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A Common Fixed Point Theorem for Four Maps under (ψ-ϕ) Contractive Condition of Integral Type in Ordered Partial Metric Spaces

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Abstract: In this paper, we obtain a common fixed point theorem for four self maps satisfing (ψ-ϕ) contractive condition of integral type in ordered partial metric spaces.

Keywords: Partial metric, weakly compatible maps, weakly contractive maps

1 Introduction

The notion of partial metric space was introduced by Matthews [18] as a part of the study of denotational semantics of data flow networks. In fact, it is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation [4, 6, 14, 20, 19, 24, 25, 26, 27, 28, 31, 32].

Matthews [18], Oltra and Valero [19], Romaguera [24] and Altun, Sola and Simsek [6] proved fixed point theorems in partial metric spaces for a single map.

In this paper, we obtain a common fixed point theorem for four self mappings satisfying a generalized (ψ-ϕ) contractive condition of integral type in ordered partial metric spaces. First we recall some definitions and lemmas in partial metric spaces.

Definition 1.1. A partial order set consists of a set X and a binary relation $\preceq$ on X which satisfies the following conditions:

(i) $x \preceq x$ (reflexive);
(ii) if $x \preceq y$ and $y \preceq x$ then $x = y$ (antisymmetry);
(iii) if $x \preceq y$ and $y \preceq z$ then $x \preceq z$ (transitivity);
for all x, y and z in X. The relation $\preceq$ is called a partial order for the set X.

A set with a partial order $\preceq$ is called partially ordered set.

Definition 1.2. Any two elements x and y of a set X, which is partially ordered by a binary relation $\preceq$, are either comparable or incomparable. Specifically, the elements x and y are comparable if and only if $x \preceq y$ or $y \preceq x$. Otherwise, x and y are incomparable.

Definition 1.3. [18] A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x,y,z \in X$:

(p$_1$) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,
(p$_2$) $p(x,x) \le p(x,y)$, $p(y,y) \le p(x,y)$,
(p$_3$) $p(x,y) = p(y,x)$,
(p$_4$) $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$.
In this case (X, p) is called a partial metric space.

It is clear that $|p(x,y) - p(y,z)| \le p(x,z)\ \forall x,y,z \in X$. Also clear that $p(x,y) = 0$ implies $x = y$ from (p_1) and (p_2). But if $x = y$, $p(x,y)$ may not be zero. A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x,y) = \max\{x,y\}$ for all $x,y \in \mathbb{R}^+$. Each partial metric p on X generates τ_0 topology τ_p on X which has a base the family of open p - balls $\{B_p(x,\varepsilon) \mid x \in X, \varepsilon > 0\}$ for all $x \in X$ and $\varepsilon > 0$, where $B_p(x,\varepsilon) = \{y \in X \mid p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}^+$ given by $p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ is a metric on X.

Definition 1.4. [18] Let (X, p) be a partial metric space.

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(i) A sequence $\{x_n\}$ in (X,p) is said to converge to a point $x \in X$ if and only if $\lim_{n\to\infty} p(x,x_n) = p(x,x)$.

(ii) A sequence $\{x_n\}$ in (X,p) is said to be Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite .

(iii) (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, w.r.to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

Lemma 1.5. [18] Let (X,p) be a partial metric space.

(a) $\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s).

(b) (X,p) is complete iff the metric space (X,p^s) is complete. Furthermore, $\lim_{n\to\infty} p^s(x_n,x) = 0$ iff $p(x,x) = \lim_{n\to\infty} p(x_n,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

Lemma 1.6.[4] Let (X,p) be a partial metric space and $x_n \to z$ as $n \to \infty$ in (X,p) such that $p(z,z) = 0$. Then $\lim_{n\to\infty} p(x_n,y) = p(z,y)$ for every $y \in X$.

Definition 1.7.[15] Two self maps F and f of a nonempty set X are said to be weakly compatible if $fFx = Ffx$ whenever $fx = Fx$ for some $x \in X$.

In 1997, Alber and Guerre-Delabriere [5] introduced the following concept of weakly contractive mapping in Hilbert spaces.

Definition 1.8.[5] A mapping $T : X \to X$ is said to be a weakly contractive mapping if

$$d(Tx,Ty) \leq d(x,y) - \phi(d(x,y)),$$

for all $x,y \in X$ and some $\phi \in \Omega$, where

$$\Omega = \{\phi \mid \phi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is continuous, non-decreasing and}$$
$$\phi(t) = 0 \Leftrightarrow t = 0\}$$

Rhoades [22] extended the results of Alber and Guerre-Delabriere to complete metric spaces.

Theorem 1.9.[22] Let (X,d) be a complete metric space and T a weakly contractive mapping. Then T has a unique fixed point.

Dutta and Choudhury [13] introduced a new generalization of contraction mapping and proved the following theorem.

Theorem 1.10.[13] Let (X,d) be a complete metric space and $T : X \to X$ be a mapping satisfying

$$\psi(d(Tx,Ty)) \leq \psi(d(x,y)) - \phi(d(x,y))$$

for all $x,y \in X$, where $\psi,\phi \in \Omega$. Then T has a unique fixed point.

Recently several authors are using the $(\psi$-$\phi)$ contractive condition on maps to prove fixed and common fixed point theorems (for example, see [1,2,12,33]).

In [8], Branciari obtained a fixed point result for a single mapping an analogue of Banach's contraction principle for an integral type inequality. Later several authors proved fixed and common fixed point theorems for mappings satisfying integral type condition (for instance, see [3,7,10,11,16,21,23,29,30]).

Recently Cai et.al [9] proved the following theorem which is a generalization of theorem of Luong and Thuan [17].

Theorem 1.11.[9] Let (X,d) be a complete metric space and $T,S : X \to X$ be mappings such that for all $x,y \in X$,

$$\psi\left(\int_0^{d(Tx,Sy)} \varphi(t)dt\right) \leq \psi\left(\int_0^{M(x,y)} \varphi(t)dt\right) - \phi\left(\int_0^{M(x,y)} \varphi(t)dt\right)$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a Lebesgue-integrable mapping which is summable, non-negative and such that $\int_0^{\varepsilon} \varphi(t)dt > 0$ for each $\varepsilon > 0$, $\psi \in \Omega$, $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is lower semicontinuous, non-decreasing and $\phi(t) = 0 \Leftrightarrow t = 0$ and

$$M(x,y) = max\{d(x,y), d(Tx,x), d(Sy,y), \tfrac{1}{2}[d(y,Tx) + d(x,Sy)]\}.$$

Then T and S have a unique common fixed point in X.

Here afterwards, assume the following:

(i) $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous and non-decreasing function.

(ii) $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a lower semi-continuous and $\phi(t) > 0$ for each $t > 0$.

(iii) $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a Lebesgue-integrable mapping which is summable, non-negative and such that $\int_0^{\varepsilon} \varphi(t)dt > 0$ for each $\varepsilon > 0$.

Remark 1.12.
If $\psi\left(\int_0^{\varepsilon} \varphi(t)dt\right) \leq \psi\left(\int_0^{\varepsilon} \varphi(t)dt\right) - \phi\left(\int_0^{\varepsilon} \varphi(t)dt\right)$, then $\varepsilon = 0$.

Now,we give our main result.

2 Main Result

Theorem 2.1. Let $(X,\preceq,p)$ be an ordered partial metric space and let $F,G,f,g : X \to X$ be mappings satisfying

$$\psi\left(\int_0^{p(Fx,Gy)} \varphi(t)dt\right) \leq \psi\left(\int_0^{M(x,y)} \varphi(t)dt\right) - \phi\left(\int_0^{M(x,y)} \varphi(t)dt\right), \quad (1)$$

for all comparable elements x,y in X, where

$$M(x,y) = \max\left\{\begin{array}{l} p(fx,gy), p(fx,Fx), p(gy,Gy), \\ \frac{1}{2}[p(fx,Gy) + p(gy,Fx)] \end{array}\right\},$$

$$F(X) \subseteq g(X), G(X) \subseteq f(X), \quad (2)$$

$$\text{the pairs } (f,F) \text{ and } (g,G) \text{ are weakly compatible,} \quad (3)$$

$$\text{if } Fx = gu \text{ then } x \preceq u, \text{ if } Gy = fv \text{ then } y \preceq v, \quad (4)$$

and one of the following:

(a) if $f(X)$ is complete and $y_n = Gx_n$ be such that $y_n \to y = fv \in f(X)$, then $x_n \preceq v$ and $x_n \preceq y$ for all n

(b) if $g(X)$ is complete and $y_n = Fx_n$ be such that $y_n \to y = gv \in g(X)$, then $x_n \preceq v$ and $x_n \preceq y$ for all n,

Then F, G, f and g have a common fixed point in X.

Proof. Let $x_0 \in X$. From (2), there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n} = Fx_{2n} = gx_{2n+1}, y_{2n+1} = Gx_{2n+1} = fx_{2n+2}, n = 0, 1, 2, \ldots.$$

From (4), it follows that $x_0 \preceq x_1 \preceq x_2 \preceq \ldots$.

Case (i). Suppose $y_{2m} = y_{2m+1}$ for some m. We have

$$M(x_{2m+2}, x_{2m+1}) = \max\left\{\begin{array}{c} p(y_{2m+1}, y_{2m}), p(y_{2m+1}, y_{2m+2}), \\ p(y_{2m}, y_{2m+1}), \\ \frac{1}{2}\left[\begin{array}{c} p(y_{2m+1}, y_{2m+1}) \\ +p(y_{2m}, y_{2m+2}) \end{array}\right] \end{array}\right\}.$$

We have, from (p_2) $p(y_{2m+1}, y_{2m}) = p(y_{2m+1}, y_{2m+1}) \leq p(y_{2m+1}, y_{2m+2})$ and from (p_4)

$$\frac{1}{2}\left[\begin{array}{c} p(y_{2m+1}, y_{2m+1}) \\ +p(y_{2m}, y_{2m+2}) \end{array}\right] \leq \frac{1}{2}[p(y_{2m}, y_{2m+1}) + p(y_{2m+1}, y_{2m+2})] \leq p(y_{2m+1}, y_{2m+2})$$

hence we have $M(x_{2m+2}, x_{2m+1}) = p(y_{2m+1}, y_{2m+2})$. Therefore, from (1),we have

$$\psi\left(\int_0^{p(y_{2m+2}, y_{2m+1})} \varphi(t)dt\right) = \psi\left(\int_0^{p(Fx_{2m+2}, Gx_{2m+1})} \varphi(t)dt\right) \leq \psi\left(\int_0^{p(y_{2m+2}, y_{2m+1})} \varphi(t)dt\right) - \phi\left(\int_0^{p(y_{2m+2}, y_{2m+1})} \varphi(t)dt\right).$$

From Remark 1.12, we have $y_{2m+1} = y_{2m+2}$. Continuing in this way, we can conclude that $y_n = y_{n+k}$ for all $k > 0$. Thus $\{y_n\}$ is a Cauchy sequence.

Case (ii) Assume that $y_n \neq y_{n+1}$ for all n. Denote $p_n = p(y_n, y_{n+1})$. Now

$$\begin{aligned}\psi\left(\int_0^{p_{2n}} \varphi(t)dt\right) &= \psi\left(\int_0^{p(y_{2n}, y_{2n+1})} \varphi(t)dt\right) \\ &= \psi\left(\int_0^{p(Fx_{2n}, Gx_{2n+1})} \varphi(t)dt\right) \\ &\leq \psi\left(\int_0^{M(x_{2n}, x_{2n+1})} \varphi(t)dt\right) - \phi\left(\int_0^{M(x_{2n}, x_{2n+1})} \varphi(t)dt\right)\end{aligned}$$

where

$$M(x_{2n}, x_{2n+1}) = \max\left\{\begin{array}{c} p(y_{2n-1}, y_{2n}), p(y_{2n-1}, y_{2n}), \\ p(y_{2n}, y_{2n+1}), \\ \frac{1}{2}\left[\begin{array}{c} p(y_{2n-1}, y_{2n+1}) \\ +p(y_{2n}, y_{2n}) \end{array}\right] \end{array}\right\} = \max\{p_{2n-1}, p_{2n}\}.$$

If p_{2n} is maximum, then

$$\psi\left(\int_0^{p_{2n}} \varphi(t)dt\right) \leq \psi\left(\int_0^{p_{2n}} \varphi(t)dt\right) - \phi\left(\int_0^{p_{2n}} \varphi(t)dt\right) < \psi\left(\int_0^{p_{2n}} \varphi(t)dt\right),$$

which is a contradiction. Hence $M(x_{2n}, x_{2n+1}) = p_{2n-1}$ and thus

$$\psi\left(\int_0^{p_{2n}} \varphi(t)dt\right) \leq \psi\left(\int_0^{p_{2n-1}} \varphi(t)dt\right) - \phi\left(\int_0^{p_{2n-1}} \varphi(t)dt\right) \leq \psi\left(\int_0^{p_{2n-1}} \varphi(t)dt\right). \quad (5)$$

Since ψ is non-decreasing, we have

$$\int_0^{p_{2n}} \varphi(t)dt \leq \int_0^{p_{2n-1}} \varphi(t)dt.$$

Similarly we can show that

$$\int_0^{p_{2n-1}} \varphi(t)dt \leq \int_0^{p_{2n-2}} \varphi(t)dt.$$

Thus $\left\{\int_0^{p_n} \varphi(t)dt\right\}$ is a monotone decreasing sequence of non-negative real numbers and must converge to a real number, say, $r \geq 0$. Taking lim sup on both sides of (5), we have

$$\psi(r) \leq \psi(r) - \phi(r),$$

which implies that $\phi(r) \leq 0$. By the property of ϕ, we have $r = 0$, that is,

$$\lim_{n\to\infty}\left(\int_0^{p(y_n, y_{n+1})} \varphi(t)dt\right) = 0.$$

Hence

$$\lim_{n\to\infty} p(y_n, y_{n+1}) = 0. \quad (6)$$

From (p_2),

$$\lim_{n\to\infty} p(y_n, y_n) = 0. \quad (7)$$

From (6) and (7) and from the definition of p^s, we have

$$\lim_{n\to\infty} p^s(y_n, y_{n+1}) = 0. \quad (8)$$

Now we prove that $\{y_{2n}\}$ is a Cauchy sequence in (X, p^s). On contrary suppose that $\{y_{2n}\}$ is not Cauchy. Then there exists an $\varepsilon > 0$ and monotone increasing sequences of natural numbers $\{2m_k\}$ and $\{2n_k\}$ such that $n_k > m_k$,

$$p^s(y_{2m_k}, y_{2n_k}) \geq \varepsilon \quad (9)$$

and

$$p^s(y_{2m_k}, y_{2n_k-2}) < \varepsilon. \quad (10)$$

From (9) and (10), we have

$$\begin{aligned}\varepsilon &\leq p^s(y_{2m_k}, y_{2n_k}) \\ &\leq p^s(y_{2m_k}, y_{2n_k-2}) + p^s(y_{2n_k-2}, y_{2n_k-1}) + p^s(y_{2n_k-1}, y_{2n_k}) \\ &< \varepsilon + p^s(y_{2n_k-2}, y_{2n_k-1}) + p^s(y_{2n_k-1}, y_{2n_k}).\end{aligned}$$

Letting $k \to \infty$ and using (8), we have

$$\lim_{k\to\infty} p^s(y_{2m_k}, y_{2n_k}) = \varepsilon. \quad (11)$$

Hence from definition of p^s and from (7),we have

$$\lim_{k\to\infty} p(y_{2m_k}, y_{2n_k}) = \frac{\varepsilon}{2}. \quad (12)$$

Letting $k \to \infty$ and using (11) and (8) in

$$|p^s(y_{2n_k+1}, y_{2m_k}) - p^s(y_{2m_k}, y_{2n_k})| \le p^s(y_{2n_k+1}, y_{2n_k})$$

we get

$$\lim_{k\to\infty} p^s(y_{2n_k+1}, y_{2m_k}) = \varepsilon. \quad (13)$$

Hence we have

$$\lim_{k\to\infty} p(y_{2n_k+1}, y_{2m_k}) = \frac{\varepsilon}{2}. \quad (14)$$

Letting $k \to \infty$ and using (11) and (8) in

$$|p^s(y_{2n_k}, y_{2m_k-1}) - p^s(y_{2n_k}, y_{2m_k})| \le p^s(y_{2m_k-1}, y_{2m_k})$$

we get

$$\lim_{k\to\infty} p^s(y_{2n_k}, y_{2m_k-1}) = \varepsilon. \quad (15)$$

Hence we have

$$\lim_{k\to\infty} p(y_{2n_k}, y_{2m_k-1}) = \frac{\varepsilon}{2}. \quad (16)$$

Letting $k \to \infty$ and using (15) and (8) in

$$|p^s(y_{2m_k-1}, y_{2n_k+1}) - p^s(y_{2m_k-1}, y_{2n_k})| \le p^s(y_{2n_k+1}, y_{2n_k})$$

we get

$$\lim_{k\to\infty} p^s(y_{2m_k-1}, y_{2n_k+1}) = \varepsilon. \quad (17)$$

Hence we have

$$\lim_{k\to\infty} p(y_{2m_k-1}, y_{2n_k+1}) = \frac{\varepsilon}{2}. \quad (18)$$

From (6), (12), (16) and (18) we have

$$M(x_{2m_k}, x_{2n_k+1}) = \max\left\{ \begin{array}{c} p(y_{2m_k-1}, y_{2n_k}), p(y_{2m_k-1}, y_{2m_k}), \\ p(y_{2n_k}, y_{2n_k+1}), \\ \frac{1}{2}\left[\begin{array}{c} p(y_{2m_k-1}, y_{2n_k+1}) \\ + p(y_{2n_k}, y_{2m_k}) \end{array}\right] \end{array} \right\}$$
$$\to \frac{\varepsilon}{2} \text{ as } k \to \infty.$$

Now from (1), we have

$$\begin{aligned}\psi\left(\int_0^{p(y_{2m_k}, y_{2n_k+1})} \varphi(t)dt\right) &= \psi\left(\int_0^{p(Fx_{2m_k}, Gx_{2n_k+1})} \varphi(t)dt\right) \\ &\le \psi\left(\int_0^{M(x_{2m_k}, x_{2n_k+1})} \varphi(t)dt\right) \\ &\quad -\phi\left(\int_0^{M(x_{2m_k}, x_{2n_k+1})} \varphi(t)dt\right)\end{aligned}$$

and letting $k \to \infty$, we get

$$\psi\left(\int_0^{\frac{\varepsilon}{2}} \varphi(t)dt\right) \le \psi\left(\int_0^{\frac{\varepsilon}{2}} \varphi(t)dt\right) - \phi\left(\int_0^{\frac{\varepsilon}{2}} \varphi(t)dt\right).$$

From Remark 1.12, we have $\varepsilon = 0$. It is a contradiction. Hence $\{y_{2n}\}$ is Cauchy. Letting $n, m \to \infty$ in $|p^s(y_{2n+1}, y_{2m+1}) - p^s(y_{2n}, y_{2m})| \le p^s(y_{2n+1}, y_{2n}) + p^s(y_{2m}, y_{2m+1})$ we get $\lim_{m,n\to\infty} p^s(y_{2n+1}, y_{2m+1}) = 0$. Hence $\{y_{2n+1}\}$ is Cauchy. Thus $\{y_n\}$ is a Cauchy sequence in (X, p^s). Hence, we have $\lim_{m,n\to\infty} p^s(y_n, y_m) = 0$. Now, from the definition of p^s and from (7), we have

$$\lim_{n,m\to\infty} p(y_n, y_m) = 0. \quad (19)$$

Suppose (a) holds. Since $\{y_{2n+1}\} = \{fx_{2n}\} \subseteq f(X)$ is a Cauchy sequence in the complete metric space $(f(X), p^s)$, it follows that $\{y_{2n+1}\}$ converges in $(f(X), p^s)$. Thus $\lim_{n\to\infty} p^s(y_{2n+1}, v) = 0$ for some $v \in f(X)$. There exists $t \in X$ such that $v = f(t)$. From (a), it is clear that $x_{2n+1} \preceq t$ and $x_{2n+1} \preceq v$ for all n. Since $\{y_n\}$ is Cauchy in X and $\{y_{2n+1}\} \to v$, it follows that $\{y_{2n}\} \to v$. From Lemma 1.5, we have

$$p(v,v) = \lim_{n\to\infty} p(y_{2n+1}, v) = \lim_{n\to\infty} p(y_{2n}, v) = \lim_{n,m\to\infty} p(y_n, y_m). \quad (20)$$

From (19) and (20), we have

$$p(v,v) = \lim_{n\to\infty} p(y_{2n+1}, v) = \lim_{n\to\infty} p(y_{2n}, v) = 0. \quad (21)$$

Considering Lemma 1.6 we have $\lim_{n\to\infty} p(Ft, y_{2n}) = p(Ft, v)$ and $\lim_{n\to\infty} p(Ft, y_{2n+1}) = p(Ft, v)$. From (7) and (21) we have

$$M(t, x_{2n+1}) = \max\left\{ \begin{array}{c} p(v, y_{2n}), p(v, Ft), p(y_{2n}, y_{2n+1}), \\ \frac{1}{2}[p(v, y_{2n+1}) + p(y_{2n}, Ft)] \end{array}\right\}$$
$$\to p(Ft, v) \text{ as } n \to \infty.$$

Therefore, we obtain

$$\begin{aligned}\psi\left(\int_0^{p(Ft, y_{2n+1})} \varphi(t)dt\right) &= \psi\left(\int_0^{p(Ft, Gx_{2n+1})} \varphi(t)dt\right) \\ &\le \psi\left(\int_0^{M(t, x_{2n+1})} \varphi(t)dt\right) \\ &\quad - \phi\left(\int_0^{M(t, x_{2n+1})} \varphi(t)dt\right)\end{aligned}$$

and letting $n \to \infty$, we get

$$\psi\left(\int_0^{p(Ft,v)} \varphi(t)dt\right) \le \psi\left(\int_0^{p(Ft,v)} \varphi(t)dt\right) - \phi\left(\int_0^{p(Ft,v)} \varphi(t)dt\right).$$

From Remark 1.12, we have $p(Ft, v) = 0$ so that $v = Ft$. Thus $ft = v = Ft$. Since the pair (f, F) is weakly compatible, we have $fv = Fv$. Again using Lemma 1.6 we have

$$\lim_{n\to\infty} p(Fv,y_{2n}) = \lim_{n\to\infty} p(Fv,y_{2n+1}) = p(Fv,v).$$

From (p_2) we have

$$\begin{aligned}\lim_{n\to\infty} M(v,x_{2n+1}) &= \lim_{n\to\infty} \max\left\{\begin{array}{l} p(Fv,y_{2n}), p(Fv,Fv), \\ p(y_{2n},y_{2n+1}), \\ \frac{1}{2}[p(Fv,y_{2n+1})+p(y_{2n},Fv)] \end{array}\right\} \\ &= \max\left\{\begin{array}{l} p(Fv,v), p(Fv,Fv), 0, \\ \frac{1}{2}[p(Fv,v)+p(v,Fv)] \end{array}\right\} \\ &= p(Fv,v)\end{aligned}$$

Therefore,

$$\begin{aligned}\psi\left(\int_0^{p(Fv,y_{2n+1})}\varphi(t)dt\right) &= \psi\left(\int_0^{p(Fv,Gx_{2n+1})}\varphi(t)dt\right) \\ &\leq \psi\left(\int_0^{M(v,x_{2n+1})}\varphi(t)dt\right) \\ &\quad -\phi\left(\int_0^{M(v,x_{2n+1})}\varphi(t)dt\right)\end{aligned}$$

and so letting $n\to\infty$, we get

$$\psi\left(\int_0^{p(Fv,v)}\varphi(t)dt\right) \leq \psi\left(\int_0^{p(Fv,v)}\varphi(t)dt\right) - \phi\left(\int_0^{p(Fv,v)}\varphi(t)dt\right).$$

From Remark 1.12, we have $Fv = v$. Thus

$$fv = Fv = v. \tag{22}$$

Since $F(X)\subseteq g(X)$, there exists $w\in X$ such that $v = Fv = gw$. From (4), it is clear that $v \preceq w$,

$$\begin{aligned}M(v,w) &= \max\left\{\begin{array}{l} p(v,v), p(v,v), p(v,Gw), \\ \frac{1}{2}[p(v,Gw)+p(v,v)] \end{array}\right\} \\ &= p(v,Gw)\end{aligned}$$

and so

$$\begin{aligned}\psi\left(\int_0^{p(v,Gw)}\varphi(t)dt\right) &= \psi\left(\int_0^{p(Fv,Gw)}\varphi(t)dt\right) \\ &\leq \psi\left(\int_0^{M(v,w)}\varphi(t)dt\right) \\ &\quad -\phi\left(\int_0^{M(v,w)}\varphi(t)dt\right) \\ &= \psi\left(\int_0^{p(v,Gw)}\varphi(t)dt\right) \\ &\quad -\phi\left(\int_0^{p(v,Gw)}\varphi(t)dt\right).\end{aligned}$$

From Remark 1.12, we have $p(v,Gw) = 0$ so that $v = Gw$. Thus $gw = v = Gw$. Since (g,G) is weakly compatible pair, we have $gv = Gv$. Thus

$$\begin{aligned}M(v,v) &= \max\left\{\begin{array}{l} p(v,Gv), p(v,v), p(Gv,Gv), \\ \frac{1}{2}[p(v,Gv)+p(Gv,v)] \end{array}\right\} \\ &= p(v,Gv)\end{aligned}$$

and so

$$\begin{aligned}\psi\left(\int_0^{p(v,Gv)}\varphi(t)dt\right) &= \psi\left(\int_0^{p(Fv,Gv)}\varphi(t)dt\right) \\ &\leq \psi\left(\int_0^{M(v,v)}\varphi(t)dt\right) - \phi\left(\int_0^{M(v,v)}\varphi(t)dt\right) \\ &= \psi\left(\int_0^{p(v,Gv)}\varphi(t)dt\right) - \phi\left(\int_0^{p(v,Gv)}\varphi(t)dt\right).\end{aligned}$$

From Remark 1.12, we have $p(v,Gv) = 0$ so that $Gv = v$. Thus

$$gv = Gv = v. \tag{23}$$

From (22) and (23) it follows that v is a common fixed point of F, G, f and g. Similarly, we can prove the theorem if (b) holds.

Remark 2.2. Theorem 2.1 is a generalization and improvement of Theorem 2.3 of [17] ,Theorem 2.1 of [9] and Theorem 5 of [4].

References

[1] M. Abbas and D. Doric, Common fixed point theorem for four mappings satisfying generalized weak contractive condition, Filomat, 24 (2010), 1-10.

[2] M. Abbas and M. A. Khan, Common fixed point theorem of two mappings satisfying a generalized weak contractive condition, Int. J. Math. Math. Sci., (2009), Article ID 131068, 9 pages.

[3] M. Abbas and B. E. Rhoades, Common fixed point theorems for hybrid pairs of occasionally weakly compatible mappings satisfying generalized contractive condition of integral type, Fixed Point Theory and Applications, (2007), Article ID 54101, 9 pages.

[4] T. Abdeljawad, E. Karapinar and K. Tas, Existence and uniqueness of a common fixed point on partial metric spaces, Appl. Math. Letters, 24 (2011), 1900-1904.

[5] Ya. I. Alber and S. Guerre-Delabriere, Principle of weakly contractive maps in Hilbert spaces, New results in Operator Theory and its Applications (Eds.I.Gohberg and Y.Lyubich), Vol.98 of Operator Theory: Advances and Applications, (1997), pp.7-22, Birkhauser, Basel, Switzerland.

[6] I. Altun, F. Sola and H. Simsek, Generalized contractions on partial metric spaces, Topology and its Applications, 157 (2010), 2778-2785.

[7] I. Altun, D. Turkoglu and B. E. Rhoades, Fixed points of weakly compatible maps satisfying a general contractive condition of integral type, Fixed Point Theory and Applications, (2007), Article ID 17301, 9 pages.

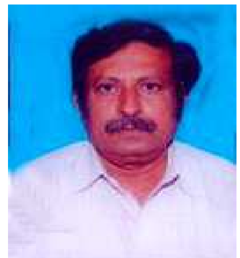
[8] A. Branciari, A fixed point theroem for mappings satisfying a general contractive condition of integral type, Int. J. Math. Math. Sci., 29 (2002), 531-536.

[9] Y. Cai, L. Hu, Y. Tang and L. Hu, Common fixed point for generalized $\psi_{\int\varphi}$-weakly contractive mappings in metric spaces, Int. Math. Forum, 6 (2011), 1803-1814.

[10] A. Djoudi and A. Aliouche, Common fixed point theorems of Gregus type for weakly compatible mappings satisfying contractive conditions of integral type, J. Math. Anal. Appl., 329 (2007), 31-45.

[11] A. Djoudi and F. Merghadi, Common fixed point theorems for maps under a contractive condition of integral type, J. Math. Anal. Appl., 341 (2008), 953-960.

[12] D. Doric, Common fixed point for generalized $(\psi$-$\varphi)$-weak contractions, Appl. Math. Letters, 22 (2009),1896-1900.
[13] P. N. Dutta and B. S. Choudhury, A generalization of contraction principle in metric spaces, Fixed Point Theory and Applications, (2008), Article ID 406368,(2008), 8 pages.
[14] R. Heckmann, Approximation of metric spaces by partial metric spaces, Appl. Categ. Structures, 7 (1999), 71-83.
[15] G. Jungck and B. E. Rhoades, Fixed point for set-valued functions without continuity, Indian J. Pur. Appl. Math., 29 (1998), 227-238.
[16] S. Kumar, R. Chugh and R. Kumar, Fixed point theorem for compatible mappings satisfying a general contractive condition of integral type, Soochow Journal of Mathematics, 33 (2007), 181-185.
[17] N. V. Luong and N. X. Thuan, A fixed point theorem for $\psi_{\int\varphi}$-weakly contractive mapping in metric spaces, Int. Journal of Math. Anal., 4 (2010), 233-242.
[18] S. G. Matthews, Partial metric topology, in: Proc. 8th Summer Conference on General Topology and Applications, in: Ann. New York Acad. Sci., 728 (1994), 183-197.
[19] S. Oltra and O. Valero, Banach's fixed point theorem for partial metric spaces, Rend. Istit. Mat. Univ. Trieste, XXXVI (2004), 17-26.
[20] S. J. O'Neill, Partial metrics, valuations and domain theory, Proc. 11th Summer Conference on General Topology and Applications, vol 806 of Annals of the New York Academy of Sciences, (1996), pp. 304-315, The New York Academy of Sciences, New York, NY, USA.
[21] H. K. Pathak, R.T iwari and M. S. Khan, A common fixed point theorem satisfying integral type implicit relations, Applied Mathematics E-Notes, 7 (2007), 222-228.
[22] B. E. Rhoades, Some theorems on weakly contractive maps, Nonlinear Anal.,47 (2001), 2683-2693.
[23] B. E. Rhoades, Two fixed point theorems for mappings satisfying a general contractive condition of integral type, Int. J. Math. Math. Sci., 63 (2003), 4007-4013.
[24] S. Romaguera, A Kirk type characterization of completeness for partial metric spaces, Fixed Point Theory and Applications, (2010), Article ID 493298, 6 pages.
[25] S. Romaguera and M. Schellekens, Partial metric monoids and semi valuation sapces, Topology and Applications, 153, (2005), 948-962.
[26] S. Romaguera and O. Valero, A quantiative computational modal for complete partial metric space via formal balls, Mathematical Structures in Computer Sciences, 19 (2009), 541-563.
[27] M. Schellekens, The Smth comletion: a common foundation for denotational semantics and complexity analysis, Electronic Notes in Theoretical Computer Science, 1 (1995), 535-556.
[28] M. Schellekens, A characterization of partial metrizebility:domains are quantifiable, Theoretical Computer Sciences, 305 (2003), 409-432.
[29] T. Suzuki, Meir-Keeler contractions of integrals are still Meir-Keeler contractions, Int. J. Math. Math. Sci., (2007), Article ID 39281, 6 pages.
[30] P. Vijayaraju, B. E. Rhoades and R. Mohanraj, A fixed point theorem for a pair of maps satisfying a general contractive condition of integral type, Int. J. Math. Math. Sci., 15 (2005), 2359-2364.
[31] P. Waszkiewicz, Quantitative continuous domains, Applied Categorical Structures, 11 (2003), 41-67.
[32] P. Waszkiewicz, Partial metrizebility of continuous posets, Mathematical Structures in Computer Sciences, 16 (2006), 359-372.
[33] Q. Zhang and Y. Song, Fixed point theory for generalized ψ-weak contractions, Appl. Math. Letters, 22, (2009), 75-78.



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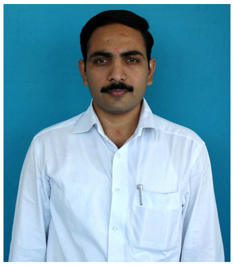

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