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Double Ramadan Group Integral Transform: Definition and Properties with Applications to Partial Differential Equations

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Abstract: The main focus of this paper is to introduce the definition of double Ramadan Group integral$(RGT)_2$transform as an extension to RG introduced recently by the authors. The convergence theorem of the transform is stated and proved and some basic properties of the double RG transform are discussed. The relationship of this double transform to the double Laplace and Sumudu transforms is obtained. The applicability of this relatively new double transform is demonstrated to solve initial and boundary value problems in applied mathematics, and mathematical physics.

Keywords: Ramadan group transform; Laplace transform; Sumudu transform; partial differential equations

1 Introduction

Partial differential equations have paramount importance in mathematics as well as in other branches of science. Therefore, it is very important to know of the methods to solve such partial differential equations. One of the most known methods to solve partial differential equations is the integral transform method presented by Duff [1], and by Estrin and Higgins [2]. Eltayeb and Kilicman, (see [3]), Kilicman and Eltayeb [4] have established and studied the relationship between the double Sumudu transform and the double Laplace transform, as well as their applications to differential equations. Singh and Mandia [5] have established the relation between the double Laplace transform and the double Mellin transform and discussed their applications. Quite recently, Eltayeb and Kilicman [6] have applied the double Laplace transform to solve the general linear telegraph and partial integro-differential equations.

This paper is organized as follows: in section 2, we first introduce the definition of the double Ramadan group integral $(RGT)_2$transform as an extension to RG which is introduced recently by Raslan et al. [7] and is applied for solving Nonlinear Partial Differential Equations, (see Ramadan and Hadhoud [8]) The convolution theorem is investigated for RG integral transform, see Ramadan [9]. In section 3, the convergence of the proposed definition, or double RG transform, is investigated, and some of its basic properties are discussed. The relationship of this double transform to the double Laplace and Sumudu transforms is obtained and presented in section 4. In section 5, $(RGT)_2$ for integral of functions of two variables are investigated and developed. The kernel of this study centers on how to implement $(RGT)_2$ for double integrals. In section 6, the applicability of this relatively new double transform is demonstrated to solve initial and boundary value problems in applied mathematics, and mathematical physics.

2 Definition of double RG integral transform

A new integral RG transform defined for functions of exponential order, is proclaimed [8]. We consider functions in the set$A$, defined by:

$$A=\left\{ f(x,t) such that \exists M, \tau_1, \tau_2>0, |f(x,t)|\leq Me^{\frac{x+t}{\tau_i^2}}, \right.$$
$$\left. i=1,2 \, if \, f(x,t)\in R_+^2 \right\}$$

Definition 1.*The double Ramadan group integral transform of the function $f(x,t)$ which is defined in the*

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set A in the positive quadrant of the xt- plane is denoted by $(RGT)_2$ *and defined by:*

$$\begin{aligned} K(s,p,u,v) &= RG_2[f(x,t);(s,p,u,v)] \\ &= \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{pt}{v}\right)} f(x,t)\,dxdt, \end{aligned} \tag{1}$$

where s,p,u*and* v*are complex variables with* s*and* p*are the transform variables for* x*and*t*, respectively and* $u,v \in (-\tau_1,\tau_2)$*where* $\tau_1,\tau_2 > 0$*and* $Re(s), Re(p) > 0$.

The above definition can be written in detail as:

$$\begin{aligned} K(s,p,u,v) &= RG_2[f(x,t);(s,p,u,v)] \\ &= RG[RG[f(x,t);(s,p,u,v)];(p,v)] \\ &= RG\left[\frac{1}{u}\int_0^\infty e^{-\left(\frac{sx}{u}\right)} f(x,t)\,dx;(p,v)\right] \\ &= \frac{1}{v}\int_0^\infty e^{-\left(\frac{pt}{v}\right)}\left[\frac{1}{u}\int_0^\infty e^{-\left(\frac{sx}{u}\right)} f(x,t)\,dxdt\right] \\ &= \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{pt}{v}\right)} f(x,t)\,dxdt, \end{aligned}$$

whenever the integer exists.

Considering the definition of double Laplace (L_2) introduced by Dhunde and Waghmare [10] and double Sumudu (S_2) presented by Tchunche and Mbare [11] given respectively as:

$$L_2[f(x,t);(s,p)] = F(s,p) = \int_0^\infty\int_0^\infty e^{-(sx+pt)} f(x,t)dxdt$$

and

$$S_2[f(x,t);(s,p)] = G(s,p) = \frac{1}{sp}\int_0^\infty\int_0^\infty e^{-\left(\frac{x}{s}+\frac{t}{p}\right)} f(x,t)dxdt$$

and $(RGT)_2$ of the function $f(x,t)$ the following duality relations can be proved easily.

1.$K(s,p,u,v) = \frac{1}{uv}F\left(\frac{s}{u},\frac{p}{v}\right)$.

*Proof.*We have

$$\begin{aligned} F(s,p) &= L_2[f(x,t);(s,p)] \\ &= \int_0^\infty\int_0^\infty e^{-(sx+pt)} f(x,t)\,dxdt, \end{aligned}$$

then,

$$\begin{aligned} F\left(\frac{s}{u},\frac{p}{v}\right) &= \int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{pt}{v}\right)} f(x,t)\,dxdt \\ &= uv\,\frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{pt}{v}\right)} f(x,t)\,dxdt \\ &= uvRG_2[f(x,t);(s,p,u,v)] = uvK(s,p,u,v). \end{aligned}$$

2.$K(1,1,u,v) = G(u,v)$.

Proof.

$$K(s,p,u,v) = \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{py}{v}\right)} f(x,y)\,dxdy$$

Setting $s = p = 1$, we get

$$K(1,1,u,v) = \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\left(\frac{x}{u}+\frac{y}{v}\right)} f(x,y)\,dxdy,$$

Let $\frac{x}{u} = r; \frac{y}{v} = t$, We get

$$x = ur, dx = udr, y = vt, dy = udt$$

Then

$$\begin{aligned} K(1,1,u,v) &= \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-(r+t)} f(ur,vt)\,uvdrdt \\ &= \int_0^\infty\int_0^\infty e^{-(r+t)} f(ur,vt)\,drdt. \end{aligned}$$

Thus $K(1,1,u,v) = G(u,v)$.

3 The Convergence Theorem of $(RGT)_2$ Transform

In this theorem we prove the convergence of double*RG* integral transform defined in (1).

Theorem 1.*Convergence of Double Ramadan Group Integral* $(RGT)_2$ *Transform.*

Let $g(x,t)$ *be a function of two variables continuous in the first quadrant of the xt- plane. If the integral, defined for* $(RGT)_2$*, of the form*

$$\frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{pt}{v}\right)} g(x,t)\,dxdt, \tag{2}$$

converges at $s = s_0, u = u_0, p = p_0, v = v_0$*,then the integral converges for* $s > s_0, u > u_0, p > p_0$*and* $v > v_0$*,where we assume* $\frac{s}{u} - \frac{s_0}{u_0} > 0$*and* $\frac{p}{p_0} - \frac{v}{v_0} > 0$.

*Proof.*We rewrite the integral (2) as

$$\begin{aligned} &\frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{pt}{v}\right)} g(x,t)\,dxdt \\ &= \frac{1}{u}\int_0^\infty e^{-\left(\frac{sx}{u}\right)}\left[\frac{1}{v}\int_0^\infty e^{-\left(\frac{pt}{v}\right)} g(x,t)\,dt\right]dx \\ &= \frac{1}{u}\int_0^\infty e^{-\left(\frac{sx}{u}\right)} h(x,s)\,dx \end{aligned} \tag{3}$$

where

$$h(x,s) = \frac{1}{v}\int_0^\infty e^{-\left(\frac{pt}{v}\right)} g(x,t)\,dt \tag{4}$$

First, we need to prove that the integral (4) converges for$p > p_0, v > v_0$, where $\frac{p}{p_0} - \frac{v}{v_0} > 0$. Assume that integral (3) converges at $p = p_0$ and $v = v_0$. Now, set

$$\varphi(x,t) = \frac{1}{v_0}\int_0^\infty e^{-\left(\frac{p_0 r}{v_0}\right)} g(x,r)\,dr, 0 < t < \infty.$$

It is clear that $\varphi(x,0)=0$ and $\lim_{t\to\infty}\varphi(x,t)$ exists.

Since the integral $\frac{1}{v}\int_0^\infty e^{-\left(\frac{pt}{v}\right)}g(x,t)\,dt$ converges at $p=p_0$ and $v=v_0$.

Using the fundamental theorem of calculus, we have

$$\varphi_t(x,t)=\frac{1}{v_0}e^{-\left(\frac{p_0t}{v_0}\right)}g(x,r). \tag{5}$$

Choose $\varepsilon$and λ so that $0<\varepsilon<\lambda$, then from (4), we get

$$\begin{aligned}\frac{1}{v}\int_0^\infty e^{-\left(\frac{pt}{v}\right)}g(x,t)\,dt&=\frac{1}{v}\int_\varepsilon^\lambda e^{-\left(\frac{pt}{v}\right)}e^{\left(\frac{p_0t}{v_0}\right)}\varphi_t(x,t)\,dt\\&=\frac{1}{v}\int_\varepsilon^\lambda e^{-\left(\frac{p}{v}-\frac{p_0}{v_0}\right)t}\varphi_t(x,t)\,dt\end{aligned}$$

Applying the integration by parts, we obtain

$$\begin{aligned}&\frac{1}{v}\int_\varepsilon^\lambda e^{-\left(\frac{pt}{v}\right)}g(x,t)\,dt\\&=\frac{1}{v}\left[\left[e^{-\left(\frac{p}{v}-\frac{p_0}{v_0}\right)t}\varphi(x,t)\right]_{t=\varepsilon}^{\lambda}\right.\\&\qquad\left.-\int_\varepsilon^\lambda-\left(\frac{p}{v}-\frac{p_0}{v_0}\right)e^{-\left(\frac{p}{v}-\frac{p_0}{v_0}\right)t}\varphi(x,t)\,dt\right]\\&=\frac{1}{v}\left[e^{-\left(\frac{p}{v}-\frac{p_0}{v_0}\right)\lambda}\varphi(x,\lambda)-e^{-\left(\frac{p}{v}-\frac{p_0}{v_0}\right)\varepsilon}\varphi(x,\varepsilon)\right.\\&\qquad\left.+\left(\frac{p}{v}-\frac{p_0}{v_0}\right)\int_\varepsilon^\lambda e^{-\left(\frac{p}{v}-\frac{p_0}{v_0}\right)t}\varphi(x,t)\,dt\right].\end{aligned}$$

Now, as$\varepsilon\to0$, the second term on the right hand side vanishes and we get

$$\begin{aligned}\frac{1}{v}\int_0^\lambda e^{-\left(\frac{pt}{v}\right)}g(x,t)\,dt=&\frac{1}{v}\left[e^{-\left(\frac{p}{v}-\frac{p_0}{v_0}\right)\lambda}\varphi(x,\lambda)\right.\\&\left.+\left(\frac{p}{v}-\frac{p_0}{v_0}\right)\int_0^\lambda e^{-\left(\frac{p}{v}-\frac{p_0}{v_0}\right)t}\varphi(x,t)\,dt\right].\end{aligned} \tag{6}$$

As $\lambda\to\infty$, it is clear that the first term on the right hand side of equation (6) approaches the zero since from assumption $\frac{p}{v}-\frac{p_0}{v_0}>0$.

Finally, we have

$$\frac{1}{v}\int_0^\infty e^{-\left(\frac{pt}{v}\right)}g(x,t)\,dt=\left(\frac{p}{v}-\frac{p_0}{v_0}\right)\int_0^\infty e^{-\left(\frac{p}{v}-\frac{p_0}{v_0}\right)t}\varphi(x,t)\,dt. \tag{7}$$

Using the limit test for converges; see Widder [[12], pages 370-371], the integral on the right converges, since from this test one has

$$\begin{aligned}\lim_{t\to\infty}t^2e^{-\left(\frac{p}{v}-\frac{p_0}{v_0}\right)t}\varphi(x,t)&=\lim_{t\to\infty}t^2\frac{1}{e^{\left(\frac{p}{v}-\frac{p_0}{v_0}\right)t}}\lim_{t\to\infty}\varphi(x,t)\\&=0*\lim_{t\to\infty}\varphi(x,t)\\&=0=\text{finite}.\end{aligned}$$

Hence, the integral (4) converges for $p>p_0, v>v_0$, with $\frac{p}{p_0}-\frac{v}{v_0}>0$.

Similarly, we can prove that the integral $\frac{1}{u}\int_0^\infty e^{-\left(\frac{sx}{u}\right)}h(x,t)\,dx$ converges for$u>u_0, s>s_0$, with $\frac{s}{s_0}-\frac{u}{u_0}>0$.

Thus, the integral on the right hand side of (3) converges for $s>s_0, u>u_0, p>p_0$ and $v>v_0$, with the assumptions $\frac{s}{u}-\frac{s_0}{u_0}>0$ and $\frac{p}{p_0}-\frac{v}{v_0}>0$.

Therefore, the $(RGT)_2$ transform defined by (1) converges and the proof of the theorem is then complete.

4 The application of the double Ramadan group integral $(RGT)_2$ transform of partial differential derivatives

First let $f(x,t)$ be a function defined in the positive quadrant of the xt- plane. The double $(RGT)_2$ transform of the first and second partial derivatives of $f(x,t)$ are given by

$$RG_2\left[\frac{\partial f(x,t)}{\partial x};(s,p,u,v)\right]=\frac{s}{u}K(s,p,u,v)-\frac{1}{u}K(0,p,0,v). \tag{8}$$

where s,p,u and v are complex variables with s and p are the transform variables for x and t, respectively and $u,v\in(-\tau_1,\tau_2)$ where $\tau_1,\tau_2>0$ and $Re(s), Re(p)>0$, and $K(0,p,0,v)$is defined as

$$K(0,p,0,v)=\frac{1}{v}\int_0^\infty e^{-\left(\frac{pt}{v}\right)}f(0,t)\,dt,$$

To prove (8), we use the definition of (RG_2) transform defined in (1) from which we have

$$\begin{aligned}&RG_2\left[\frac{\partial f(x,t)}{\partial x};(s,p,u,v)\right]=RG\left[RG\left[\frac{\partial f(x,t)}{\partial x};(s,u)\right];(p,v)\right]\\&=RG\left[\frac{1}{u}\int_0^\infty e^{-\left(\frac{sx}{u}\right)}\frac{f(x,t)}{\partial x}dx;(p,v)\right]\\&=RG\left[\frac{1}{u}\left[\left[e^{-\left(\frac{sx}{u}\right)}f(x,t)\right]_{x=0}^\infty+\frac{s}{u}\int_0^\infty e^{-\left(\frac{sx}{u}\right)}f(x,t)\,dx\right];(p,v)\right]\\&=RG\left[\frac{1}{u}\left[-f(0,t);(p,v)\right]+RG\left[\frac{s}{u^2}\int_0^\infty e^{-\left(\frac{sx}{u}\right)}f(x,t)\,dx;(p,v)\right]\right]\\&=-\frac{1}{u}\frac{1}{v}\int_0^\infty e^{-\left(\frac{pt}{v}\right)}f(0,t)\,dt+\frac{s}{vu^2}\int_0^\infty\int_0^\infty e^{-\left(\frac{pt}{v}\right)}e^{-\left(\frac{sx}{u}\right)}f(x,t)\,dxdt\\&=-\frac{1}{u}\bar{K}(0,P,0,v)+\frac{s}{u}\left[\frac{1}{vu}\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{pt}{v}\right)}f(x,t)\,dxdt\right]\\&=-\frac{1}{u}\bar{K}(0,P,0,v)+\frac{s}{u}K(s,p,u,v).\end{aligned}$$

Then ,

$$RG_2\left[\frac{\partial f(x,t)}{\partial x};(s,p,u,v)\right]=\frac{s}{u}K(s,p,u,v)-\frac{1}{u}\bar{K}(0,p,0,v).$$

Similarly, we can show that

$$RG_2\left[\frac{\partial f(x,t)}{\partial t};(s,p,u,v)\right]=\frac{p}{v}K(s,p,u,v)-\frac{1}{v}\bar{K}(s,0,u,0).$$

Now for the second partial derivatives $\frac{\partial^2 f(x,t)}{\partial x^2}$ one can prove easily that

$$RG_2\left[\frac{\partial^2 f(x,t)}{\partial x^2};(s,p,u,v)\right] = \frac{s^2}{u^2}K(s,p,u,v) - \frac{s}{u^2}K(0,p,0,v) - \frac{s}{u}\frac{\partial K(0,p,0,v)}{\partial x}.$$

In general, we can conclude for the n^{th} partial derivatives $\frac{\partial^n f(x,t)}{\partial x^n}$, its double RG transform is as:

$$RG_2[\frac{\partial^n f(x,t)}{\partial x^n};(s,p,u,v)] = \frac{s^n}{u^n}K(s,p,u,v) - \frac{s^{n-1}}{u^n}K(0,p,0,v) - \frac{s^{n-2}}{u^{n-1}}\frac{\partial}{\partial x}K(0,p,0,v) - \cdots\cdots\cdots\cdots - \frac{1}{u}\frac{\partial^n}{\partial x^n}K(0,p,o,v).$$

5 Double Ramadan group integral transform of double integral

In this section, we investigate and develop $(RGT)_2$ for the integral of functions of two variables. Mainly, we study how to implement this integral transform for double integrals. We state and prove the following theorem which will be useful for applying $(RGT)_2$ to solve integro-partial differential equations.

Theorem 2.*Suppose that the double integral*

$$\int_0^x \int_0^t f(\beta,w)dwd\beta$$

is well defined, then the double Ramadan group integral $(RGT)_2$ transform of this double integral is given by

$$(RGT)_2\left[\int_0^x \int_0^t f(x,t)dxdt;(p,s,u,v)\right] = \left(\frac{uv}{ps}\right)K(s,p,u,v),$$

where

$$K(s,p,u,v) = (RGT)_2\left[f(x,t);(s,p,u,v)\right].$$

*Proof.*Denote to the integral

$$\int_0^x \int_0^t f(\beta,w)dwd\beta,$$

by the function

$$\varphi(x,t) = \int_0^x \int_0^t f(\beta,w)dwd\beta. \quad (9)$$

Also, denote to this integral

$$\int_0^t f(x,w)dw,$$

by the function

$$\psi(x,t) = \int_0^t f(x,w)dw. \quad (10)$$

Take the derivative of both sides of Eq. (10) with respect to the variable t, we get

$$\frac{\partial\psi(x,t)}{\partial t} = f(x,t), \quad (11)$$

where it is clear that

$$\psi(x,0) = 0. \quad (12)$$

$$\left(\frac{p}{v}\right)\overline{\psi}(s,p,u,v) - \left(\frac{1}{v}\right)\overline{\psi}(s,0,u,0) = K(s,p,u,v). \quad (13)$$

Take also the single RGT of Eq. (12), and then we have

$$\overline{\psi}(s,0,u,0) = 0. \quad (14)$$

Hence, from (13) and (14), we get

$$\overline{\psi}(s,p,u,v) = \left(\frac{v}{p}\right)K(s,p,u,v). \quad (15)$$

Also, from (10), one care further writes Eq. (9) as:

$$\varphi(x,t) = \int_0^x \psi(\beta,t)d\beta,$$

and from which, we have

$$\frac{\partial\varphi(x,t)}{\partial x} = \psi(x,t), \quad (16)$$

with

$$\varphi(0,t) = 0. \quad (17)$$

Similarly, take $(RGT)_2$ of Eq. (16) and single RGT of Eq. (17), we get respectively

$$\left(\frac{s}{u}\right)\overline{\varphi}(p,s,u,v) - \left(\frac{1}{u}\right)\overline{\varphi}(0,p,0,v) = \overline{\psi}(s,p,u,v), \quad (18)$$

and

$$\overline{\varphi}(0,p,0,v) = 0. \quad (19)$$

Substituting from Eq. (19) in Eq. (18), we obtain

$$\overline{\varphi}(s,p,u,v) = \left(\frac{u}{v}\right)\overline{\psi}(s,p,u,v). \quad (20)$$

Finally, from Eq. (15) and (20), we get

$$\bar{\varphi}(s,p,u,v) = (\frac{uv}{sp})K(s,p,u,v)$$

That is,

$$(RGT)_2\left[\int_0^x \int_0^t f(\beta,w)dwd\beta;(s,p,u,v)\right] = \left(\frac{uv}{ps}\right)\overline{K}(s,p,u,v),$$

and the theorem is thus proved.

6 Application of Double Ramadan group integral $(RGT)_2$ Transform

As stated in the introduction of this paper, the double Ramadan Group transform can be used as an effective tool for solving partial differential and integro-partial differential equations. Hence it will be illustrated by some numerical examples. It is well known that in order to obtain the solution of partial differential equations by integral transform methods we need the following two steps:

Firstly, we transform the partial differential equations to algebraic equations by using double Ramadan Group transform method.

Secondly, on using inverse double Ramadan Group transform we get the solution of PDEs.

*Example 1.*Consider the linear homogenous telegraph equation in the form:

$$U_{xx} - U_{tt} - U_t - U = 0 \tag{21}$$

with initial conditions

$$U(x,0) = e^x,\, U_t(x,0) = -e^x \tag{22}$$

and boundary conditions:

$$U(0,t) = e^{-t},\, U_t(0,t) = -e^x \tag{23}$$

Solution

Take the double Ramadan group integral transform $(\mathrm{RGT})_2$ of Eq.(21) and single RGT of the conditions (22) , (23), then we have:

$$\frac{s^2}{u^2}K(s,p,u,v) - \frac{s}{u^2}K(0,p,0,v) - \frac{1}{u}\frac{\partial}{\partial x}K(0,p,0,v) - \frac{p^2}{v^2}K(s,p,u,v) + \frac{p}{v^2}K(s,0,u,0) + \frac{1}{v}\frac{\partial}{\partial t}K(s,0,u,0) - \frac{p}{v}K(s,p,u,v) + \frac{1}{v}K(s,0,u,0) - K(s,p,u,v) = 0 \tag{24}$$

where, from initial and boundary condition

$$\begin{aligned} K(0,p,0,v) &= (RG)\left[U(0,t);(0,p,0,v)\right] \\ &= \frac{1}{v}\int_0^\infty e^{-\frac{p}{v}t}U(0,t)\,dt \\ &= \frac{1}{v}\int_0^\infty e^{-\frac{p}{v}t}e^{-t}dt = \frac{1}{v(\frac{p}{v}+1)} \end{aligned} \tag{25}$$

and

$$\begin{aligned} K(s,0,u,0) &= (RG)\left[U(x,0);(s,0,u,0)\right] \\ &= \frac{1}{u}\int_0^\infty e^{-\frac{s}{u}x}U(x,0)dx \\ &= \frac{1}{u}\int_0^\infty e^{-\frac{s}{u}x}e^x dx \\ &= \frac{1}{u}\int_0^\infty e^{-(\frac{s}{u}-1)x}dx = \frac{1}{u(\frac{s}{u}-1)} \end{aligned} \tag{26}$$

while,

$$\begin{aligned} \frac{\partial}{\partial x}K(0,p,0,v) &= \frac{1}{v}\int_0^\infty e^{-\frac{p}{v}t}\frac{\partial}{\partial x}U(0,t)dt \\ &= \frac{1}{v}\int_0^\infty e^{-\frac{p}{v}t}e^{-t}dt \\ &= \frac{1}{v}\int_0^\infty e^{-(\frac{p}{v}+1)t}dt = \frac{1}{v(\frac{p}{v}+1)} \end{aligned} \tag{27}$$

and

$$\begin{aligned} \frac{\partial}{\partial t}K(s,0,u,0) &= \frac{1}{u}\int_0^\infty e^{-\frac{s}{u}x}U_t(x,0)dx \\ &= -\frac{1}{u}\int_0^\infty e^{-\frac{s}{u}x}e^x dx \\ &= -\frac{1}{u}\int_0^\infty e^{-(\frac{s}{u}-1)x}dx = -\frac{1}{u(\frac{s}{u}-1)} \end{aligned} \tag{28}$$

Now, substituting Eqns. (25)-(28) in Eq. (24) we get

$$\begin{aligned} &\left(\frac{s^2}{u^2} - \frac{p^2}{v^2} - \frac{p}{v} - 1\right)K(s,p,u,v) \\ &= \frac{s}{u^2}K(0,p,0,v) + \frac{1}{u}\frac{\partial}{\partial x}K(0,p,0,v) \\ &- \frac{p}{v^2}K(s,0,u,0) - \frac{1}{v}\frac{\partial}{\partial t}K(s,0,u,0) - \frac{1}{v}K(s,0,u,0) \\ &= \frac{s}{u^2v(\frac{p}{v}+1)} + \frac{1}{uv(\frac{p}{v}+1)} \\ &- \frac{p}{uv^2(\frac{s}{u}-1)} + \frac{1}{uv(\frac{s}{u}-1)} - \frac{1}{uv(\frac{s}{u}-1)} \\ &= \frac{\frac{s^2}{u^2} - \frac{p^2}{v^2} - \frac{p}{v} - 1}{uv(\frac{p}{v}+1)(\frac{s}{u}-1)} \end{aligned}$$

Hence,

$$K(s,p,u,v) = \frac{1}{uv(\frac{p}{v}+1)(\frac{s}{u}-1)} = \frac{1}{v(\frac{p}{v}+1)}.\frac{1}{u(\frac{s}{u}-1)} \tag{29}$$

By taking inverse of the double Ramadan group integral transform for (29), we get the solution of (21) in the following form $U(x,t) = e^{x-t}$.

*Example 2.*Consider the wave equation in the form:

$$U_{tt} - U_{xx} = 3\left(e^{x+2t} - e^{2x+t}\right) \qquad (x,t) \in R_+^2 \tag{30}$$

With initial conditions

$$U(x,0) = e^{2x} + e^x,\, U_t(x,0) = e^{2x} + 2e^x \tag{31}$$

and boundary conditions

$$U(0,t) = e^t + e^{2t},\, U_x(0,t) = 2e^t + e^{2t} \tag{32}$$

Solution

Take the double Ramadan group integral transform (RGT)2 of Eq.(31) and single RGT of the conditions (32), (33), then we have

$$\begin{aligned}&\frac{p^2}{v^2}K(s,p,u,v)-\frac{p}{v^2}K(s,0,u,0)-\frac{1}{v}\frac{\partial}{\partial t}K(s,0,u,0)\\&-\frac{s^2}{u^2}K(s,p,u,v)+\frac{s}{u^2}K(0,p,0,v)+\frac{1}{u}\frac{\partial}{\partial x}K(0,p,0,v)\\&=(RGT)_2 3\left[e^{x+2t}-e^{2x+t}\right]\end{aligned} \tag{33}$$

where, from initial and boundary conditions we have

$$\begin{aligned}K(s,0,u,0)&=(RG)\,[U(x,0);(s,0,u,0)]\\&=\frac{1}{u}\int_0^\infty e^{-\frac{s}{u}x}U(x,0)dx\\&=\frac{1}{u}\int_0^\infty e^{-\frac{s}{u}x}e^{2x+x}dx\\&=\frac{1}{u}[\int_0^\infty e^{-(\frac{s}{u}-2)x}dx+\int_0^\infty e^{-(\frac{s}{u}-1)x}dx]\\&=\frac{1}{u(\frac{s}{u}-2)}+\frac{1}{u(\frac{s}{u}-1)}\end{aligned} \tag{34}$$

Also, we have

$$\begin{aligned}K(0,p,0,v)&=(RG)\,[U(0,t);(0,p,0,v)]\\&=\frac{1}{v}\int_0^\infty e^{-\frac{p}{v}t}U(0,t)dt\\&=\frac{1}{v}\int_0^\infty e^{-\frac{p}{v}t}[e^t+e^{2t}]dt\\&=\frac{1}{v}\left[\int_0^\infty e^{-(\frac{p}{v}-1)t}dt-\int_0^\infty e^{-(\frac{p}{v}-2)t}dt\right]\\&=\frac{1}{v(\frac{p}{v}-1)}+\frac{1}{v(\frac{p}{v}-2)}\end{aligned} \tag{35}$$

Moreover, we have

$$\begin{aligned}\frac{\partial}{\partial t}K(s,0,u,0)&=\frac{1}{u}\int_0^\infty e^{-\frac{s}{u}x}U_t(x,0)dx\\&=-\frac{1}{u}\int_0^\infty e^{-\frac{s}{u}x}[e^{2x}+2e^x]dx\\&=\frac{1}{u}\int_0^\infty e^{-(\frac{s}{u}-2)x}dx+\frac{2}{u}\int_0^\infty e^{-(\frac{s}{u}-1)x}dx\\&=\frac{1}{u(\frac{s}{u}-2)}+\frac{2}{u(\frac{s}{u}-1)}\end{aligned} \tag{36}$$

and

$$\begin{aligned}\frac{\partial}{\partial x}K(0,p,0,v)&=\frac{1}{v}\int_0^\infty e^{-\frac{p}{v}t}U_x(0,t)dt\\&=\frac{1}{v}\int_0^\infty e^{-\frac{p}{v}t}[2e^t+e^{2t}]dt\\&=\frac{2}{v}\int_0^\infty e^{-(\frac{p}{v}-1)t}dt+\frac{1}{v}\int_0^\infty e^{-(\frac{p}{v}-2)t}dt\\&=\frac{2}{v(\frac{p}{v}-1)}+\frac{1}{v(\frac{p}{v}-2)}\end{aligned} \tag{37}$$

where

$$(RGT)_2 3[e^{x+2t}-e^{2x+t}]=3(RGT)_2[e^{x+2t}]-3\,(RGT)_2\,[e^{2x+t}]$$

$$\begin{aligned}3(RGT)_2[e^{x+2t}]&=\frac{3}{uv}\int_0^\infty e^{-(\frac{s}{u}x+\frac{p}{v}t)}[e^{x+2t}]dxdt\\&=\frac{3}{uv}\left[-\frac{1}{(\frac{s}{u}-1)}[e^{-(\frac{s}{u}-1)x}]_0^\infty\right.\\&\qquad\left.-\frac{1}{(\frac{p}{v}-2)}[e^{-(\frac{p}{v}-2)t}]_0^\infty\right]\\&=\frac{3}{uv(\frac{s}{u}-1)(\frac{p}{v}-2)}-3(RGT)_2[e^{2x+t}]\\&=\frac{3}{uv}\int_0^\infty e^{-(\frac{s}{u}x+\frac{p}{v}t)}[e^{2x+t}]dxdt\\&=\frac{-3}{uv}\left[-\frac{1}{(\frac{s}{u}-2)}[e^{-(\frac{s}{u}-2)x}]_0^\infty\right.\\&\qquad\left.-\frac{1}{(\frac{p}{v}-1)}[e^{-(\frac{p}{v}-1)t}]_0^\infty\right]\\&=\frac{-3}{uv(\frac{s}{u}-2)(\frac{p}{v}-1)}\end{aligned}$$

Then,

$$(RGT)_2 3[e^{x+2t}-e^{2x+t}]=\frac{3}{uv(\frac{s}{u}-1)(\frac{p}{v}-2)}+\frac{-3}{uv(\frac{s}{u}-2)(\frac{p}{v}-1)} \tag{38}$$

Substituting Eqns. (34)-(38) and (38) we get

$$\begin{aligned}(\frac{p^2}{v^2}-\frac{s^2}{u^2})K(s,p,u,v)=&\frac{p}{uv^2(\frac{s}{u}-2)}+\frac{p}{uv^2(\frac{s}{u}-1)}+\frac{1}{uv(\frac{s}{u}-2)}\\&+\frac{2}{uv(\frac{s}{u}-1)}-\frac{s}{u^2v(\frac{p}{v}-1)}\\&-\frac{s}{u^2v(\frac{p}{v}-2)}-\frac{2}{uv(\frac{p}{v}-1)}-\frac{1}{uv(\frac{p}{v}-2)}\\&+\frac{3}{uv(\frac{s}{u}-1)(\frac{p}{v}-2)}-\frac{3}{uv(\frac{s}{u}-2)(\frac{p}{v}-1)}\\&=\frac{(\frac{p^2}{v^2}-\frac{s^2}{u^2})}{uv(\frac{s}{u}-2)(\frac{p}{v}-1)}+\frac{(\frac{p^2}{v^2}-\frac{s^2}{u^2})}{uv(\frac{s}{u}-1)(\frac{p}{v}-2)}.\end{aligned}$$

Hence,

$$K(s,p,u,v)=\frac{1}{uv(\frac{s}{u}-2)(\frac{p}{v}-1)}+\frac{1}{uv(\frac{s}{u}-1)(\frac{p}{v}-2)} \tag{39}$$

By taking inverse of double Ramadan group integral transform for Eq. (39), you get the solution of (30) in the following form

$$U(x,t)=e^{2x+t}+e^{x+2t}$$

*Example 3.*Consider the following Volterra integro- partial differential equation

$$U_x+U_t=-1+e^x+e^t+e^{x+t}+\int_0^x\int_0^t U(r,y)drdy\,(x,t)\in R_+^2 \quad (40)$$

Subject to the initial conditions

$$U(x,0)=e^x \quad , \quad U(0,t)=e^t \quad (41)$$

Solution

Take the double Ramadan group integral transform of Eq. (40) and apply Theorem 5 for the integral part of the equation you get

$$\frac{s}{u}K(s,p,u,v)-\frac{1}{u}K(o,p,o,v)+\frac{p}{v}K(s,p,u,v)$$
$$-\frac{1}{v}K(s,0,u,0)=-\frac{1}{sp}+\frac{1}{p(s-u)}+\frac{1}{s(p-v)}$$
$$+\frac{1}{(s-u)(p-v)}+\frac{uv}{sp}K(s,p,u,v)$$

Rearrange the above equation to get

$$(\frac{s}{u}+\frac{p}{v}-\frac{uv}{sp})K(s,p,u,v)=-\frac{1}{sp}+\frac{1}{p(s-u)}$$
$$+\frac{1}{s(p-v)}+\frac{1}{(s-u)(p-v)}\frac{1}{u}K(o,p,o,v)+-\frac{1}{v}K(s,0,u,0) \quad (42)$$

Now, take single Ramadan group transform to the initial conditions we have

$$\begin{aligned}K(s,o,u,o)&=RG[U(x,0);(s,0,u,0)\\&=RG[e^x;(s,0,u,0)\\&=\frac{1}{u}\int_0^\infty e^{-\frac{s}{u}x}U(x,0)dx\\&=\frac{1}{u}\int_0^\infty e^{-\frac{s}{u}x}e^xdx\\&=\frac{1}{u(\frac{s}{u}-1)}=\frac{1}{s-u}\end{aligned}$$

Similarly, for the second part of the initial condition we obtain

$$\begin{aligned}K(0,p,0,v)=&RG[U(0,t);(0,p,0,v)\\&=RG[e^t;(0,p,0,v)]\\&=\frac{1}{v}\int_0^\infty e^{-\frac{p}{v}t}U(0,t)dt\\&=\frac{1}{v}\int_0^\infty e^{-\frac{p}{v}t}e^tdt\\&=\frac{1}{v(\frac{p}{v}-1)}\\&=\frac{1}{p-v}\end{aligned}$$

Now, substituting to eq. (42) we get

$$\begin{aligned}(\frac{s}{u}+\frac{p}{v}-\frac{uv}{sp})K(s,p,u,v)=&-\frac{1}{sp}+\frac{1}{p(s-u)}+\frac{1}{s(p-v)}\\&+\frac{1}{(s-u)(p-v)}-\frac{1}{u}\frac{1}{(p-v)}\\&+-\frac{1}{v}\frac{1}{(s-u)}\end{aligned}$$

After simplifying, we obtain

$$K(s,p,u,v)=\frac{1}{(s-u)(p-v)}.$$

By using double inverse Ramadan group transform we obtain the solution of (40) as follows:

$$U(x,t)=e^{x+t}.$$

Competing interests

The authors confirm that there is no conflict of interest in this paper.

Authors' contributions

All authors have contributed equally in this paper; they read and approved the final manuscript.

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