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A Reverse Exponential-Generalized Truncated Logarithmic (Rev-EGTL) Distribution For Ordered Spacing Statistics

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Abstract: This paper extends the exponential-generalized truncated logarithmic (EGTL) distribution to model reverse ordered statistics. Our procedure generalizes the exponential-logarithmic (EL) distribution to a distribution more appropriate for modelling the m^{th}-largest value of lifetime instead of the maximum lifetime. We give a general form of the new family of spacing distribution appropriate for descending ordered statistics. General forms of the pdf and the failure rate function as well as their properties are presented for some special cases. The parameters' estimation is attained by the maximum likelihood (ML) and the expectation maximization (EM) algorithms. The application study is illustrated based on four real data sets.

Keywords: Lifetime distributions; order statistics; exponential distribution; truncated logarithmic distribution; survivor function.

1 Introduction

The distributions of ordered spacing statistics are of great interest in many areas of statistics, in particular, detection of outliers, quality control, auction theory, reliability and life testing models. By spacings we refer to the distances or gaps between two successive points on a line. For example, in the reliability analysis spacings correspond to the lifetime between successive failures of components in a system. For industrial accidents, they refer to the intervals of time between successive accidents occurring in a given period (for more details on spacings we refer to [1,2,3,4,5,6]). Ordered random variables are already known for their ascending order. The concept of dual generalized ordered statistics, introduced by Burkschat et al. [7], enables a common approach to descending ordered spacings like reverse ordered statistics and lower record values.

Reverse order statistics have also a wide range of applications in economics such as providing diverse distributive criteria in assessing welfare and inequalities of incomes and wealth [8,9]. The k^{th}-reverse order statistics corresponds to the $(n-k+1)^{th}$-order statistics. Ordered spacings may provide information about a sequence of ordered intervals of varying lengths. For instance, in the analysis of the intervals of time between industrial accidents, testing for change with time is more important than studying only accident frequencies in relatively long fixed interval of times [10]. If the expectation of accidents per unit of time is constant, then the time-intervals between successive accidents are exponentially distributed. Exponential distribution is often used to model spacings such as system reliability at a component level, assuming the failure rate is constant [11,12]. In recent years, several compound distributions have been introduced as extensions of the exponential distribution, following Adamidis and Loukas [13] and Kuş [14], by using the mixture of count data and spacing distributions. The most common count data distributions are the Poisson, binomial and negative-binomial models arising for events randomly and independently occurred in time. They are the distributions of non-negative counts in a number of trials with a probability of occurrence of outcomes under observation. However, in the analysis of spacings between successive events, the count variable includes at least one observation i.e. positive counts

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[15]. The logarithmic series density has support positive integers, which make more sense to our model since it is defined for the positive valued count random variables. Furthermore, the logarithmic distribution is used generally to model the number of occurrence of events in a specified period of time. For instance, Chatfield [16] noted that purchases of a given consumer in successive intervals of time is a Poisson distribution while logarithmic series distribution should be used to model the purchasing behavior in a single time-period. The logarithmic series distribution is a one parameter for positive integers and has a long positive tail. It was introduced by Fisher et al. [17] to investigate the relationship between the number of individuals and the number of species in a random sample of animal population.

The purpose of this article is to extend the two-parameter EGTL distribution for reverse order statistics, proposed by Rahmouni and Orabi [18] to model the k^{th}-smallest value of lifetime. Our procedure generalizes the exponential-logarithmic distribution modelling the reliability of systems by use of the first-order concepts, where the maximum lifetime of a system is considered. The approach comes from the idea of modelling a system reliability based on the reliability of its components. However, the maximum lifetime, in simple series, is not often adequate to model reliability of real systems. We assume that a system fails if a given number m of the components fails and then, we consider any m^{th}-largest value of lifetime instead of the maximum lifetime [18,19,20]. Let $Y=(Y_1,Y_2,...,Y_Z)$ a random sample of independent and identically distributed (iid) random variable following an exponential distribution. We generalize the distributions modelling the first or last spacings or lifetime, to a distribution more appropriate for modelling any m^{th} order statistic i.e. the m^{th}-largest value of lifetime instead of the maximum lifetime $X_{(1)} = max\{Y_i\}_{i=1}^{Z}$. Let the reverse (descending) order statistics $X_{(1)} > X_{(2)} > ... > X_{(Z)}$. The joint probability density is determined by compounding a truncated at m logarithmic series distribution and the probability density function (pdf) of the $(z-m)^{th}$ order statistic ($m=0,2,...,z$). We include in the sample only $(z-m)$ individuals who have experienced the event.

The rest of the paper is structured as follows. In section 2, we present the new family of spacing distribution and the pdf for some special cases. The moment generating function, the survival and hazard rate functions are discussed in this section. The estimation of parameters for this new family of distributions will be discussed in section 3. The estimation of the parameters is attained by ML and EM algorithms. The application study is illustrated based on four real data sets in the last section.

2 Properties of the distribution

2.1 Distribution

Let $Y=(Y_1,Y_2,...,Y_Z)$ be iid exponential random variable with scale parameter $\theta > 0$ and pdf given by: $f(y) = \theta e^{-\theta y}$, for $y>0$, where Z is a discrete random variable following a logarithmic series distribution with parameter $0<p<1$ and a probability mass function $P(Z=z)$ of a $Log(p)$-distributed random variable given by:

$$P(Z=z) = \frac{1}{-\ln(1-p)}\frac{p^z}{z}; z \in \{1,2,3,\dots\} \tag{1}$$

If the discrete random variable Z follows a truncated at m logarithmic distribution with parameter p, then the probability function $P_m(Z=z)$ will be given by the following equation:

$$P_m(Z=z) = \frac{1}{A(p,m)}\frac{p^z}{z}; m=0,1,2,\dots,z \text{ and } z=m+1,m+2,\dots \tag{2}$$

where,

$$A(p,m) = \begin{cases} -\ln(1-p) & \text{if } m=0 \\ -\ln(1-p) - \sum_{j=1}^{m}\frac{p^j}{j} & \text{if } m=1,2,...,z \end{cases}$$

The probability density function distribution (pdf) for the proposed Rev-EGTL distribution is:

$$g_m(x/p,\theta,m) = \frac{\theta p^{m+1} e^{-\theta(m+1)x}}{A(p,m)[1-p(1-e^{-\theta x})]^{m+1}}; \quad x \in [0,\infty) \tag{3}$$

Where $0<p<1$ and θ are the shape and the scale parameters, respectively. This distribution is more appropriate for modelling any $(z-m)^{th}$ order statistic (z^{th}, $(z-1)^{th}$ or any $(z-m)^{th}$ lifetime). The maximum lifetime distribution is a special case for $m=0$. It is the exponential-logarithmic (EL) distribution for modelling the last value $X_{(z)} = max\{Y_i\}_{i=1}^{Z}$:

$$g_0(x/p,\theta) = \frac{p\theta e^{-\theta x}}{-\ln(1-p)[1-p(1-e^{-\theta x})]}$$

This pdf decreases strictly in x and tends to 0 as $x \to \infty$. The modal value of the EL distribution at $x = 0$ is given by $\frac{\theta p^{m+1}}{A(p,m)}$. The function is concave upward on $[0,\infty)$. The graphs of the density resemble those of exponential and Pareto II distributions (see, Figure 1).

$$\lim_{x\to 0} g_m(x/p,\theta,m) = \frac{\theta p^{m+1}}{A(p,m)}$$

$$\lim_{x\to\infty} g_m(x/p,\theta,m) = 0$$

Proof.:

Suppose $Y_{(1)}, Y_{(2)}, ..., Y_{(Z)}$ be the order statistics of Z observations and $X = Y_{(k)}$ is the k^{th} order statistics. Then, the pdf of X given Z is given by:

$$g_k(x/z,\theta) = \frac{\theta\Gamma(z+1)}{\Gamma(k)\Gamma(z-k+1)} e^{-\theta(z-k+1)x}(1-e^{-\theta x})^k \; ; \theta, x > 0 \tag{4}$$

Let $m = z-k$ for the m^{th} reverse order statistics (the m^{th}-largest value of lifetime). Therefore, from equations (2) and (4) the joint probability density of X and Z is given by:

$$g_m(x,z/p,\theta) = \frac{\Gamma(z)}{\Gamma(z-m)\Gamma(m+1)} \frac{\theta p^z e^{-\theta(m+1)x}(1-e^{-\theta x})^{z-m-1}}{A(p,m)} \tag{5}$$

Then, the marginal pdf of X is given by:

$$g_m(x/p,\theta,m) = \frac{\theta(pe^{-\theta x})^{m+1}}{A(p,m)[1-p(1-e^{-\theta x})]^{m+1}} \; ; \quad x \in [0,\infty) \tag{6}$$

Also, the cumulative distribution function (cdf) of X corresponding to the pdf in equation (3) is given by:

$$G_m(x/p,\theta,m) = \int_0^x g_m(t/p,\theta,m)\partial t$$

$$G_m(x/p,\theta,m) = 1 - \frac{A(pw,m)}{A(p,m)} \tag{7}$$

where,

$$w = \frac{e^{-\theta x}}{1-p(1-e^{-\theta x})}$$

2.2 *Moment generating function and r^{th} moment*

Suppose X has the pdf in equation (3), then the moment generating function is given by:

$$E(e^{tx}) = \frac{1}{A(p,m)} \sum_{s=0}^{\infty} C_s^{m+s} p^{m+s+1} \beta(s+1, m-\frac{t}{\theta}+1) \tag{8}$$

where,

$$\beta(i,j) = \int_0^1 t^{i-1}(1-t)^{j-1}\partial t$$

The r^{th} moment is given by:

$$E(X^r) = \frac{1}{A(p,m)\theta^r} \sum_{s=0}^{\infty} \sum_{i=0}^{i=s} C_s^{m+s} C_i^s p^{m+1}(m+i+1)^{-(r+1)} \tag{9}$$

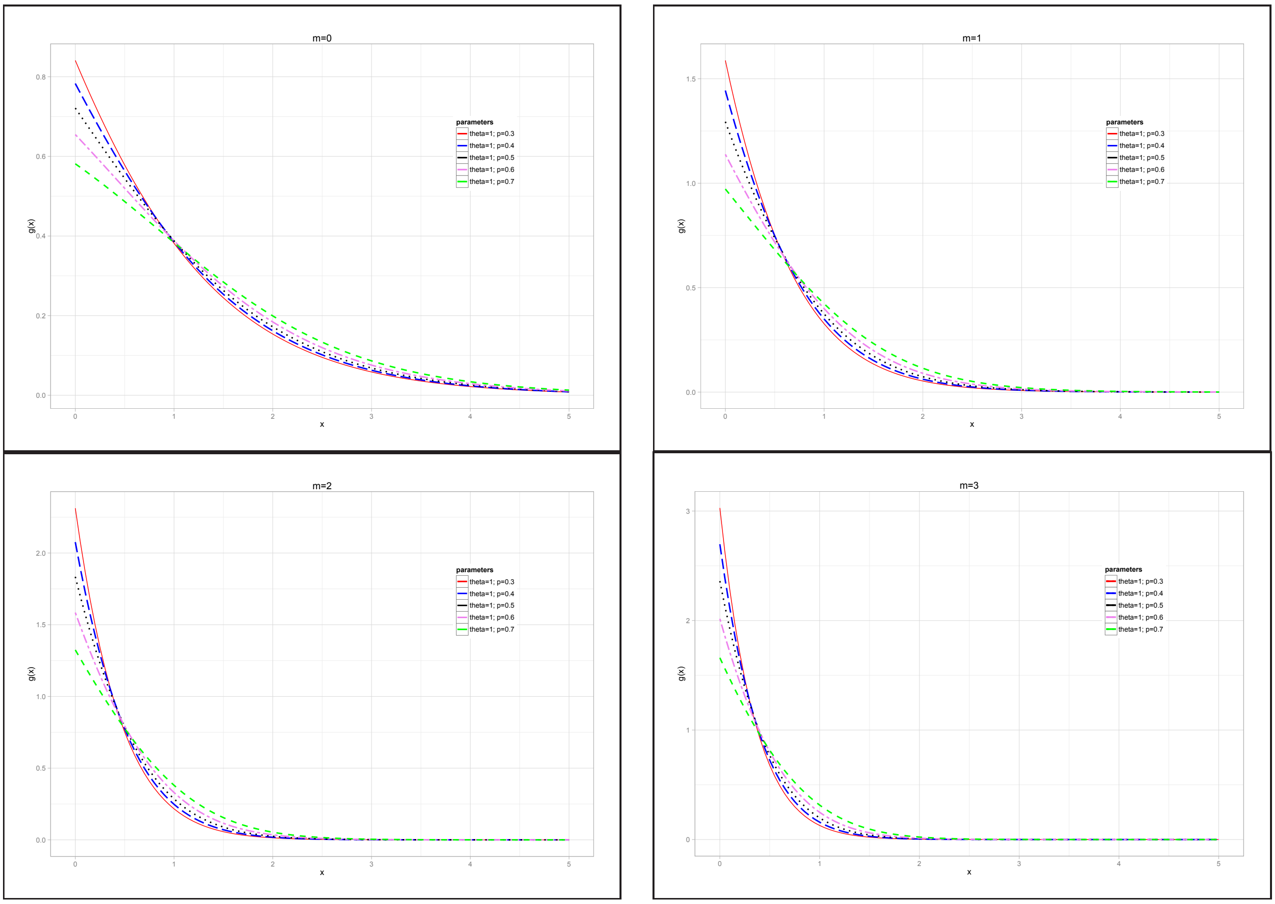

Fig. 1: The pdf of the new distribution for $m = 0,1,2,3$

2.3 The survival and hazard functions

The survival or reliability function is the probability of being alive just before duration x. It is given by $S(x) = Pr(X \geq x) = 1 - G(x) = \int_x^\infty f(t)\partial t$ which is the probability that the event of interest does not occur by duration x. In other words, reliability $S(x)$ is the probability that a system will be successful in the interval from time 0 to time x, where the random variable X denotes the time-to-failure or failure time [12,21,22]. The survival function, corresponding to the pdf in equation (3), is given in equation (10) and it is presented in Table (1) for some special cases.

$$S_m(x/p,\theta,m) = \frac{A(pw,m)}{A(p,m)} \tag{10}$$

The hazard rate $h(x)$, known as failure rate function, is the instantaneous rate of occurrence of the event of interest at duration x (i.e. the rate of event occurrence per unit of time). Mathematically, it equals to the pdf of events at x, divided by the probability of surviving to that duration without experiencing the event. Thus, we define a failure rate function as in Bakouch et al. [23] by $h(x) = g(x)/S(x)$. The hazard function for some special cases is given in the last column of Table (1).

$$h_m(x/p,\theta,m) = \frac{\theta(pw)^{m+1}}{A(pw,m)} \tag{11}$$

The hazard rate function is analytically related to the time-failure probability distribution. It leads to the examination of increasing failure rate (IFR) or decreasing failure rate (DFR) properties of life-length distributions. G is an IFR distribution, if $h(x)$ increases for all X such that $G(X) < 1$. The motivation of the new lifetime distribution is the realistic features of the hazard rate in many real-life physical and non-physical systems, which is not monotonically increasing,

Table 1: Survivor and hazard functions for some special cases

order statistic	k	S(x)	h(x)
last	$m=0$	$\frac{\ln(1-pw)}{\ln(1-p)}$	$\frac{-\theta pw}{\ln(1-pw)}$
last-1	$m=1$	$\frac{\ln(1-pw)+pw}{\ln(1-p)+p}$	$\frac{-\theta(pw)^2}{\ln(1-pw)+pw}$
last-2	$m=2$	$\frac{\ln(1-pw)+pw+\frac{(pw)^2}{2}}{\ln(1-p)+p+\frac{p^2}{2}}$	$\frac{-\theta(pw)^3}{\ln(1-pw)+pw+\frac{(pw)^2}{2}}$
last-3	$m=3$	$\frac{\ln(1-pw)+pw+\frac{(pw)^2}{2}+\frac{(pw)^3}{3}}{\ln(1-p)+p+\frac{p^2}{2}+\frac{p^3}{3}}$	$\frac{-\theta(pw)^4}{\ln(1-pw)+pw+\frac{(pw)^2}{2}+\frac{(pw)^3}{3}}$

decreasing or constant hazard rate. From Figure 2 we observe that the hazard rate function is increasing. In fact, if $x \to 0$ then $h(x/p,\theta,m) = \frac{\theta p^{m+1}}{A(p,m)}$ and if $x \to \infty$ then $h(x/p,\theta,m) \to \theta(m+1)$.

$$\lim_{x\to 0} h(x/p,\theta,m) = \frac{\theta p^{m+1}}{A(p,m)}$$
$$\lim_{x\to\infty} h(x/p,\theta,m) = \theta(m+1)$$

2.4 Random variables generation

A random variable can be generated from the cdf of X and the standard uniform distribution of U and by solving the nonlinear equation in w:

$$A(p,m)(1-U) = A(pw,m)$$

Solve the equation in w:

$$X = -\frac{1}{\theta}\ln\frac{w(1-p)}{1-pw}$$

We can determine the quantiles by dividing the set of observations into equal sized groups. The quantile function is given by $X = q(U)$. The median is computed by letting $U = 0.5$. At $m = 0$,

$$q(U) = -\frac{1}{\theta}\ln\left(\frac{[1-(1-p)^{1-U}](1-p)}{1+p-(1-p)^{1-U}}\right)$$

Then, the median is:

$$\tilde{X} = q(\frac{1}{2}) = -\frac{1}{\theta}\ln\left(\frac{[1-(1-p)^{0.5}](1-p)}{1+p-(1-p)^{0.5}}\right)$$

3 Estimation of the parameters

3.1 Maximum Likelihood Estimation

In this section, we will determine the estimates of the parameters p and θ for our new family of distributions. Let $(X_1,X_2,\ldots,X_n)$ be a random sample with observed values $(x_1,x_2,\ldots,x_n)$ from this distribution with pdf in equation (3). The log-likelihood function given the observed values, $x_{obs} = (x_1,x_2,\ldots,x_n)$, is:

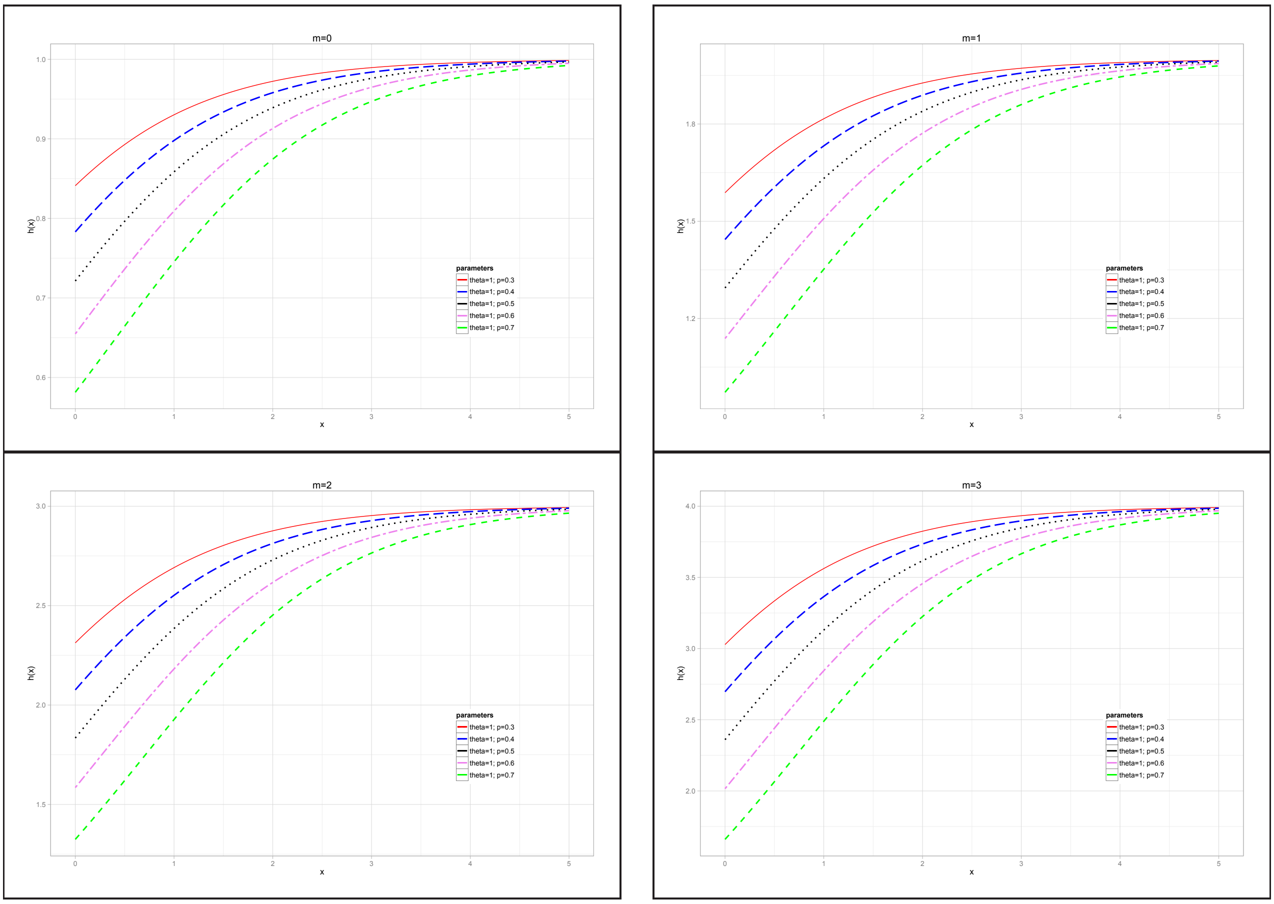

Fig. 2: Hazard functions of the new distribution for $m = 0, 1, 2, 3$

$$\ln L \;\propto\; n\ln(\theta) + n(m+1)\ln(p) - \theta(m+1)\sum_{i=1}^{n} x_i - n\ln\sum_{j=m+1}^{\infty}\frac{p^j}{j} - (m+1)\sum_{i=1}^{n}\ln[1-p(1-e^{-\theta x_i})] \quad (12)$$

The associated gradients are given by:

$$\frac{\partial \ln L}{\partial p} = \frac{-np^m}{(1-p)\sum_{j=m+1}^{\infty}\frac{p^j}{j}} + \frac{n(m+1)}{p} + (m+1)\sum_{i=1}^{n}\frac{1-e^{-\theta x_i}}{1-p(1-e^{-\theta x_i})}$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta} - (m+1)\sum_{i=1}^{n} x_i + (m+1)\sum_{i=1}^{n}\frac{\theta p e^{-\theta x_i}}{1-p(1-e^{-\theta x_i})}$$

We need the Fisher information matrix for interval estimation and tests of hypotheses on the parameters. The Fisher information matrix related to the second derivative of the log-likelihood function is:

$$\mathscr{I}(\widehat{p},\widehat{\theta}) = -\begin{pmatrix} E\left(\frac{\partial^2 \ln L}{\partial p^2}\right) & E\left(\frac{\partial^2 \ln L}{\partial p \partial \theta}\right) \\ E\left(\frac{\partial^2 \ln L}{\partial \theta \partial p}\right) & E\left(\frac{\partial^2 \ln L}{\partial \theta^2}\right) \end{pmatrix}\Bigg|_{\theta=(\widehat{p},\widehat{\theta})}$$

The maximum likelihood estimates (MLEs) $\widehat{p}$ and $\widehat{\theta}$ of the parameters p and θ, respectively, can be found using the iterative EM algorithm to handle the "incomplete data problem" [24,25]. The iterative method consists on repeatedly updating the parameter estimates by replacing the "missing data" with the new estimated values. The standard method used to determine the MLEs is the Newton-Raphson algorithm. To employ this algorithm, second derivatives of the log-likelihood are required for all iterations. However, when the amount of information in the "missing data" is relatively large, EM algorithm will converge reliably but rather slowly than the Newton-Raphson method [26,27,28,29,13]. Newton-Raphson is required for the M-step of the EM algorithm. To start the algorithm, hypothetical distribution of complete-data is defined with the joint probability density $g_m(x,z/p,\theta)$ and drives the conditional mass function as:

$$p(z/x,p,\theta)=\frac{(z-1)!}{(z-m-1)!m!}p^{z-m-1}(1-e^{-\theta x})^{z-m-1}[1-p(1-e^{-\theta x})]^{m+1} \tag{13}$$

E-step:

$$E(z/x,p,\theta)=\frac{m+1}{1-p(1-e^{-\theta x})} \tag{14}$$

M-step:

$$\widehat{p}^{(r+1)}=\frac{-\left(1-p^{(r+1)}\right)\sum_{j=m+1}^{\infty}\frac{\left(p^{(r+1)}\right)^j}{j}}{n\left(p^{(r+1)}\right)^m}\sum_{i=1}^{n}\frac{m+1}{1-p^{(r)}(1-e^{-\theta^{(r)}x_i})} \tag{15}$$

$$\widehat{\theta}^{(r+1)}=\frac{n}{m+1}\left[\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}\frac{x_i p^{(r)}(1-e^{-\theta^{(r)}x_i})e^{-\theta^{(r+1)}x_i}}{\left[1-p^{(r)}(1-e^{-\theta^{(r)}x_i})\right](1-e^{-\theta^{(r+1)}x_i})}\right]^{-1} \tag{16}$$

3.2 Bayesian estimation

Let $X=(X_1,X_2,...,X_n)$ is a sample from the distribution given in equation (3). The likelihood function is given by:

$$L(p,\theta|x)=[A(p,m)]^{-n}\theta^n p^{n(m+1)}e^{-\theta(m+1)\sum x_i}\prod_{i=1}^{n}[1-p(1-e^{-\theta x_i})]^{-(m+1)} \tag{17}$$

In the Bayesian approach inferences are expressed in a posterior distribution for the parameters which is, according to Bayes' theorem, given in terms of the likelihood and a prior density function by:

$$P_m(p,\theta/x_1,x_2,...,x_n)=\frac{g_m(x/p,\theta).\pi_m(p,\theta)}{K} \tag{18}$$

where, $\pi_m(p,\theta)$ is a prior probability distribution function and $g_m(x/p,\theta)$ is the likelihood of observations $(x_1,x_2,...,x_n)$. Note that K is the normalizing constant for the function $\pi_m(p,\theta)g_m(x/p,\theta)$ given by:

$$\int\int \pi_m(p,\theta)g_m(x/p,\theta)dpd\theta \tag{19}$$

We suppose that the conjugate prior for the parameter p is the Beta distribution:

$$\pi_1(p)=\frac{1}{B(a,b)}p^{a-1}(1-p)^{b-1}$$

and the conjugate prior for the parameter θ is the Gamma distribution given by:

$$\pi_2(\theta)=\frac{d^c}{\Gamma(c)}\theta^{c-1}e^{-d\theta}$$

The posterior distribution is given by:

$$\pi(p,\theta|x)=\frac{1}{K[A(p,m)]^n}\theta^{n+c-1}p^{n(m+1)+a-1}(1-p)^{b-1}e^{-\theta[(m+1)\sum x_i+d]}\prod[1-p(1-e^{-\theta x_i})]^{-(m+1)} \tag{20}$$

Bayesian estimates of the parameters can be found numerically through the use of the Markov Chain Monte Carlo (MCMC) method.

Table 2: ”Waiting times of 100 bank customers” [30]

0.8	0.8	1.3	1.5	1.8	1.9	1.9	2.1	2.6	2.7	2.9	3.1	3.2	3.3	3.5
3.6	4.0	4.1	4.2	4.2	4.3	4.3	4.4	4.4	4.6	4.7	4.7	4.8	4.9	4.9
5.0	5.3	5.5	5.7	5.7	6.1	6.2	6.2	6.2	6.3	6.7	6.9	7.1	7.1	7.1
7.1	7.4	7.6	7.7	8.0	8.2	8.6	8.6	8.6	8.8	8.8	8.9	8.9	9.5	9.6
9.7	9.8	10.7	10.9	11.0	11.0	11.1	11.2	11.2	11.5	11.9	12.4	12.5	12.9	13.0
13.1	13.3	13.6	13.7	13.9	14.1	15.4	15.4	17.3	17.3	18.1	18.2	18.4	18.9	19.0
19.9	20.6	21.3	21.4	21.9	23.0	27.0	31.6	33.1	38.5					

scriptsize

Table 3: ”Ordered Failure Times (in hours) of 107 Right Rear Brakes on D9G-66A Caterpillar Tractors” [31,32]

56	753	1153	1586	2150	2624	3826	83	763	1154	1599	2156	2675	3995	104
806	1193	1608	2160	2701	4007	116	834	1201	1723	2190	2755	4159	244	838
1253	1769	2210	2877	4300	305	862	1313	1795	2220	2879	4487	429	897	1329
1927	2248	2922	5074	452	904	1347	1957	2285	2986	5579	453	981	1454	2005
2325	3092	5623	503	1007	1464	2010	2337	3160	6869	552	1008	1490	2016	2351
3185	7739	614	1049	1491	2022	2437	3191	661	1069	1532	2037	2454	3439	673
1107	1549	2065	2546	3617	683	1125	1568	2096	2565	3685	685	1141	1574	2139
2584	3756													

4 Application examples

In this section, we fit the distribution to four real data sets. The first set of data is used by Ghitany et al. [30] and represents ”the waiting times (in minutes) before service of 100 Bank customers” (Table 2).

The second set (Table 3) consists of ”107 failure times for right rear brakes on D9G-66A caterpillar tractors”, reproduced from Barlow and Campo [31] and also used by Chang and Rao [32].

The third data set is used by Bakouch et al. [33] to fit the binomial exponential-2 (BE2) distribution lifetime model with increasing failure rate. Data represent ”final examination marks of 48 slow space students in Mathematics in the final examination of the Indian Institute of Technology, Kanpur in year 2003” [34]. This data set has the values in Table 4.

Table 4: Finals marks [34]

29	25	50	15	13	27	15	18	7	7	8	19	12	18	5	21
15	86	21	15	14	39	15	14	70	44	6	23	58	19	50	23
11	6	34	18	28	34	12	37	4	60	2	0 23	40	65	19	31

The fourth set of data involves 100 observations (Table 5) of ”the results from an experiment concerning the tensile fatigue characteristics of a polyester/viscose yarn. The observations were obtained on the cycles to failure of a 100 cm yarn sample put to test under 2.3% strain level” [27].

Table 5: Results of Model Selection Program on Yarn Data [35]

86	146	251	653	98	249	400	292	131	169	175	176	76	264	15
364	195	262	88	264	157	220	42	321	180	198	38	20	61	121
282	224	149	180	325	250	196	90	229	166	38	337	65	151	341
40	40	135	597	246	211	180	93	315	353	571	124	279	81	186
497	182	423	185	229	400	338	290	398	71	246	185	188	568	55
55	61	244	20	284	393	396	203	829	239	286	194	277	143	198
264	105	203	124	137	135	350	193	188	236					

Table 6 shows the fitted parameters, calculated values of Kolmogorov-Smirnov (K-S) and their respective *p-values* for the four data sets. We estimate some special cases of the distribution at 5% significant level. The K-S test shows that our new distribution is an attractive alternative to the other ones and it generalizes them to any $(z-m)^{th}$ order statistics. The new lifetime model provides good fit to the data set. The results show that *p-values* are significant for all the cases. Indeed, the data exhibit increasing failure rates and the hazard rate function is increasing. We also conducted the MCMC study, with 5000 replications, on Maximum likelihood method to empirically see the better performance of parameter estimation methods.

Our proposed distribution is compared with the known gamma and Weibull distributions with respective densities $f_1(x,\lambda_1,\beta_1) = \lambda_1^{\beta_1} x^{\beta_1 - 1} exp(-\lambda_1 x)\Gamma(\beta_1)^{-1}$ and $f_2(x,\lambda_2,\beta_2) = \beta_2 \lambda_2^{\beta_2} x^{\beta_2 - 1} exp(-\lambda_2 x)^{\beta_2}$. Using Barlow and Campo's data set [31] (n = 107), Rahmouni & Orabi [18] showed the estimated parameters for the Gamma distribution are $(0.943; 1.908)$ with corresponding K–S $value$ equals to 0.0680 and $p - value = 0.7343$. For the Weibull distribution the obtained parameters are $(0.447; 1.486)$ with K–S $value = 0.0490$ and $p - value = 0.9999$. Also, Using the Quesenberry and Kent's data set [35], the estimated results for the Gamma distribution are $(1.008; 2.239)$ with corresponding K–S $value = 0.0950$ and $p - value$ equals to 0.3118. For the Weibull distribution the obtained parameters are $(0.403; 1.604)$ with K–S $value = 0.0760$ and $p - value = 0.6080$.

Table 6: The Goodness of Fit for some Special Cases

	MLE				MCMC			
Distributions	$\widehat{p}$	$\widehat{\theta}$	K-S value	p-value	$\widehat{p}$ (mean)	s.e.	$\widehat{\theta}$ (mean)	s.e.
Ghitany et al.'s data set [30]:								
last order (m=0)	0.5597	0.1149	0.1116	0.1655	0.5997	2.87×10^{-4}	0.2001	5.13×10^{-4}
last-1 order (m=1)	0.7658	0.0903	0.1119	0.1630	0.6012	9.63×10^{-4}	4.63×10^{-4}	4.62×10^{-4}
last-2 order (m=2)	0.7682	0.0698	0.1167	0.1308	0.5999	5.53×10^{-4}	0.2009	6.38×10^{-4}
last-3 order (m=3)	0.8199	0.0651	0.1148	0.1427	0.5999	3.81×10^{-4}	0.2029	5.12×10^{-4}
last-4 order (m=4)	0.8637	0.0646	0.1113	0.1678	0.5987	9.97×10^{-4}	0.1998	2.97×10^{-4}
Barlow and Campo's data set [31]:								
last order (m=0)	0.9654	1.09×10^{-3}	0.0823	0.4623	0.5995	3.33×10^{-4}	0.1908	4.08×10^{-4}
last-1 order (m=1)	0.9392	6.93×10^{-4}	0.0888	0.9186	0.5990	5.13×10^{-4}	0.1955	9.12×10^{-4}
last-2 order (m=2)	0.9349	5.59×10^{-4}	0.0917	0.3285	0.6010	3.78×10^{-4}	0.1977	1.63×10^{-3}
last-3 order (m=3)	0.9372	4.92×10^{-4}	0.0934	0.3080	0.5988	8.94×10^{-4}	0.1998	4.93×10^{-4}
last-4 order (m=4)	0.9412	4.52×10^{-4}	0.0944	0.2957	0.6008	4.09×10^{-4}	0.1978	1.09×10^{-3}
Gupta and Kundu's data set [34]:								
last order (m=0)	0.9289	0.0731	0.1086	0.6232	0.5999	3.55×10^{-4}	0.1997	2.77×10^{-4}
last-1 order (m=1)	0.9142	0.0484	0.1096	0.6106	0.6012	2.15×10^{-4}	0.2001	3.12×10^{-4}
last-2 order (m=2)	0.9177	0.0399	0.1112	0.5926	0.5981	3.12×10^{-4}	0.1974	5.43×10^{-4}
last-3 order (m=3)	0.8968	0.0319	0.1225	0.4665	0.5992	4.13×10^{-4}	0.1997	3.28×10^{-4}
last-4 order (m=4)	0.5314	0.0130	0.1697	0.1258	0.6098	6.13×10^{-4}	0.1987	5.14×10^{-4}
Quesenberry and Kent's data set [35]:								
last order (m=0)	0.9796	0.0111	0.1026	0.2430	0.5991	7.58×10^{-4}	0.1985	6.33×10^{-4}
last-1 order (m=1)	0.9610	7.22×10^{-3}	0.1077	0.1961	0.6120	6.98×10^{-4}	0.2013	7.01×10^{-4}
last-2 order (m=2)	0.9571	5.89×10^{-3}	0.1102	0.1759	0.5997	5.93×10^{-4}	0.1924	6.81×10^{-4}
last-3 order (m=3)	0.9688	5.78×10^{-3}	0.1082	0.1921	0.6009	6.08×10^{-4}	0.2070	5.93×10^{-4}
last-4 order (m=4)	0.9604	4.82×10^{-3}	0.1125	0.1586	0.5972	5.21×10^{-4}	0.1929	6.10×10^{-4}

5 Conclusion

In this paper, we present a reverse exponential-generalized truncated logarithmic (Rev-EGTL) distribution for ordered spacing statistics. We extended Rahmouni & Orabi [18] EGTL distribution to the concept of dual generalized ordered statistics, introduced by Burkschat et al. [7], that enables a common approach to the descending ordered spacings like the reverse ordered statistics and the lower record values. Also, our procedure generalizes the EL distribution proposed by [36]. We derive some mathematical properties and we present the plots of the pdf for some special cases. The estimation of the parameters is attained by the maximum likelihood and the Bayesian approach, with MCMC numerical sampling. The application study is illustrated based on six real data sets used in many applications of reliability.

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References

[1] Barton D, David F. Some notes on ordered random intervals. Journal of the Royal Statistical Society Series B (Methodological). 1956;p. 79–94.
[2] Weiss L. The limiting joint distribution of the largest and smallest sample spacings. The Annals of Mathematical Statistics. 1959;30(2):590–593.
[3] Pyke R. Spacings. Journal of the Royal Statistical Society Series B (Methodological). 1965;p. 395–449.
[4] Weiss L. The joint asymptotic distribution of the k-smallest sample spacings. Journal of Applied Probability. 1969;6(2):442–448.
[5] Hall P. Limit theorems for sums of general functions of m-spacings. In: Mathematical Proceedings of the Cambridge Philosophical Society. vol. 96. Cambridge Univ Press; 1984. p. 517–532.
[6] Bairamov I, Berred A, Stepanov A. Limit results for ordered uniform spacings. Statistical Papers. 2010;51(1):227–240.
[7] Burkschat M, Cramer E, Kamps U. Dual generalized order statistics. Metron. 2003;61(1):13–26.
[8] Weymark JA. Generalized Gini inequality indices. Mathematical Social Sciences. 1981;1(4):409–430.
[9] Parrado-Gallardo EM, Imedio-Olmedo LJ, Bárcena-Martín E. Inequality, welfare and order statistics. In: Bishop JA, Rodríguez JG, editors. Economic Well-being and Inequality: Papers from the Fifth ECINEQ Meeting (Research on Economic Inequality). vol. 22. Emerald Group Publishing Limited; 2014. p. 383–399.
[10] Maguire BA, Pearson E, Wynn A. The time intervals between industrial accidents. Biometrika. 1952;39(1/2):168–180.
[11] Balakrishnan N, Basu A. The Exponential Distribution: Theory. Methods and Applications, Gordon and Breach Publishers. 1995;.
[12] Barlow RE, Proschan F. Statistical theory of reliability and life testing: Probability models. Holt, Rinehart and Winston, New York; 1975.
[13] Adamidis K, Loukas S. A lifetime distribution with decreasing failure rate. Statistics & Probability Letters. 1998;39(1):35–42.
[14] Kuş C. A new lifetime distribution. Computational Statistics & Data Analysis. 2007;51(9):4497–4509.
[15] Winkelmann R. Econometric analysis of count data. Springer Science & Business Media; 2008.
[16] Chatfield C. On estimating the parameters of the logarithmic series and negative binomial distributions. Biometrika. 1969;56(2):411–414.
[17] Fisher RA, Corbet AS, Williams CB. The relation between the number of species and the number of individuals in a random sample of an animal population. The Journal of Animal Ecology. 1943;p. 42–58.
[18] Rahmouni M, Orabi A. A generalization of the exponential-logarithmic distribution for reliability and life data analysis. Life Cycle Reliability and Safety Engineering. 2018;7(3):159–171.
[19] Rahmouni M, Orabi A. The exponential-generalized truncated geometric (EGTG) distribution: a new lifetime distribution. International Journal of Statistics and Probability. 2018;7(1):1–20.
[20] Rahmouni M, Orabi A. A New Generalization of the Exponential-Poison Distribution Using Order Statistics. International Journal of Applied Mathematics & Statistical Sciences (IJAMSS). 2017;6(6):27–36.
[21] Barlow RE, Proschan F. Statistical Theory of Reliability and Life Testing: Probability Models; 1981.
[22] Basu AP. Multivariate exponential distributions and their applications in reliability. Handbook of Statistics. 1988;7:467–477.
[23] Barlow RE, Marshall AW, Proschan F. Properties of probability distributions with monotone hazard rate. The Annals of Mathematical Statistics. 1963;p. 375–389.
[24] Dempster AP, Laird NM, Rubin DB. Maximum likelihood from incomplete data via the EM algorithm. Journal of the royal statistical society Series B (methodological). 1977;p. 1–38.
[25] Krishnan T, McLachlan G. The EM algorithm and extensions. Wiley. 1997;1(997):58–60.
[26] Little RJ, Rubin DB. On jointly estimating parameters and missing data by maximizing the complete-data likelihood. The American Statistician. 1983;37(3):218–220.
[27] Adamidis K, Dimitrakopoulou T, Loukas S. On an extension of the exponential-geometric distribution. Statistics & probability letters. 2005;73(3):259–269.
[28] Karlis D, Xekalaki E. Choosing initial values for the EM algorithm for finite mixtures. Computational Statistics & Data Analysis. 2003;41(3):577–590.
[29] Ng H, Chan P, Balakrishnan N. Estimation of parameters from progressively censored data using EM algorithm. Computational Statistics & Data Analysis. 2002;39(4):371–386.
[30] Ghitany M, Atieh B, Nadarajah S. Lindley distribution and its application. Mathematics and computers in simulation. 2008;78(4):493–506.
[31] Barlow RE, Campo R. Total Time on Test Procedures and Applications to Failure Data Analysis. In: Barlow RE, Fussel R, Singpurwalla ND, editors. Reliability and Fault Tree Analysis. SIAM , Philadelphia, PA; 1975. p. 451–481.
[32] Chang MN, Rao PV. Improved estimation of survival functions in the new-better-than-used class. Technometrics. 1993;35(2):192–203.
[33] Bakouch HS, Jazi MA, Nadarajah S, Dolati A, Roozegar R. A lifetime model with increasing failure rate. Applied Mathematical Modelling. 2014;38(23):5392–5406.
[34] Gupta RD, Kundu D. A new class of weighted exponential distributions. Statistics. 2009;43(6):621–634.
[35] Quesenberry C, Kent J. Selecting among probability distributions used in reliability. Technometrics. 1982;24(1):59–65.
[36] Tahmasbi R, Rezaei S. A two-parameter lifetime distribution with decreasing failure rate. Computational Statistics & Data Analysis. 2008;52(8):3889–3901.