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Efficient Method for Fractional Lévy-Feller Advection-Dispersion Equation Using Jacobi Polynomials

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Abstract: This paper presents a practical formula expressing the fractional-order derivatives, in the sense of Riesz-Feller operator, of Jacobi polynomials. Jacobi spectral collocation method and the trapezoidal rule are used to reduce the fractional Lévy-Feller advection-dispersion equation (LFADE) to a system of algebraic equations which simplifies solving such fractional differential equation. Numerical simulations with some comparisons are introduced to confirm the effectiveness and reliability of the proposed technique for the Lévy-Feller fractional partial differential equations.

Keywords: Riesz-Feller fractional operator, Lévy-Feller advection-dispersion equation, spectral method, Jacobi polynomials, trapezoidal rule.

1 Introduction

Recently, the notion of fractional calculus and differential operators of fractional order has gained great popularity because of its engaging applications in different fields, such as engineering, finance, system control, hydrology, viscoelasticity, and physics [1,2,3,4,5]. The fractional models often can be described as ordinary or partial fractional differential equations. Differential equations are more suitable than standard integer-order differential equations to describe the memorial and genetic characteristic of several phenomena and materials [6].

Analytic solutions of most fractional differential equations can not always be obtained explicitly, so the utilization of various numerical methods for solving like these equations is very necessary. Some of the numerical approaches proposed for solving such equations are: finite difference methods [7,8], finite element methods [9], semi-analytic methods [10,11, 12], spectral methods [13,14], higher order numerical methods [15,16].

It is well known that the spectral method is a class of approaches in which the numerical approximation is expressed in terms of some certain "basis functions". These basis functions can be orthogonal polynomials (see, for example [17, 18] and the references therein). Collocation methods, one of the three well-known kinds of spectral methods, have turned out increasingly to be common used in solving ordinary and partial differential equations. Despite when utilizing a little number of grids, these spectral collocation methods are valuable in finding high accurate approximation solutions for linear and nonlinear differential equations because they have an exponential convergence rate. Choosing collocation nodes has a very important role in the stability and convergence of these spectral collocation methods [19]. Spectral collocation methods were applied to approximate the solutions of both the linear and nonlinear fractional partial differential equations [13,14,20] and the fractional integro-differential equations [21,22].

The Jacobi polynomials, which we denote by $(J_k^{\beta,\gamma}(x),\ k \geqslant 0,\ \beta > -1, \gamma > -1)$, have been extensively used in mathematical analysis and practical applications. They play an important role in the analysis and implementation of spectral methods. Using of the Jacobi polynomials benefit in getting the approximation solutions in terms of the parameters β and γ. Hence, it is very helpful to carry out a systematic study with general indexes on the Jacobi polynomials instead of developing approximation results for each specific pair of indexes. Accordingly, we can directly apply Jacobi polynomials to other applications.

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In physics, anomalous diffusion phenomena are modeled using fractional derivatives, such that the particles spreading differently from the standard motion of Brownian type [2] follow Lévy stable motion [23]. The Fokker-Planck equations, Reaction-diffusion equations, diffusion advection equations, and Kinetic equations of the diffusion can be applications of the phenomenon of anomalous diffusion. The Fractional Advection-Dispersion Equation (FADE), i.e. the fractional kinetic equation [24], proved to be an amplification of the time-continuous random walk model. FADE is utilized in the hydrology of the groundwater for describing the transfer of passive traces held by fluid flow in a permeable environment [25].

Considerable approximating methods have been proposed to solve FADE. Roop [26] proposed a finite element method to find numerical solutions of FADE in two spatial dimensions. Meerschaert *et al.* [27] developed a workable approximation technique to resolve the space FADE on a finite domain in one dimensional with variable coefficients. Liu *et al.* [25] suggested a technique to find the solution of the Lévy-Feller advection-dispersion model depending on using a finite difference method and random walk. EI-Sayed *et al.* [28] solved the advection-dispersion equation with the fractional Caputo derivative using the Adomian's decomposition technique. Golbabai *et al.* [10] used the homotopy perturbation approach to find analytic solutions to FADE. In [29] Shen *et al.* utilized the fractional finite difference approximating schema with a weight factor for FADE with the Riesz operator. Bhrawy *et al.* in [30] offered method depending on the operational matrices to solve the two sided FADE. Recently, Feng *et al.*, depending on the Shifted Grünwald Difference model with a weight factor, gave approximation to the Riesz fractional operator for finding approximation solution of the space fractional FADE when the fractional derivative is defined in sense of Riesz derivative in [31].

The present paper addresses the Lévy-Feller Advection-Dispersion Equation (LFADE) [25], with source term, which is obtained by replacing the second-order space derivative with the Riesz-Feller fractional derivative of order α and skewness θ, ($|\theta| \leq min\{\alpha, 2-\alpha\}$) in the standard advection-dispersion equation. LFADE takes the following form:

$$\frac{\partial u(x,t)}{\partial t} = dD^{\alpha}_{\theta}u(x,t) - e\frac{\partial u(x,t)}{\partial x} + s(x,t), \quad d > 0, \;\; e \geq 0, \;\; t \geq 0, \quad x \in \mathbb{R}, \tag{1}$$

with initial condition:

$$u(x,0) = f(x), \tag{2}$$

where the operator D^{α}_{θ} is the Riesz-Feller fractional derivative operator of order α and skewness θ. The constants e and d represent the average fluid velocity and the dispersion coefficient respectively and $s(x,t)$ is the source term. The fundamental solution of (1) and (2) has been derived using the Fourier transform [32] as:

$$G'_{\alpha}(k,t;\theta) = exp(-td|k|^{\alpha}e^{isign(k)\theta\pi/2} + itek), \quad k \in \mathbb{R}. \tag{3}$$

Numerical studies of Eq.(1) have been commonly obtained by the finite difference methods (FDMs) (see [25]) with their limited accuracy, because they FDMs have a local character. However, fractional derivatives are essentially global differential operators. Hence, global schemes such as spectral methods may be more appropriate for discretizing fractional operators.

The present paper aims to construct an accurate numerical technique for solving equations (1) and (2) using Jacobi spectral collocation (JSC) method combined with the trapezoidal rule (Crank-Nicolson method) in $\Omega : a \leq x \leq b$ (one-dimensional domain) with specific Dirichlet conditions of the boundary as follows:

$$u(a,t) = 0, \; u(b,t) = 0, \tag{4}$$

Implementing JSC method to the spatial variable of the fractional advection-dispersion equation and using the boundary conditions reduce the problem to solve, with respect to the time, a system of ordinary differential equations. Then this system is solved using the trapezoidal rule to reduce the problem to solve system of algebraic equations which are far easier to be solved. This is a generalization of the previous authors' work in [14].

Few pieces of literature to approximate the numerical solution of the fractional differential equations using the Riesz-Feller fractional deferential operator to describe the derivatives (see [23,25,33,34,35], most of them used FDM). We used in the previous work [14] Chebyshev-Legendre collocation technique with the first-order Euler manner to solve Lévy-Feller diffusion equation numerically. To the authors' knowledge, No paper has used spectral method to solve Lévy-Feller advection-dispersion equations. Thus, we have been motivated to conduct this study.

The present paper is outlined as follows: The following section involves some definitions relevant to calculus of fractional order, some attributes of Jacobi polynomial. In Section 3, we propose and prove an explicit formula to the derivative of the Jacobi polynomials where the derivative is Riesz-Feller fractional operator. In Section 4 we , we apply JSC method to solve (1, 2, 4) on Ω and change this model to a finite system of ordinary differential equations, which is solved using the trapezoidal rule. Section 5 presents a few numerical outcomes to show the applicability and the accuracy of the introduced technique. Section 6is dedicated to conclusion.

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2 Definitions and fundamentals

This section comprises the necessary mathematical preliminaries properties of the calculus of the fractional derivative.

2.1 *Some properties of fractional calculus*

Definition 2.1

For $0<\alpha<2$, $\alpha\neq 1$ and $|\theta|\leq min\{\alpha,2-\alpha\}$, the Riesz-Feller fractional operator D^{α}_{θ} is represented in the following form (see e.g. [23,33,34,36])

$$D^{\alpha}_{\theta}f(x)=-(c_{+}D^{\alpha}_{+}+c_{-}D^{\alpha}_{-})f(x), \tag{5}$$

where the coefficients $c_{\pm}$ are given by

$$c_{+}=c_{+}(\alpha,\theta)=\frac{sin((\alpha-\theta)\pi/2)}{sin(\alpha\pi)},\;\; c_{-}=c_{-}(\alpha,\theta)=\frac{sin((\alpha+\theta)\pi/2)}{sin(\alpha\pi)}, \tag{6}$$

and

$$(D^{\alpha}_{+}f)(x)=(\frac{d}{dx})^{n}(I^{n-\alpha}_{+}f)(x),\;\; (D^{\alpha}_{-}f)(x)=(-\frac{d}{dx})^{n}(I^{n-\alpha}_{-}f)(x), \tag{7}$$

are the right side and left side of the Liouville derivatives of fractional order where $x\in\mathbb{R}$ and $n-1<\alpha\leq n,\ n=1,\ 2.$

The integral operators $I^{n-\alpha}_{\pm}$, in expressions (7) are known as the right side and left side of Liouville fractional integrals. These integrals are specified by the following definition:

Definition 2.2 For $\alpha>0$,

$$(I^{\alpha}_{+}f)(x)=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}\frac{f(\xi)}{(x-\xi)^{1-\alpha}}d\xi,\;\; (I^{\alpha}_{-}f)(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{+\infty}\frac{f(\xi)}{(\xi-x)^{1-\alpha}}d\xi. \tag{8}$$

For $\alpha=1$, the representation (5) is not valid and has to be replaced by the formula

$$D^{1}_{\theta}f(x)=[cos(\theta\pi/2)D^{1}_{0}-sin(\theta\pi/2)D]f(x), \tag{9}$$

where D refers to the first standard derivative and the operator D^{1}_{0} is related to the Hilbert transform as first noted by Feller in his pioneering paper [37]

$$D^{1}_{0}=\frac{1}{\pi}\frac{d}{dx}\int_{-\infty}^{+\infty}\frac{f(\xi)}{x-\xi}d\xi$$

.

For $\alpha=2$ (such that, $\theta=0$), $D^{\alpha}_{\theta}f(x)=\frac{d^2f(x)}{dx^2}$.

We conclude depending on the definition (2.1) that the Riesz-Feller fractional operator is a linear combination of right side and left side of Riemann-Liouville fractional operator, so:

$$D^{\alpha}_{\theta}(\lambda f(x)+\gamma g(x))=\lambda D^{\alpha}_{\theta}f(x)+\gamma D^{\alpha}_{\theta}g(x).$$

Recalling the following helpful merit of the right side and left side Riemann-Liouville fractional operator from [38]. Assuming $x\in[a,b],\ a,b\in\mathbb{R}$ and $f(a)=f(b)=0$, therefor we have

$${}_{a}D^{p}_{x}\,{}_{a}I^{p}_{x}f(x)=f(x),\qquad {}_{x}D^{p}_{b}\,{}_{x}I^{p}_{b}f(x)=f(x), \tag{10}$$

for $0<p,q\leq 1$.

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2.2 Some relevant characteristics of Jacobi polynomials

We introduce in the present section a few requisite properties of (shifted) Jacobi polynomials ($J_k^{\beta,\gamma}(x),\quad k \geqslant 0,\ \beta > -1,\ \gamma > -1$) that are most relevant to the proposed spectral collocation approximations [39,40,41,42].

The following recurrence relation can give all Jacobi polynomials

$$J_{k+1}^{\beta,\gamma}(x) = (a_k^{\beta,\gamma}x - b_k^{\beta,\gamma})J_k^{\beta,\gamma}(x) - c_k^{\beta,\gamma}J_{k-1}^{\beta,\gamma}(x), k = 1,\ 2,\ ...,$$

$$J_0^{\beta,\gamma}(x) = 1, \qquad J_1^{\beta,\gamma}(x) = \frac{(\beta+\gamma+2)x+\beta-\gamma}{2},$$

where

$$a_k^{\beta,\gamma} = \frac{(2k+\beta+\gamma+1)(2k+\beta+\gamma+2)}{2(k+1)(k+\beta+\gamma+1)},$$

$$b_k^{\beta,\gamma} = \frac{(2k+\beta+\gamma+1)(\gamma^2-\beta^2)}{2(k+1)(k+\beta+\gamma+1)(2k+\beta+\gamma)},$$

$$c_k^{\beta,\gamma} = \frac{(k+\beta)(k+\gamma)(2k+\beta+\gamma+2)}{(k+1)(k+\beta+\gamma+1)(2k+\beta+\gamma)}.$$

Here, we recall here some important properties of Jacobi polynomials

$$J_k^{\beta,\gamma}(-x) = (-1)^k J_k^{\gamma,\beta}(x), \qquad J_k^{\beta,\gamma}(1) = \frac{\Gamma(k+\beta+1)}{k!\Gamma(\beta+1)}, \tag{11}$$

$$\frac{d^m}{dx^m}J_k^{\beta,\gamma}(x) = \frac{\Gamma(m+k+\beta+\gamma+1)}{2^m\Gamma(k+\beta+\gamma+1)}J_{k-m}^{\beta+m,\gamma+m}(x). \tag{12}$$

The requisite property of the Jacobi polynomials is: they consider the eigenfunctions of the Sturm-Liouville singular equation:

$$(1-x^2)\phi''(x) + [\gamma-\beta+(\beta+\gamma+2)x]\phi'(x) + k(k+\beta+\gamma+1)\phi(x) = 0.$$

In order to use Jacobi polynomials on the interval $[0,L],\ L>0$, we mention the shifted Jacobi polynomials $J_{L,k}^{\beta,\gamma}(x) = J_k^{\beta,\gamma}(\frac{2x-L}{L})$.

The analytic form of the shifted Jacobi polynomials $J_{L,k}^{\beta,\gamma}(x)$ of degree k (k integer) is given by

$$J_{L,k}^{\beta,\gamma}(x) = \sum_{i=0}^{k}(-1)^{k-i}\frac{\Gamma(k+\gamma+1)\Gamma(k+i+\beta+\gamma+1)}{\Gamma(i+\gamma+1)\Gamma(k+\beta+\gamma+1)(k-i)!i!L^i}x^i, \tag{13}$$

or,

$$J_{L,k}^{\beta,\gamma}(x) = \sum_{i=0}^{k}\frac{\Gamma(k+\beta+1)\Gamma(k+i+\beta+\gamma+1)}{\Gamma(i+\beta+1)\Gamma(k+\beta+\gamma+1)(k-i)!i!L^i}(x-L)^i, \tag{14}$$

where

$$J_{L,k}^{\beta,\gamma}(0) = (-1)^k\frac{\Gamma(k+\gamma+1)}{\Gamma(\gamma+1)k!},$$

and

$$J_{L,k}^{\beta,\gamma}(L) = \frac{\Gamma(k+\beta+1)}{\Gamma(\beta+1)k!}.$$

The orthogonality condition of shifted Jacobi polynomials is

$$\int_0^L J_{L,k}^{\beta,\gamma}(x)J_{L,j}^{\beta,\gamma}(x)w_L^{\beta,\gamma}(x)dx = h_k,$$

where

$$w_L^{\beta,\gamma}(x) = x^\gamma(L-x)^\beta \ \text{ and } h_k = \begin{cases} \frac{L^{\beta+\gamma+1}\Gamma(k+\beta+1)\Gamma(k+\gamma+1)}{(2k+\beta+\gamma+1)k!\Gamma(k+\beta+\gamma+1)}, & k=j, \\ 0, & k \neq j. \end{cases}$$

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The expansion of x^i and $(x-L)^i$ in terms of shifted Jacobi polynomials are given, respectively, by:

$$x^i = \frac{(\gamma+1)_i}{(\beta+\gamma+2)_i}\sum_{k=0}^{i}\frac{(-1)^k L^i(-i)_k(\beta+\gamma+2k+1)(\beta+\gamma+2)_{k-1}}{(1+\gamma)_k(\beta+\gamma+i+2)_k}J_{L,k}^{\beta,\gamma}(x),$$

$$(x-L)^i = \frac{(\beta+1)_i}{(\beta+\gamma+2)_i}\sum_{k=0}^{i}\frac{L^i(-i)_k(\beta+\gamma+2k+1)(\beta+\gamma+2)_{k-1}}{(1+\beta)_k(\beta+\gamma+i+2)_k}J_{L,k}^{\beta,\gamma}(x),$$

where $(.)_k$ is Pochhammer's symbol.

Assume $f(t)\in L^2_{w_L^{\beta,\gamma}(x)}(0,L)$, then this function can be expanded by means of the shifted Jacobi polynomials as the following form [43]:

$$f(x) = \sum_{j=0}^{\infty} c_j J_{L,j}^{\beta,\gamma}(x), \tag{15}$$

where

$$c_j = \frac{1}{h_k}\int_0^L w_L^{\beta,\gamma}(x) f(x) J_{L,j}^{\beta,\gamma}(x)dx, \quad j=0,1,2,\cdots.$$

If we approximate $f(x)$ by the first M term, then we have the following approximation

$$f(x) = \sum_{j=0}^{M} c_j J_{L,j}^{\beta,\gamma}(x). \tag{16}$$

Here, we recall that the ultraspherical, Legendre, and Chebyshev polynomials are particular cases of the Jacobi polynomials.

3 Riesz-Feller fractional derivative of the shifted Jacobi polynomials

For $1<\alpha<2$, depending on definition of Riemann-Liouville fractional derivatives on $[0,L]$,

$${}_0D_x^{\alpha}(x)^k = \frac{\Gamma(k+1)}{\Gamma(k-p+1)}(x)^{k-\alpha}, \quad k>-1, \tag{17}$$

$${}_xD_L^{\alpha}(x-L)^k = \frac{(-1)^k\Gamma(k+1)}{\Gamma(k-\alpha+1)}(L-x)^{k-\alpha}, \quad k>-1, \tag{18}$$

Theorem 3.1 The analytic form of the left-side Riemann-Liouville fractional derivative of the shifted Jacobi polynomial on $[0,L]$ is written in the following form:

$${}_0D_x^{\alpha}J_{L,j}^{\beta,\gamma}(x) = \sum_{k=0}^{j}\sum_{i=0}^{k}\Theta_{i,j,k}^{\alpha,\beta,\gamma}\times\Upsilon_{i,k}^{\alpha,\beta,\gamma}\times x^{-\alpha}\times J_{L,j}^{\beta,\gamma}(x), \tag{19}$$

where,

$$\Theta_{i,j,k}^{\alpha,\beta,\gamma} = \frac{(-1)^{(i+j+k)}\Gamma(1+\beta+\gamma+j+k)\Gamma(1+\gamma+j)\Gamma(1+k)}{\Gamma(1+\beta+\gamma+j)\Gamma(1+\gamma+k)\Gamma(1+k-\alpha)(j-k)!k!},$$
$$\Upsilon_{i,k}^{\alpha,\beta,\gamma} = \frac{(-k)_i(1+\gamma)_k(2+\beta+\gamma)_{i-1}(1+\beta+\gamma+2i)}{(1+\gamma)_i(2+\beta+\gamma)_k(2+k+\beta+\gamma)_i}. \tag{20}$$

Proof. See [13]. The proof is driven depending on linearity of Riemann-Liouville fractional operator, relation (17) and the expansion of x^k in terms of shifted Jacobi polynomials.

Theorem 3.2 The analytic form of the right-side Riemann-Liouville fractional derivative of the shifted Jacobi polynomial on $[0,L]$ is given by:

$${}_xD_L^{\alpha}J_{L,j}^{\beta,\gamma}(x) = \sum_{k=0}^{j}\sum_{i=0}^{k}\overline{\Theta}_{j,k}^{\alpha,\beta,\gamma}\times\overline{\Upsilon}_{i,k}^{\beta,\gamma}\times(L-x)^{-\alpha}\times J_{L,j}^{\beta,\gamma}(x), \tag{21}$$

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where,

$$\overline{\Theta}_{j,k}^{\alpha,\beta,\gamma} = \frac{(-1)^{(k)}\Gamma(1+\beta+\gamma+j+k)\Gamma(1+\beta+j)\Gamma(1+k)}{\Gamma(1+\beta+\gamma+j)\Gamma(1+\beta+k)\Gamma(1+k-\alpha)(j-k)!k!},$$
$$\overline{\Upsilon}_{i,k}^{\beta,\gamma} = \frac{(-k)_i(1+\beta)_k(2+\beta+\gamma)_{i-1}(1+\beta+\gamma+2i)}{(1+\beta)_i(2+\beta+\gamma)_k(2+k+\beta+\gamma)_i}. \quad (22)$$

Proof. See [13]. The proof is driven depending on linearity of Riemann-Liouville fractional operator, relation (18) and the expansion of $(x-L)^k$ in terms of shifted Jacobi polynomials.

Theorem 3.3 The analytic form of the fractional derivative of the shifted Jacobi polynomial on $[0,L]$ in sense of Riesz-Feller fractional operator is as follows:

$$\begin{aligned}
D_\theta^\alpha J_{L,j}^{\beta,\gamma}(x) &= -(c_+\, {}_0D_x^\alpha J_{L,j}^{\beta,\gamma}(x) + c_-\, {}_xD_L^\alpha J_{L,j}^{\beta,\gamma}(x)),\\
&= -\Big[c_+ \sum_{k=0}^{j}\sum_{i=0}^{k} \Theta_{i,j,k}^{\alpha,\beta,\gamma} \times \Upsilon_{i,k}^{\alpha,\beta,\gamma} \times x^{-\alpha} \times J_{L,j}^{\beta,\gamma}(x)\\
&\quad + c_- \sum_{k=0}^{j}\sum_{i=0}^{k} \overline{\Theta}_{j,k}^{\alpha,\beta,\gamma} \times \overline{\Upsilon}_{i,k}^{\beta,\gamma} \times (L-x)^{-\alpha} \times J_{L,j}^{\beta,\gamma}(x)\Big],\\
&= -\sum_{k=0}^{j}\sum_{i=0}^{k}[c_+\, \Theta_{i,j,k}^{\alpha,\beta,\gamma} \times \Upsilon_{i,k}^{\alpha,\beta,\gamma} \times x^{-\alpha} + c_-\, \overline{\Theta}_{j,k}^{\alpha,\beta,\gamma} \times \overline{\Upsilon}_{i,k}^{\beta,\gamma} \times (L-x)^{-\alpha}] J_{L,j}^{\beta,\gamma}(x),\\
&= \sum_{k=0}^{j}\sum_{i=0}^{k} \Psi_{i,j,k}^{\alpha,\beta,\gamma} \times J_{L,j}^{\beta,\gamma}(x). \quad (23)
\end{aligned}$$

Where

$$\Psi_{i,j,k}^{\alpha,\beta,\gamma} = -[c_+\, \Theta_{i,j,k}^{\alpha,\beta,\gamma} \times \Upsilon_{i,k}^{\alpha,\beta,\gamma} \times x^{-\alpha} + c_-\, \overline{\Theta}_{j,k}^{\alpha,\beta,\gamma} \times \overline{\Upsilon}_{i,k}^{\beta,\gamma} \times (L-x)^{-\alpha}]. \quad (24)$$

4 Proceedings of solution for the Lévy-Feller advection-dispersion equation

This section presents numerical algorithm to approximate the solution of the Lévy-Feller advection-dispersion equation which given as follows:

$$\begin{cases}
\frac{\partial u(x,t)}{\partial t} = dD_\theta^\alpha u(x,t) - e\frac{\partial u(x,t)}{\partial x} + s(x,t), & t>0,\ 0<x<L,\ 1<\alpha\le 2,\\
u(0,t)=0, \qquad u(L,t)=0, & t>0,\\
u(x,0)=f(x), \qquad 0\leqslant x\leqslant L,
\end{cases} \quad (25)$$

assuming $u(x,t)=0$ for $x\in\mathbb{R}\backslash[0,L]$.
For using Jacobi spectral collocation method, we usually write $u(x,t)$ as follows:

$$u_m(x,t) = \sum_{j=0}^{m} u_j(t) J_{L,j}^{\beta,\gamma}(x). \quad (26)$$

Using properties of Riesz-Feller fractional derivatives:

$$\begin{aligned}
D_\theta^\alpha u_m(x,t) &= \sum_{j=0}^{m} u_j(t) D_\theta^\alpha J_{L,j}^{\beta,\gamma}(x),\\
&= \sum_{j=0}^{m} u_j(t) \sum_{k=0}^{j}\sum_{i=0}^{k} \Psi_{i,j,k}^{\alpha,\beta,\gamma} \times J_{L,j}^{\beta,\gamma}(x),\\
&= \sum_{j=0}^{m}\sum_{k=0}^{j}\sum_{i=0}^{k} u_j(t) \times \Psi_{i,j,k}^{\alpha,\beta,\gamma} \times J_{L,j}^{\beta,\gamma}(x).
\end{aligned}$$

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Thus, Eq. (25) takes the following form:

$$\sum_{j=0}^{m}\frac{du_j(t)}{dt}J_{L,j}^{\beta,\gamma}(x)=d\sum_{j=0}^{m}\sum_{k=0}^{j}\sum_{i=0}^{k}u_j(t)\times\Psi_{i,j,k}^{\alpha,\beta,\gamma}\times J_{L,j}^{\beta,\gamma}(x)-e\sum_{j=0}^{m}u_j(t)\frac{d}{dx}J_{L,j}^{\beta,\gamma}(x)+s(x,t). \tag{27}$$

We now collocate equation. (27) at $(m-1)$ points $x_q,\ \ q=1\ ,2\ ,...,\ m-1,\ \ (a<x_q<b)$ as follows:

$$\sum_{j=0}^{m}\frac{du_j(t)}{dt}J_{L,j}^{\beta,\gamma}(x_q)=d\sum_{j=0}^{m}\sum_{k=0}^{j}\sum_{i=0}^{k}u_j(t)\times\Psi_{i,j,k}^{\alpha,\beta,\gamma}\big|_{x=x_q}\times J_{L,j}^{\beta,\gamma}(x_q)-e\sum_{j=0}^{m}u_j(t)\frac{d}{dx}J_{L,j}^{\beta,\gamma}(x)\big|_{x=x_q}+s(x_q,t). \tag{28}$$

Substituting Eq. (26) in both the initial condition and the boundary conditions respectively gives us the constants u_i in case of $t=0$ and the following two equations:

$$\sum_{j=0}^{m}(-1)^j\frac{\Gamma(j+\gamma+1)}{\Gamma(\gamma+1)j!}u_j(t)=0,\qquad\sum_{j=0}^{m}\frac{\Gamma(j+\beta+1)}{\Gamma(\beta+1)j!}u_j(t)=0, \tag{29}$$

The ordinary differential equations (28) and (29) model a system of $(m+1)$ equation. All these equations are in the unknown $u_j,\ j=0,1,...,m$. The trapezoidal rule is used to solve this system (which is implicit, stable and second-order method) as follows:
Let $0<t_n<T_{final}$ and let $\triangle t=\frac{T_{final}}{N},\ \ t_n=n\triangle t,\ for\ n=0,1,2,...,N$, then we build the next system of algebraic equations:

$$\begin{cases}\sum\limits_{j=0}^{m}(-1)^j\frac{\Gamma(j+\gamma+1)}{\Gamma(\gamma+1)j!}u_j^n=0,\\ \sum\limits_{j=0}^{m}\frac{u_j^n-u_j^{n-1}}{\triangle t}J_{L,j}^{\beta,\gamma}(x_q)=\frac{1}{2}\Bigg[\Bigg(d\sum\limits_{j=0}^{m}\sum\limits_{k=0}^{j}\sum\limits_{i=0}^{k}u_j^n\times\Psi_{i,j,k}^{\alpha,\beta,\gamma}\big|_{x=x_q}\times J_{L,j}^{\beta,\gamma}(x_q)\\ \qquad\qquad-e\sum\limits_{j=0}^{m}u_j^n\frac{d}{dx}J_{L,j}^{\beta,\gamma}(x)\big|_{x=x_q}+s_q^n\Bigg)\\ \qquad+\Bigg(d\sum\limits_{j=0}^{m}\sum\limits_{k=0}^{j}\sum\limits_{i=0}^{k}u_j^{n-1}\times\Psi_{i,j,k}^{\alpha,\beta,\gamma}\big|_{x=x_q}\times J_{L,j}^{\beta,\gamma}(x_q)\\ \qquad\qquad-e\sum\limits_{j=0}^{m}u_j^{n-1}\frac{d}{dx}J_{L,j}^{\beta,\gamma}(x)\big|_{x=x_q}+s_q^{n-1}\Bigg)\Bigg],\\ \qquad q=1,2,...,m-1,\\ \sum\limits_{i=0}^{m}\frac{\Gamma(j+\beta+1)}{\Gamma(\beta+1)j!}u_j^n=0,\end{cases} \tag{30}$$

with the initial conditions:

$$\sum_{j=0}^{m}u_j^0J_{L,j,q}^{\beta,\gamma}=f_q,\quad q=0,1,2,...,m,$$

where $u_j^n=u_j(t_n),\quad J_{L,j,q}^{\beta,\gamma}=J_{L,j}^{\beta,\gamma}(x_q),\quad s_q^n=s(x_q,t_n)\quad and\quad f_q=f(x_q).$
System (30) can be written in a matrix form as follows:

$$(\boldsymbol{J_1}-\boldsymbol{A})\boldsymbol{U}^n=(\boldsymbol{J_0}+\boldsymbol{A})\boldsymbol{U}^{n-1}+\frac{1}{2}\triangle t(\boldsymbol{S}^n+\boldsymbol{S}^{n-1}), \tag{31}$$

such that:

$$\boldsymbol{U}^n=(u_0^n,u_1^n,...,u_m^n)^T,$$

$$\boldsymbol{S}^n=(0,s_1^n,s_2^n...s_{m-1}^n,0)^T,$$

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$$\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ \Omega_{0,1} & \Omega_{1,1} & \Omega_{2,1} & \Omega_{3,1} & \cdots & \Omega_{m,1} \\ \Omega_{0,2} & \Omega_{1,2} & \Omega_{2,2} & \Omega_{3,2} & \cdots & \Omega_{m,2} \\ \Omega_{0,3} & \Omega_{1,3} & \Omega_{2,3} & \Omega_{3,3} & \cdots & \Omega_{m,3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \Omega_{0,m-1} & \Omega_{1,m-1} & \Omega_{2,m-1} & \Omega_{3,m-1} & \cdots & \Omega_{m,m-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{m+1},$$

$$\boldsymbol{J_1} = \begin{pmatrix} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1)} & -\frac{\Gamma(1+\gamma+1)}{\Gamma(\gamma+1)1!} & \frac{\Gamma(2+\gamma+1)}{\Gamma(\gamma+1)2!} & -\frac{\Gamma(3+\gamma+1)}{\Gamma(\gamma+1)3!} & \cdots & (-1)^m\frac{\Gamma(m+\gamma+1)}{\Gamma(\gamma+1)m!} \\ J^{\beta,\gamma}_{L,0,1} & J^{\beta,\gamma}_{L,1,1} & J^{\beta,\gamma}_{L,2,1} & J^{\beta,\gamma}_{L,3,1} & \cdots & J^{\beta,\gamma}_{L,m,1} \\ J^{\beta,\gamma}_{L,0,2} & J^{\beta,\gamma}_{L,1,2} & J^{\beta,\gamma}_{L,2,2} & J^{\beta,\gamma}_{L,3,2} & \cdots & J^{\beta,\gamma}_{L,m,2} \\ J^{\beta,\gamma}_{L,0,3} & J^{\beta,\gamma}_{L,1,3} & J^{\beta,\gamma}_{L,2,3} & J^{\beta,\gamma}_{L,3,3} & \cdots & J^{\beta,\gamma}_{L,m,3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ J^{\beta,\gamma}_{L,0,m-1} & J^{\beta,\gamma}_{L,1,m-1} & J^{\beta,\gamma}_{L,2,m-1} & J^{\beta,\gamma}_{L,3,m-1} & \cdots & J^{\beta,\gamma}_{L,m,m-1} \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1)} & \frac{\Gamma(1+\beta+1)}{\Gamma(\beta+1)1!} & \frac{\Gamma(2+\beta+1)}{\Gamma(\beta+1)2!} & \frac{\Gamma(3+\beta+1)}{\Gamma(\beta+1)3!} & \cdots & \frac{\Gamma(m+\beta+1)}{\Gamma(\beta+1)m!} \end{pmatrix}_{m+1},$$

$$\boldsymbol{J_0} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ J^{\beta,\gamma}_{L,0,1} & J^{\beta,\gamma}_{L,1,1} & J^{\beta,\gamma}_{L,2,1} & J^{\beta,\gamma}_{L,3,1} & \cdots & J^{\beta,\gamma}_{L,m,1} \\ J^{\beta,\gamma}_{L,0,2} & J^{\beta,\gamma}_{L,1,2} & J^{\beta,\gamma}_{L,2,2} & J^{\beta,\gamma}_{L,3,2} & \cdots & J^{\beta,\gamma}_{L,m,2} \\ J^{\beta,\gamma}_{L,0,3} & J^{\beta,\gamma}_{L,1,3} & J^{\beta,\gamma}_{L,2,3} & J^{\beta,\gamma}_{L,3,3} & \cdots & J^{\beta,\gamma}_{L,m,3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ J^{\beta,\gamma}_{L,0,m-1} & J^{\beta,\gamma}_{L,1,m-1} & J^{\beta,\gamma}_{L,2,m-1} & J^{\beta,\gamma}_{L,3,m-1} & \cdots & J^{\beta,\gamma}_{L,m,m-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{m+1},$$

and

$$\Omega_{j,q} = \frac{1}{2}(\triangle t d\sum_{k=0}^{j}\sum_{i=0}^{k} \times \Psi^{\alpha,\beta,\gamma}_{i,j,k}\big|_{x=x_q} \times J^{\beta,\gamma}_{L,j,q} + \triangle t e \frac{d}{dx} J^{\beta,\gamma}_{L,j}(x)\big|_{x=x_q}),$$
$$i = 0,1,2,\dots,m, \quad j = 1,2,\dots,m-1, \quad n = 1,2,\dots,N.$$

Substituting the computed coefficients $u_j, \ j = 0,1,2,\dots,m,$ and the Jacobi polynomials in Eq. (26) give us (u) the approximating solution of the studied model.

In the current paper we use the Jacobi Gauss-Lobatto points that are beneficial for the convergence, efficiency and stability of the Jacobi spectral collocation technique.

5 Numerical simulations

Two examine examples are demonstrated in this section to show the accuracy of the proposed technique.

Example 1. [25] We treat, in a bounded domain, the following Lévy-Feller advection-dispersion equation:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = dD^{\alpha}_{\theta}u(x,t) - e\frac{\partial u(x,t)}{\partial x}, & 0 < x < \pi, \ t > 0, \quad 1 < \alpha \leq 2, \\ u(x,0) = sin(x), & 0 \leqslant x \leqslant \pi, \\ u(0,t) = 0, \quad u(\pi,t) = 0, & t > 0. \end{cases} \tag{32}$$

Let $d = 1.5, \ e = 1, \ \alpha = 1.7, \ \theta = 0.3, \ T_{final} = 0.3$, we list in table (1) the numerical solutions obtained by the explicit finite difference approximation (EFDA) in [25] and the presented scheme JSC for problem (1).

Let $d = 1.5, \ e = 1, \ \alpha = 1.7, \ T_{final} = 0.4$ and $\triangle t = 0.008$, figure (1) exhibits the computed numerical solutions by using the presented scheme JSC $(m = 5)$ for different values of θ, which indicates the skewness.

When $\alpha = 2, \ d = 1, \ e = 0$, the exact analytic solution for example (1) is given as $u(x,t) = sin(x)e^{-t}$. Let us consider $T_{final} = 3$ and $\triangle t = 0.05$. Figure (2) exhibits this analytic exact solution and the computed approximation solutions by using the presented scheme JSC for example (1) when $m = 7$. This illustrates that the EFDA [25] is not convergent.

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Table 1: Comparison of the numerical results calculated by EFDA [25] when $h = \pi/100$ and by the presented scheme JSC when $m = 5$, and $\beta = \gamma = 0$ for example (1), where $\alpha = 1.7,\ \theta = 0.3\ and\ t = 0.3$.

(x,0.3)	EFDA in [25]	Present method JSC
0.3142	0.23041	0.21208
0.6283	0.40603	0.38590
0.9425	0.54876	0.51814
1.2566	0.64661	0.60546
1.5708	0.68848	0.64455
1.8850	0.66770	0.63208
2.1991	0.58292	0.56471
2.5133	0.43764	0.43913
2.8274	0.23952	0.25200

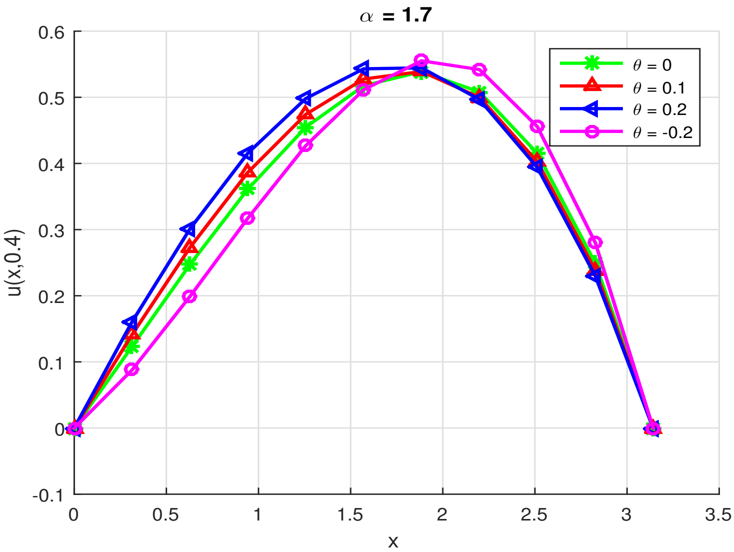

Fig. 1: Comparison between the approximation solution obtained by the introduced method JSC for example (1) at $t = 0.4$ with $\alpha = 1.7$ and different values of θ.

*Example 2.*We deal in this example with the Lévy-Feller advection-dispersion equation in form:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = dD^{\alpha}_{\theta} u(x,t) - e\frac{\partial u(x,t)}{\partial x} + s(x,t), & t > 0,\ \ 0 < x < 1, \\ u(0,t) = 0, \qquad u(1,t) = 0, \quad t > 0, \\ u(x,0) = x(1-x), \qquad 0 \leqslant x \leqslant 1, \end{cases} \tag{33}$$

where

$$d = \Gamma(3-\alpha), \qquad e = 1,$$

$$s(x,t) = \{\frac{(2-\alpha)}{sin(\alpha\pi)}\big(sin(\frac{(\alpha-\theta)\pi}{2})x^{1-\alpha} + sin(\frac{(\alpha+\theta)\pi}{2})(1-x)^{1-\alpha}\big)$$
$$-\frac{2}{sin(\alpha\pi)}\big(sin(\frac{(\alpha-\theta)\pi}{2})x^{2-\alpha} + sin(\frac{(\alpha+\theta)\pi}{2})(1-x)^{2-\alpha}\big) + \frac{3}{2}x^2 - \frac{7}{2}x + 1\}e^{-\frac{3}{2}t},$$

and the analytic exact solution of this problem, in case $1 < \alpha \le 2$, is given as following:

$$u(x,t) = x(1-x)e^{-\frac{3}{2}t}.$$

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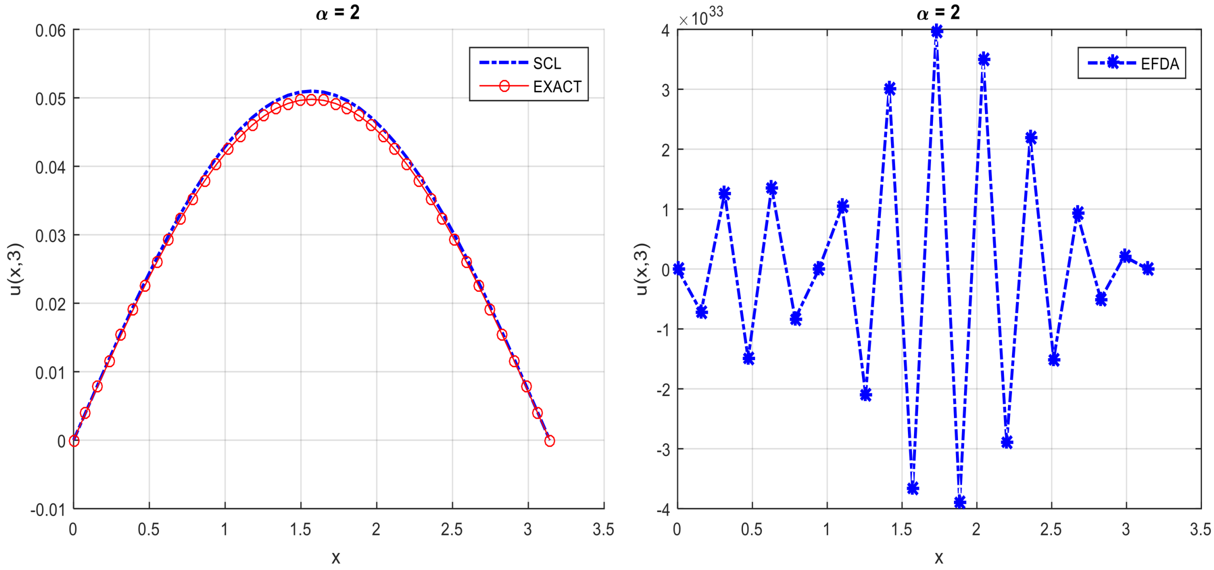

Fig. 2: The graphs show the exact solution and the numerical solution obtained by EFDA [25] and the approximation solution obtained by the introduced method JSC for example (1) at $t = 3$ with $\alpha = 2$.

The weighted difference scheme (WADS) introduced in [34] is used here to study numerically example 2, with wight factor $\sigma = 0,\ 0.5,\ 1$, and the proposed scheme JSC is also used. The absolute error which is the differences between the exact solution and the approximation solution are given by :

$$E_1(x,t) = |u_{exact}(x,t) - u_{JSC}(x,t)|, \ \ E_2(x,t) = |u_{exact}(x,t) - u_{WDS}(x,t)|,$$

where E_1 *and* E_2 are the errors of the proposed schemes JSC and WDS [34] respectively. Furthermore, the maximum absolute error is:

$$M_1 = max\{E_1(x,t) : \ a \leq x \leq b, \ \ 0 \leq t \leq T_{final}\},$$

$$M_2 = max\{E_2(x,t) : \ a \leq x \leq b, \ \ 0 \leq t \leq T_{final}\},$$

where $M_1,\ M_2$ are (respectively) the maximum absolute errors of JSC and WDS [34].

Table (2), for $T_{final} = 0.5$ *and* $\triangle t = 0.01$, shows that JSC is more accurate than WDS [34], for example (2), by comparing the errors $E_1(x,0.5)$ with $E_2(x,0.5)$ such that $\alpha = 1.4,\ \theta = -0.5$ with different values of m *and* h.

In table (3), for $T_{final} = 1$ for problem (2), and $\triangle t = 0.005$, we compare the maximum errors M_1 when $m = 3$ with M_2 when $h = 0.05$ for different values of α *and* θ.

Figure (3) displays the exact solution of Ex.(2) where $T_{final} = 2$ and displays the errors of using JSC ($m = 3$) and WDS [34] where $\triangle t = 0.002$ and $h = 0.05$ in case $(\sigma = 0)$.

Likewise, taking $T_{final} = 5$ and $\triangle t = 0.02$, figure (4) displays the exact analytic solution of Ex.(2) at $t = 5$ and the numerical solutions obtained by means of the presented scheme JSC where $m = 3$ and by the WDS $(\sigma = 0)$ [34] when $h = 0.1$.

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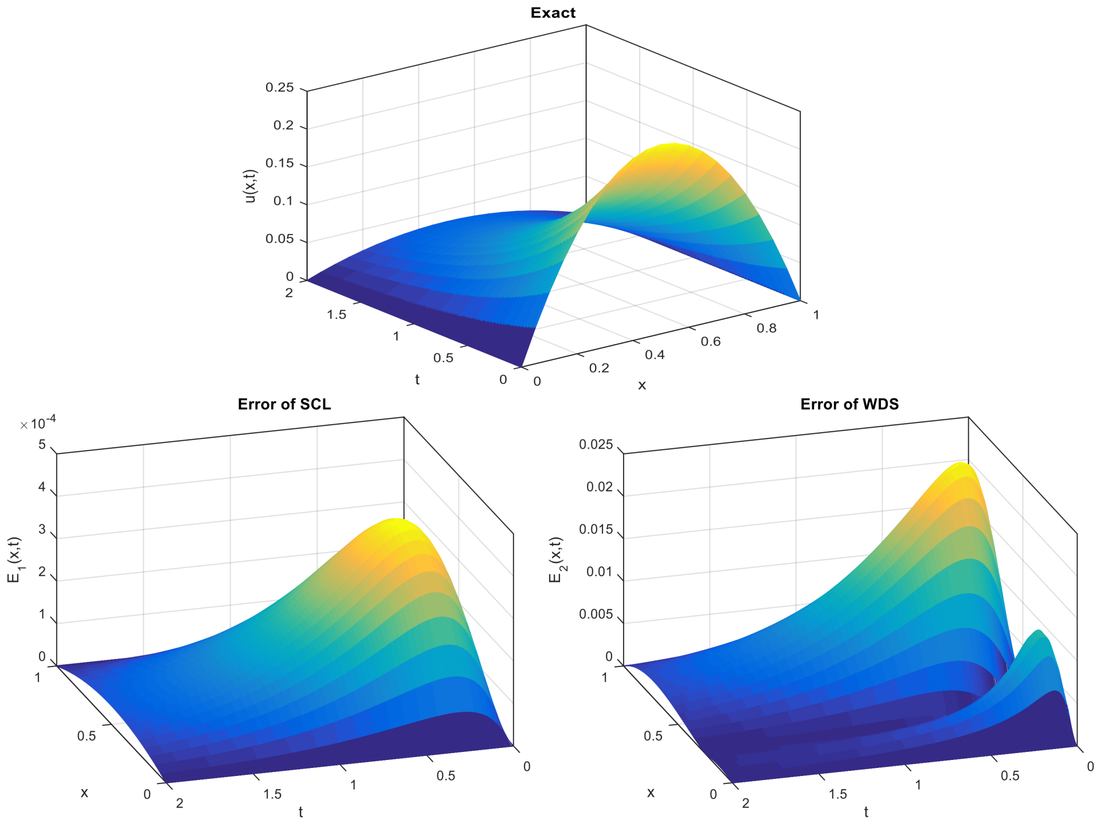

Fig. 3: Comparing between the errors that achieved by JSC $(\beta = \gamma = -0.5)$ and by WDS [34] where $T_{final} = 2$, $\alpha = 1.7$, and $\theta = 0.2$ for example (2).

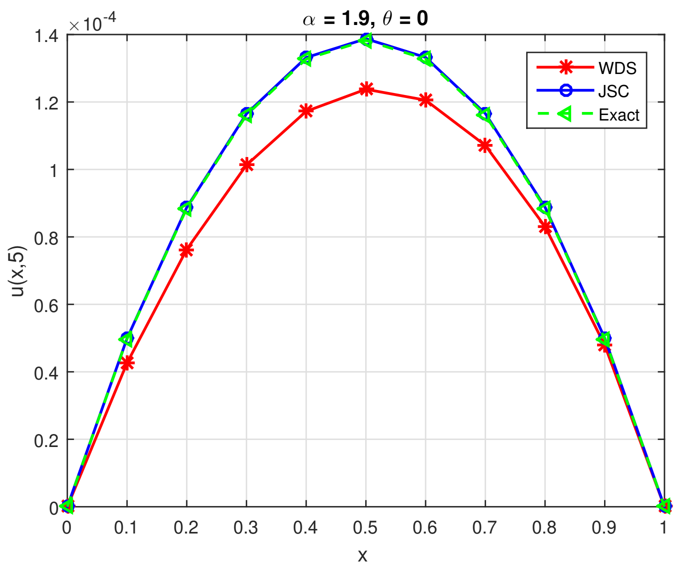

Fig. 4: Comparing between the approximating results obtained using JSC and WDS [34] at $t = 5$ with $\alpha = 1.9,\ \theta = 0$ for example (2).

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Table 2: Comparing between the errors obtained from JSC ($\beta = \gamma = 0.5$) (E_1) and from WDS [34] (E_2) with $\sigma = 0.5$ at t= 0.5, for example (2).

	$E_1(x, 0.5)$			$E_2(x, 0.5)$		
x	$m = 3$	$m = 6$	$m = 12$	$h = 0.1$	$h = 0.05$	$h = 0.025$
0	0	0	0	0	0	0
0.1	1.9827e-05	1.5831e-06	1.2334e-06	1.0946e-02	9.5111e-03	8.3772e-03
0.2	2.9278e-05	6.1593e-06	1.5369e-06	5.2330e-02	1.4747e-02	1.4856e-02
0.3	3.0590e-05	9.4183e-06	9.5057e-07	2.1783e-02	1.7541e-02	1.9247e-02
0.4	2.6003e-05	1.0927e-05	6.1780e-06	5.7803e-02	1.7939e-02	2.1571e-02
0.5	1.7757e-05	1.1467e-05	8.1340e-06	2.7705e-02	1.6039e-02	2.1855e-02
0.6	8.0902e-06	3.1435e-06	1.2858e-06	3.8135e-02	1.2109e-02	2.0173e-02
0.7	7.5812e-06	9.0309e-07	9.0856e-07	1.3635e-02	6.6847e-03	1.6691e-02
0.8	6.5487e-06	2.1818e-06	8.6676e-07	9.1295e-03	7.4279e-04	1.1747e-02
0.9	7.0424e-06	2.3546e-06	1.7558e-06	3.5132e-03	3.7610e-03	6.0224e-03
1	0	0	0	0	0	0

Table 3: Comparing between the maximum errors obtained from JSC ($\beta = \gamma = 0$) (M_1) and from WDS [34] (M_2) for example (2)

	M_1	M_2		
α, θ	m=3	$\sigma = 1$	$\sigma = 0.5$	$\sigma = 0$
$\alpha = 1.8,\ \theta = 0.1$	1.0995e-05	divergent	1.0064e-02	9.4012e-03
$\alpha = 1.6,\ \theta = 0.1$	4.5779e-05	2.0452e-02	2.0031e-02	1.9644e-02
$\alpha = 1.6,\ \theta = 0.3$	8.9812e-05	2.2966e-02	2.2429e-02	2.1961e-02
$\alpha = 1.4,\ \theta = 0.3$	7.6254e-05	3.9459e-02	3.8952e-02	1.2893e-02
$\alpha = 1.4,\ \theta = 0.5$	6.4506e-05	4.4378e-02	4.3804e-02	4.3232e-02
$\alpha = 1.2,\ \theta = 0.5$	5.2149e-05	6.3144e-02	6.2514e-02	6.1884e-02
$\alpha = 1.2,\ \theta = -0.5$	1.3202e-05	divergent	divergent	divergent
$\alpha = 1.1,\ \theta = -0.5$	4.1385e-05	divergent	divergent	divergent

6 Conclusion

An accurate numerical technique is constructed to give approximation solutions of the Lévy-Feller advection-dispersion equation. This method depends on the Jacobi collocation method in combination with the trapezoidal rule to create a system of algebraic equations of the unknown coefficients of the spectral collocation expansion. The major advantage of the proposed technique is the availability of the implementation of the fractional differential equations, including the case when the Riesz-(Feller) operator is used to define the fractional derivative.The other advantage is using a few number of terms of the proposed expansion to achieve the highly accurate numerical solutions. Comparison between our approximation results of the proposed models with the exact solutions and the approximation results obtained by other methods is introduced to highlight accuracy and validity of the proposed schema. Using the proposed approach to solve both Lévy-Feller diffusion equation and Lévy-Feller advection-dispersion equation we have found that this approach is more efficient than the finite difference method proposed in [33] and [25] for solving these two equations respectively.

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