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Soft Subnear-Rings, Soft Ideals and Soft N-Subgroups of Near-Rings

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Abstract: Molodtsov introduced the concept of soft sets, which can be seen as a new mathematical tool for dealing with uncertainty. In this paper, we introduce and study soft subnear-rings, soft ideals and soft N-subgroups of near-rings by using Molodtsov's definition of the soft sets. Some related properties are investigated and illustrated by a great deal of examples.

Keywords: Soft sets, Soft subnear-rings of near-rings, Soft ideals of near-rings, Soft N-subgroups of near-rings

1 Introduction

Researchers studying to solve complicated problems in economics, engineering, environmental science, sociology, medical science and many other fields deal with the complex problems of modeling uncertain data. While some mathematical theories such as probability theory, fuzzy set theory [24,25], rough set theory [18,19], vague set theory [9] and the interval mathematics [10] are useful approaches to describing uncertainty, each of these theories has its inherent difficulties as mentioned by Molodtsov [17]. Consequently, Molodtsov [17] proposed a completely new approach for modeling vagueness and uncertainty. This approach called *soft set theory* is free from the difficulties affecting existing methods. Soft set theory has potential applications in many fields, including the smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability theory and measurement theory. Most of these applications have already been demonstrated in Molodtsov's paper [17].

At present, works on soft set theory are progressing rapidly. Maji et al.[15] investigated the applications of soft set theory to a decision making problem. Maji [16] defined and studied several operations on soft sets. Jun [12] introduced and investigated the notion of soft BCK/BCI algebras. Jun and Park [13] discussed the applications of soft sets in ideal theory of BCK/BCI algebras. Aktaş and Çağman [2] compared the soft sets to the related concepts of fuzzy and rough sets. They also defined and studied soft groups, soft subgroups, normal soft subgroups and soft homomorphisms. Feng et al.[8] introduced and investigated soft semirings, soft subsemirings, soft ideals, idealistic soft semirings and soft semiring homomorphisms. The algebraic structure of set theories dealing with uncertainties has also been studied by some authors. Rosenfeld [21] proposed the concept of fuzzy groups in order to establish the algebraic structures of fuzzy sets. Abou-Zaid [1] introduced the notion of a fuzzy subnear-ring and studied fuzzy ideals of a near-ring. This concept is also discussed by many authors (e.g., [6,7,14,22]). Rough groups were defined by Biswas et al.[4] and some other authors (e.g., [5,11]) have studied the algebraic properties of rough sets as well.

In [23], Sezgin et al. introduced union soft subnear-rings (ideals) of a near-ring and union soft N-subgroups (N-ideals) of an N-group by using Molodtsov's definition of soft sets and investigated their related properties with respect to soft set operations, soft anti-image and lower a-inclusion of soft sets. Throughout this study, applying to soft set theory, we define the notions of soft subnear-rings, soft ideals and soft N-subgroups of near-rings, and give several illustrating examples. We also establish the bi-intersection and product operation of soft subnear-rings, soft ideals and soft N-subgroups of near-rings.

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2 Preliminaries

By a near-ring, we shall mean an algebraic system $(N,+,.)$, where

i)$(N,+)$ forms a group (not necessarily abelian)
ii)$(N,.)$ forms a semi-group and
iii)$(a+b)c = ac+bc$ for all $a,b,c \in N$ (i.e., we study on right near-rings.)

Throughout this paper, N will always denote a right near-ring. A subgroup M of N with $MM \subseteq M$ is called a subnear-ring of N. A normal subgroup I of N is called a right ideal if $IN \subseteq I$ and denoted by $I \lhd_r N$. It is called a left ideal if $n(s+i)-ns \in I$ for all $n,s \in N$ and $i \in I$ and denoted by $I \lhd_l N$. If such a normal subgroup I is both left and right ideal in N, then it is called an ideal in N and denoted by $I \lhd N$. A subgroup H of N is called a left N-subgroup of N if $NH \subseteq H$ and H is called a right N-subgroup of N if $HN \subseteq H$. For all undefined concepts and notions we refer to Pilz [20].
Molodtsov [17] defined the soft set in the following manner: Let U be an initial universe set, E be a set of parameters, $P(U)$ be the power set of U and $A \subseteq E$.

Definition 1.([17]) *A pair (F,A) is called a soft set over U, where F is a mapping given by*

$$F : A \to P(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of ε-elements of the soft set (F,A), or as the set of ε-approximate elements of the soft set. To illustrate this idea, Molodtsov considered several examples in [17]. These examples were also discussed in [12], [16]. Maji et al.[16] introduced and investigated several binary operations such as intersection, union, AND-operation, and OR-operation of soft sets. Feng et al.[8] defined the bi-intersection of two soft sets.

Definition 2.([8]) *The bi-intersection of two soft sets (F,A) and(G,B) over a common universe U is defined to be the soft set (H,C), where $C = A \cap B$ and $H : C \to P(U)$ is a mapping given by $H(x) = F(x) \cap G(x)$ for all $x \in C$. This is denoted by $(F,A)\tilde{\sqcap}(G,B) = (H,C)$.*

3 Soft Subnear-rings of Near-rings

In the sequel, let N be a near-ring and A be a nonempty set. α will refer to an arbitrary binary relation between an element of A and an element of N, that is, α is a subset of $A \times N$ without otherwise specified. A set-valued function $F : A \to P(N)$ can be defined as $F(x) = \{y \in N \mid (x,y) \in \alpha\}$ for all $x \in A$. Then the pair (F,A) is a soft set over N, which is derived from the relation α.

Definition 3.*Let M be a subnear-ring of N and let (F,M) be a soft set over N. If for all $x,y \in M$,*

s1)$F(x-y) \supseteq F(x) \cap F(y)$ and
s2)$F(xy) \supseteq F(x) \cap F(y)$,

then the soft set (F,M) is called a soft subnear-ring of N and denoted by $(F,M)\tilde{<}N$ or simply $F_M\tilde{<}N$.

Example 31(cf.,[3]). *Let the additive group $(Z_6,+)$. Under a multiplication given in the following table, $(Z_6,+,.)$ is a (right) near-ring.*

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	3	1	5	3	1	5
2	0	2	4	0	2	4
3	3	3	3	3	3	3
4	0	4	2	0	4	2
5	3	5	1	3	5	1

Let the soft set (F,N) over $N = Z_6$, where $F : N \to P(Z_6)$ is a set-valued function defined by

$$F(x) = \{y \in Z_6 \mid x\alpha y \Leftrightarrow xy \in \{0,3\}\}$$

for all $x \in N$. Then $F(0) = F(3) = Z_6$ and $F(1) = F(2) = F(4) = F(5) = \{0,3\}$. Hence, it is seen that $F_N\tilde{<}N$.
Let the subnear-ring $M = \{0,2,4\}$ of N and let the soft set (G,M) over N, where $G : M \to P(N)$ is defined by

$$G(x) = \{y \in M \mid x\alpha y \Leftrightarrow xy \in \{0,1,2\}\}$$

for all $x \in M$. Then $G(0) = \{0,2,4\}$, $G(2) = \{0,4\}$ and $G(4) = \{0,2\}$. Since $G(0-4) = G(2) = \{0,4\} \not\supseteq G(0) \cap G(4) = \{0,2\}$, (G,M) is not a soft subnear-ring of N.

For a near-ring N, the zero-symmetric part of N denoted by N_0 is defined by $N_0 = \{n \in N \mid n0 = 0\}$, and the constant part of N denoted by N_c is defined by $N_c = \{n \in N \mid n0 = n\}$. It is well known that N_0 and N_c are subnear-rings of N [20]. For a near-ring N, we can obtain at least two soft subnear-rings of N using N_0 and N_c. We give these soft subnear-rings by the following example:

Example 32*Let N be a near-ring and let $F_0 : N_0 \to P(N)$ be a set-valued function defined by $F_0(x) = \{y \in N_0 \mid xy \in N_0\}$ for all $x \in N_0$. Then (F_0,N_0) is a soft subnear-ring of N. In fact, for all $x,y \in N_0$ assume that $a \in F_0(x) \cap F_0(y)$. Then $xa \in N_0$ and $ya \in N_0$. Since N_0 is a subnear-ring of N, then $xa-ya = (x-y)a \in N_0$. Hence $F_0(x-y) \supseteq F_0(x) \cap F_0(y)$, i.e., the condition (s_1) is satisfied. Since $x \in N_0$ and $ya \in N_0$, then $((xy)a)0 = x((ya)0) = x0 = 0$, i.e. $(xy)a \in N_0$. Hence $a \in F_0(xy)$, i.e. $F_0(xy) \supseteq F_0(x) \cap F_0(y)$ and this shows us that the condition (s_2) is satisfied. Therefore $(F_0,N_0)\tilde{<}N$.*
Let $F_c : N_c \to P(N)$ be a set-valued function defined by $F_c(x) = \{y \in N_c \mid xy \in N_c\}$ for all $x \in N_c$. Then (F_c,N_c) is a soft subnear-ring of N. In fact, for all $x,y \in N_c$ assume that $a \in F_c(x) \cap F_c(y)$. Then $xa \in N_c$ and $ya \in N_c$. Since N_c is a subnear-ring of N, then $xa-ya = (x-y)a \in N_c$. Hence $F_c(x-y) \supseteq F_c(x) \cap F_c(y)$, i.e. the condition (s_1) is

satisfied. Since $ya \in N_c$, then $((xy)a)0 = x((ya)0) = x(ya) = (xy)a$, i.e., $(xy)a \in N_c$. Hence $a \in F_c(xy)$, i.e. $F_c(xy) \supseteq F_c(x) \cap F_c(y)$ and this shows us that the condition (s_2) is satisfied. Therefore $(F_c, N_c)\widetilde{<}N$.

Theorem 33 *If $F_M\widetilde{<}N$ and $G_K\widetilde{<}N$, then $F_M\widetilde{\sqcap}G_K\widetilde{<}N$.*

Proof. By Definition 2.2, let $F_M\widetilde{\sqcap}G_K = (F,M)\widetilde{\sqcap}(G,K) = (H, M\cap K)$, where $H(x) = F(x)\cap G(x)$ for all $x \in M\cap K$. Then for all $x,y \in M\cap K$,

s1) $H(x-y) = F(x-y) \cap G(x-y) \supseteq (F(x) \cap F(y)) \cap (G(x) \cap G(y)) = (F(x)\cap G(x))\cap(F(y)\cap G(y)) = H(x)\cap H(y)$ and

s2) $H(xy) = F(xy) \cap G(xy) \supseteq (F(x) \cap F(y)) \cap (G(x) \cap G(y)) = (F(x)\cap G(x))\cap(F(y)\cap G(y)) = H(x)\cap H(y)$.

Therefore $F_M\widetilde{\sqcap}G_K = H_{M\cap K}\widetilde{<}N$.

Definition 4. *Let N_1 and N_2 be near-rings and let $F_M\widetilde{<}N_1$, $G_K\widetilde{<}N_2$. The product of soft subnear-rings (F,M) and (G,K) is defined as $(F,M)\times(G,K) = (H, M\times K)$, where $H(x,y) = F(x)\times G(y)$ for all $(x,y) \in M\times K$.*

Theorem 34 *If $F_M\widetilde{<}N_1$ and $G_K\widetilde{<}N_2$, then $F_M\times G_K\widetilde{<}N_1\times N_2$.*

Proof. Since M and K are subnear-rings of N_1 and N_2 respectively, then $M\times K$ is a subnear-ring of $N_1\times N_2$. By Definition 3.5, let $F_M \times G_K = (F,M) \times (G,K) = (H, M \times K)$, where $H(x,y) = F(x)\times G(y)$ for all $(x,y) \in M\times K$. Then for all $(x_1,y_1),(x_2,y_2) \in M\times K$,

s1) $H((x_1,y_1) - (x_2,y_2)) = H(x_1 - x_2, y_1 - y_2) = F(x_1 - x_2)\times G(y_1 - y_2) \supseteq (F(x_1)\cap F(x_2))\times(G(y_1)\cap G(y_2)) = (F(x_1) \times G(y_1)) \cap (F(x_2) \times G(y_2)) = H(x_1,y_1)\cap H(x_2,y_2)$ and

s2) $H((x_1,y_1)(x_2,y_2)) = H(x_1x_2, y_1y_2) = F(x_1x_2) \times G(y_1y_2) \supseteq (F(x_1) \cap F(x_2)) \times (G(y_1) \cap G(y_2)) = (F(x_1) \times G(y_1)) \cap (F(x_2) \times G(y_2)) = H(x_1,y_1)\cap H(x_2,y_2)$.

Hence $F_M\times G_K = H_{M\times K}\widetilde{<}N_1\times N_2$.

Lemma 35 *If $F_M\widetilde{<}N$, then $F(0) \supseteq F(x)$ for all $x \in M$.*

Proof. Since (F,M) is a soft subnear-ring of N, $F(0) = F(x-x) \supseteq F(x)\cap F(x) = F(x)$ for all $x \in M$.

Proposition 36 *If $F_M\widetilde{<}N$, then $M_F = \{x \in M | F(x) = F(0)\}$ is a subnear-ring of N.*

Proof. We need to show that $x-y \in M_F$ and $xy \in M_F$ for all $x,y \in M_F$ and then to show that $F(x-y) = F(0)$ and $F(xy) = F(0)$ for all $x,y \in M_F$. Since $x,y \in M_F$, then $F(x) = F(y) = 0$. By Lemma 3.7, $F(0) \supseteq F(x-y)$ and $F(0) \supseteq F(xy)$ for all $x,y \in M$. Since (F,M) is a soft subnear-ring of N, then $F(x-y) \supseteq F(x)\cap F(y) = F(0)$ and $F(xy) \supseteq F(x)\cap F(y) = F(0)$ for all $x,y \in M_F$. Hence $F(x-y) = F(0)$ and $F(xy) = F(0)$ for all $x,y \in M_F$. Therefore M_F is a subnear-ring of N.

4 Soft Ideals of Near-rings

Definition 5. *Let $I \lhd N$ and let (F,I) be a soft set over N. If for all $x,y \in I$ and for all $n,s \in N$,*

i_1) $F(x-y) \supseteq F(x)\cap F(y)$,
i_2) $F(n+x-n) \supseteq F(x)$,
i_3) $F(xn) \supseteq F(x)$,
i_4) $F(n(s+x)-ns) \supseteq F(x)$,

then (F,I) is called a soft ideal of N and denoted by $(F,I)\widetilde{\lhd}N$ or simply $F_I\widetilde{\lhd}N$. If $I \lhd_l N$, (F,I) is a soft set over N and if the conditions i_1, i_2 and i_4 are satisfied, then (F,I) is called a soft left ideal of N and denoted by $(F,I)\widetilde{\lhd}_lN$ or simply $F_I\widetilde{\lhd}_lN$. If $I \lhd_r N$, (F,I) is a soft set over N and if the conditions i_1, i_2 and i_3 are satisfied, then (F,I) is called a soft right ideal of N and denoted by $(F,I)\widetilde{\lhd}_rN$ or simply $F_I\widetilde{\lhd}_rN$.

Example 41 *Let $N = (Z_6,+,.)$ be the near-ring given in Example 3.2 and let the ideal $I = \{0,2,4\}$ of N. Then (F,I) is a soft set over N, where $F : I \to P(N)$ is a set-valued function defined by*

$$F(x) = \{y \in I \mid xy = 0\}$$

for all $x \in I$. Then $F(0) = \{0,2,4\}$ and $F(2) = F(4) = \{0\}$. Hence it is seen that $F_I\widetilde{\lhd}N$. Let the ideal $J = \{0,3\}$ of N and let the soft set (G,J) over N, where $G : J \to P(N)$ is a set-valued function defined by

$$G(x) = \{y \in J \mid xy \in \{0,2,4\}\}$$

for all $x \in J$. Then we have $G(0) = \{0,3\}$ and $G(3) = \emptyset$. It is easily seen that $G_J\widetilde{\lhd}N$.

Example 42 *Let N be the near-ring on S_3 with two binary operations as given in table below*

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	5	4	3	2
2	2	4	0	5	1	3
3	3	5	4	0	2	1
4	4	2	3	1	5	0
5	5	3	1	2	0	4

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	1	1	1	1	1
2	1	1	3	2	2	3
3	1	1	2	3	3	2
4	0	0	5	4	4	5
5	0	0	4	5	5	4

Let the soft set (G,N) over N, where $G : N \to P(N)$ is a set-valued function defined by

$$G(x) = \{y \in N \mid x\alpha y \Leftrightarrow xy \in \{0,1\}\}$$

for all $x \in N$. Then $G(0) = G(1) = N$ and $G(2) = G(3) = G(4) = G(5) = \{0,1\}$. Since $G(4+1-4) = G(3) = \{0,1\} \not\supseteq G(1) = N$, then (G,N) is not a soft ideal of N.

Theorem 43*If $F_I \widetilde{\lhd} N$ (resp., $F_I \widetilde{\lhd_l} N$, $F_I \widetilde{\lhd_r} N$) and $G_J \widetilde{\lhd} N$ (resp., $G_J \widetilde{\lhd_l} N$, $G_J \widetilde{\lhd_r} N$), then $F_I \widetilde{\sqcap} G_J \widetilde{\lhd} N$ (resp., $F_I \widetilde{\sqcap} G_J \widetilde{\lhd_l} N$, $F_I \widetilde{\sqcap} G_J \widetilde{\lhd_r} N$).*

*Proof.*We give the proof for soft ideals; the same proof can be seen for soft left ideals and soft right ideals. Since $I, J \lhd N$, then $I \cap J \lhd N$. By Definition 2.2, $F_I \widetilde{\sqcap} G_J = (F,I) \widetilde{\sqcap} (G,J) = (H, I \cap J)$, where $H(x) = F(x) \cap G(x)$ for all $x \in I \cap J$. Then for all $x,y \in I \cap J$ and for all $n,s \in N$:

$i_1)$ $H(x-y) = F(x-y) \cap G(x-y) \supseteq (F(x) \cap F(y)) \cap (G(x) \cap G(y)) = (F(x) \cap G(x)) \cap (F(y) \cap G(y)) = H(x) \cap H(y)$,
$i_2)$ $H(n+x-n) = F(n+x-n) \cap G(n+x-n) \supseteq (F(x) \cap G(x)) = H(x)$,
$i_3)$ $H(xn) = F(xn) \cap G(xn) \supseteq (F(x) \cap G(x)) = H(x)$,
$i_4)$ $H(n(s+x)-ns) = F(n(s+x)-ns) \cap G(n(s+x)-ns) \supseteq (F(x) \cap G(x)) = H(x)$.

Therefore $F_I \widetilde{\sqcap} G_J = H_{I \cap J} \widetilde{\lhd} N$.

Definition 6.*Let N_1 and N_2 be near-rings and let $F_I \widetilde{\lhd} N_1$, $G_J \widetilde{\lhd} N_2$. The product of soft ideals (F,I) and (G,J) is defined as $(F,I) \times (G,J) = (H, I \times J)$, where $H(x,y) = F(x) \times G(y)$ for all $(x,y) \in I \times J$.*

Theorem 44*If $F_I \widetilde{\lhd} N_1$ (resp., $F_I \widetilde{\lhd_l} N_1$, $F_I \widetilde{\lhd_r} N_1$) and $G_J \widetilde{\lhd} N_2$ (resp., $G_J \widetilde{\lhd_l} N_2$, $G_J \widetilde{\lhd_r} N_2$), then $F_I \times G_J \widetilde{\lhd} N_1 \times N_2$ (resp., $F_I \times G_J \widetilde{\lhd_l} N_1 \times N_2$, $F_I \times G_J \widetilde{\lhd_r} N_1 \times N_2$).*

*Proof.*We give the proof for soft ideals; the same proof can be seen for soft left ideals and soft right ideals. Since $I \lhd N_1$ and $J \lhd N_2$, $I \times J \lhd N_1 \times N_2$. By Definition 4.5, $F_I \times G_J = (F,I) \times (G,J) = (H, I \times J)$, where $H(x,y) = F(x) \times G(y)$ for all $(x,y) \in I \times J$. Then for all $(x_1,y_1),(x_2,y_2) \in I \times J$ and for all $(n_1,n_2),(s_1,s_2) \in N_1 \times N_2$,

$i_1)$ $H((x_1,y_1)-(x_2,y_2)) = H(x_1-x_2, y_1-y_2) = F(x_1-x_2) \times G(y_1-y_2) \supseteq (F(x_1) \cap F(x_2)) \times (G(y_1) \cap G(y_2)) = (F(x_1) \times G(y_1)) \cap (F(x_2) \times G(y_2)) = H(x_1,y_1) \cap H(x_2,y_2)$,
$i_2)$ $H((n_1,n_2)+(x_1,y_1)-(n_1,n_2)) = H(n_1+x_1-n_1, n_2+y_1-n_2) = F(n_1+x_1-n_1) \times G(n_2+y_1-n_2) \supseteq F(x_1) \times G(y_1) = H(x_1,y_1)$,
$i_3)$ $H((x_1,y_1)(n_1,n_2)) = H(x_1n_1, y_1n_2) = F(x_1n_1) \times G(y_1n_2) \supseteq F(x_1) \times G(y_1) = H(x_1,y_1)$,
$i_4)$ $H((n_1,n_2)((s_1,s_2)+(x_1,y_1))-(n_1,n_2)(s_1,s_2)) = H(n_1(s_1+x_1)-n_1s_1, n_2(s_2+y_1)-n_2s_2) = F(n_1(s_1+x_1)-n_1s_1) \times G(n_2(s_2+y_1)-n_2s_2) \supseteq F(x_1) \times G(y_1) = H(x_1,y_1)$.

Therefore $F_I \times G_J = H_{I \times J} \widetilde{\lhd} N_1 \times N_2$.

Proposition 45*If $F_I \widetilde{\lhd} N$, then $I_F = \{x \in I | F(x) = F(0)\}$ is an ideal of N.*

*Proof.*We need to show that (i) $x-y \in I_F$, (ii) $n+x-n \in I_F$, (iii) $xn \in I_F$ and (iv) $n(s+x)-ns \in I_F$ for all $x,y \in I_F$ and $n,s \in N$. If $x,y \in I_F$, then $F(x) = F(y) = 0$. By Lemma 3.7, $F(0) \supseteq F(x-y)$, $F(0) \supseteq F(n+x-n)$, $F(0) \supseteq F(xn)$ and $F(0) \supseteq F(n(s+x)-ns)$ for all $x,y \in I$ and $n,s \in N$. Since (F,I) is a soft ideal of N, then for all $x,y \in I_F$ and $n,s \in N$, (i) $F(x-y) \supseteq F(x) \cap F(y) = F(0)$, (ii) $F(n+x-n) \supseteq F(x) = F(0)$, (iii) $F(xn) \supseteq F(x) = F(0)$ and (iv) $F(n(s+x)-ns) \supseteq F(x) = F(0)$. Hence $F(x-y) = F(0)$, $F(n+x-n) = F(0)$, $F(xn) = F(0)$ and $F(n(s+x)-ns) = F(0)$ for all $x,y \in I_F$ and $n,s \in N$. Therefore I_F is an ideal of N.

5 Soft N-subgroups of Near-rings

Definition 7.*Let H be a right (resp., left) N-subgroup of N and let (F,H) be a soft set over N. If for all $x,y \in H$ and $n \in N$,*

g1)$F(x-y) \supseteq F(x) \cap F(y)$ and
g2)$F(xn) \supseteq F(x)$ (resp. $F(nx) \supseteq F(x)$),

then the soft set (F,H) is called a soft right (resp., left) N-subgroup of N and denoted by $(F,H) \widetilde{<}_{N^r} N$ (resp.$(F,H) \widetilde{<}_{N^l} N$) or simply $F_H \widetilde{<}_{N^r} N$ (resp., $F_H \widetilde{<}_{N^l} N$).

Proposition 51*If (F,H) is a soft right (resp., left) N-subgroup of N, then $H_F = \{x \in H | F(x) = F(0)\}$ is a right (resp., left) N-subgroup of N.*

*Proof.*We give the proof for soft right N-subgroups of N, the same proof can be seen for soft left N-subgroups of N. It has to be showed that (i) $x-y \in H_F$ and (ii) $xn \in H_F$ for all $x,y \in H_F$ and $n \in N$. If $x,y \in H_F$, then $F(x) = F(y) = 0$. By Lemma 3.7, $F(0) \supseteq F(x-y)$ and $F(0) \supseteq F(xn)$ for all $x,y \in H$ and $n \in N$. Since (F,H) is a soft right N-subgroup of N, then for all $x,y \in H_F$ and $n \in N$, $F(x-y) \supseteq F(x) \cap F(y) = F(0)$ and $F(xn) \supseteq F(x) = F(0)$. Hence $F(x-y) = F(0)$ and $F(xn) = F(0)$ for all $x,y \in H_F$ and $n \in N$. Therefore H_F is right N-subgroup of N.

Example 52*Let the near-ring $N = (S_3,+,\cdot)$ given in Example 4.3. Let $H = \{0,1\}$. Then H is both left and right N-subgroup of N. Let the soft set (F,H) over N, where $F : H \to P(N)$ is a set-valued function defined by*

$$F(x) = \{y \in N \mid xy = 0\}$$

for all $x \in H$. Then $F(0) = N$ and $F(1) = \emptyset$. It is seen that (F,H) is a soft right N-subgroup of N. Since $F(2.0) = F(1) = \emptyset \not\supseteq F(0) = N$, (F,H) is not a soft left N-subgroup of N. To illustrate Proposition 5.2., it is seen that $H_F = \{x \in H | F(x) = F(0)\} = \{x \in H | F(x) = N\} = \{0\}$ is a right N-subgroup of N.

Theorem 53*If $F_H \widetilde{<}_{N^l} N$ (resp., $F_H \widetilde{<}_{N^r} N$) and $G_K \widetilde{<}_{N^l} N$ (resp., $G_K \widetilde{<}_{N^r} N$), then $F_H \widetilde{\sqcap} G_K \widetilde{<}_{N^l} N$ (resp., $F_H \widetilde{\sqcap} G_K \widetilde{<}_{N^r} N$).*

Proof. Since every ideal of N is also a right N-subgroup of N, the proof for soft right N-subgroups is obtained by Theorem 4.4. Let $F_H \widetilde{<}_{N^l} N$ and $G_K \widetilde{<}_{N^l} N$. Since H and K are left N-subgroups of N, it is easily seen that $H \cap K$ is a left N-subgroup of N. By Definition 2.2, $F_H \widetilde{\sqcap} G_K = (F,H) \widetilde{\sqcap} (G,K) = (T, H \cap K)$, where $T(x) = F(x) \cap G(x)$ for all $x \in H \cap K$. Then for all $x,y \in H \cap K$ and for all $n \in N$:

$g_1)$ $T(x-y) = F(x-y) \cap G(x-y) \supseteq (F(x) \cap F(y)) \cap (G(x) \cap G(y)) = (F(x) \cap G(x)) \cap (F(y) \cap G(y)) = T(x) \cap T(y)$,
$g_2)$ $T(nx) = F(nx) \cap G(nx) \supseteq (F(x) \cap G(x)) = T(x)$.

Therefore $F_H \widetilde{\sqcap} G_K = T_{H \cap K} \widetilde{<}_{N^l} N$.

Definition 8. *Let N_1 and N_2 be near-rings and let $F_H \widetilde{<}_{N_1^l} N_1$ (resp., $F_H \widetilde{<}_{N_1^r} N_1$) and $G_K \widetilde{<}_{N_2^l} N_2$ (resp., $G_K \widetilde{<}_{N_2^r} N_2$). The product of soft left N-subgroups (resp., soft right N-subgroups) (F,H) and (G,K) is defined as $(F,H) \times (G,K) = (T, H \times K)$, where $T(x,y) = F(x) \times G(y)$ for all $(x,y) \in H \times K$.*

Theorem 54 *If $F_H \widetilde{<}_{N_1^l} N_1$ (resp., $F_H \widetilde{<}_{N_1^r} N_1$) and $G_K \widetilde{<}_{N_2^l} N_2$ (resp., $G_K \widetilde{<}_{N_2^r} N_2$), then $F_H \times G_K \widetilde{<}_{(N_1 \times N_2)^l} N_1 \times N_2$ (resp., $F_H \times G_K \widetilde{<}_{(N_1 \times N_2)^r} N_1 \times N_2$).*

Proof. The proof for soft right N-subgroups is obtained by Theorem 4.6. Let $F_H \widetilde{<}_{N_1^l} N_1$ and $G_K \widetilde{<}_{N_2^l} N_2$. Since H is a left N_1-subgroup of N_1 and K is a left N_2-subgroup of N_2, it is easily seen that $H \times K$ is a left $N_1 \times N_2$-subgroup of $N_1 \times N_2$. By Definition 5.5, $F_H \times G_K = (F,H) \times (G,K) = (T, H \times K)$, where $T(x) = F(x) \times G(y)$ for all $(x,y) \in H \times K$. Then for all $(x_1,y_1),(x_2,y_2) \in H \times K$ and for all $(n,s) \in N_1 \times N_2$:

$g_1)$ $T((x_1,y_1)-(x_2,y_2)) = T(x_1-x_2, y_1-y_2) = F(x_1-x_2) \times G(y_1-y_2) \supseteq (F(x_1) \cap F(x_2)) \times (G(y_1) \cap G(y_2)) = (F(x_1) \times G(y_1)) \cap (F(x_2) \times G(y_2)) = T(x_1,y_1) \cap T(x_2,y_2)$,
$g_2)$ $T((n,s)(x_1,y_1)) = T(nx_1, sy_1) = F(nx_1) \times G(sy_1) \supseteq F(x_1) \times G(y_1) = T(x_1,y_1)$.

Therefore $F_H \times G_K = T_{H \times K} \widetilde{<}_{(N_1 \times N_2)^l} N_1 \times N_2$.

6 Conclusion

Throughout this paper, in a near-ring structure we study the algebraic properties of soft sets which were introduced by Molodtsov as a new mathematical tool for dealing with uncertainty. This work bears on soft subnear-rings, soft (left, right) ideals and soft left (right) N-subgroups of near-rings. Since every associative ring is a near-ring, the results in this study are also true for associative rings. To extend this work, one could study the soft substructures of other algebraic structures such as semirings and fields.

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