obtain

$$\frac{s^a}{u^a} K(s,u) - \frac{s^{\alpha-1}}{u^{-\alpha}} c + aK(s,u) = 0, \tag{28}$$

where $K(s,u)$ is the RGT of $y(t)$, hence we can write

$$K(s,u) = s^{\alpha-1}(s^\alpha + au^\alpha)^{-1} c$$

Inverse RGT is applied to both sides of the previous equation, yielding

$$y(t) = E_{\alpha,1}(-at^\alpha) c.$$

*Example 3.*Think about the following non-homogenous differential equation of fractional type involving Caputo fractional derivative

$$D_*^a f(x) + D_*^b f(x) = g(x) \tag{29}$$

with non-zero initial condition $f(0) = c$, where c is a constant and D_*^a, D_*^b represents fractional derivatives of order a, and b respectively in the Caputo sense, such that $0 < a < b < 1$.
Taking RGT to both sides of equation (29) and using (13), we obtain

$$\frac{s^a}{u^a} K_1(s,u) - \frac{s^{a-1}}{u^a} c + \frac{s^b}{u^b} K_1(s,u) - \frac{s^{b-1}}{u^b} c = K_2(s,u), \tag{30}$$

where $K_1(s,u)$ and $K_2(s,u)$ are RGT of $f(x)$ and $g(x)$ respectively, hence we can write

$$[\frac{s^a}{u^a} + \frac{s^b}{u^b}] K_1(s,u) = [\frac{s^{a-1}}{u^a} + \frac{s^{b-1}}{u^b}] c + K_2(s,u),$$

and so

$$\begin{aligned} K_1(s,u) &= \frac{K_2(s,u)}{[\frac{s^a}{u^a} + \frac{s^b}{u^b}]} + \frac{c}{s} \\ &= uK_2(s,u) \frac{u^{b-1} s^{-a}}{(s^{b-a} + u^{b-a})} + \frac{c}{s}, \end{aligned} \tag{31}$$

taking the inverse Ramadan Group and using (23), we obtain

$$\begin{aligned} f(x) &= g(x) \star t^{b-1} E_{b-a,b}(-t^{b-a}) + c \\ &= \int_0^x g(\tau)(t-\tau)^{b-1} E_{b-a,b}(-(t-\tau)^{b-a}) + c \end{aligned}$$

4 Conclusion

In this paper, we deduce and demonstrate the Fractional Ramadan Group transform (FRGT) of fractional derivatives, which represents a generalization of both Laplace and Sumudu fractional derivative transforms. Some key formulae for FRGT are stated and demonstrated. These formulae are used to solve non-zero initial conditions homogeneous and non-homogeneous fractional differential equations.

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