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Fractional-Order Delayed Salmonella Transmission Model: A Numerical Simulation

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Abstract: In this paper, a fractional dynamical model of Salmonella with time delay is studied numerically. The proposed model is administered by a system of fractional delay differential equations, where the fractional derivative is defined in the sense of the Caputo definition. The parameters are modified regarding to the order of the fractional derivative. The stability of the disease free equilibrium point and endemic equilibrium point is investigated for any time delay. Weighted difference numerical technique is introduced to simulate the proposed model. This scheme was unconditionally stable when the weight factor is less than 1. Numerical simulations with some comparison are introduced to show the applicability and effectivity of the proposed method to solve such stiff systems of fractional delay differential equations and to confirm the theoretical studies.

Keywords: Fractional delay model of Salmonella, weighted difference method, stability, fractional Caputo derivative.

1 Introduction

Epidemics are one of the most serious health issues facing the world today, and they must be addressed. For a long time, researchers have been studying the dynamics of epidemic diseases. The most effective approach for studying epidemic dynamics is to use models that include time derivatives and are composed of systems of ordinary differential equations. Each equation in these models represents the change in the number of bodies in various categories as determined by continuous variables ([1], [2]).

Salmonella is well known to be a major public health concern throughout the world. Every year in the United States, it causes approximately 1.4 million clinical cases, 16,000 hospitalizations, and 600 deaths. It is a major zoonotic disease that is passed from animals to humans through milk, beef, eggs, and other dairy products, or through direct contact with sick animals and their surroundings. Salmonella can be found in the intestines of humans, animals, and birds. Typically, the illness lasts 4 to 7 days, and most people recover without medical attention. However, in some cases, diarrhea is so dangerous that the patient must be hospitalized.

Due to the impact of epidemics is always delayed in time, it is necessary to incorporate memory into differential epidemics models. As a result, models in which the current state is dependent on all of its previous states rather than just the first previous one are better suited to describing the epidemic's flow. Moreover, because fractional derivatives are defined in terms of an integral form over the entire history of the plan [3], they can be used to describe the hereditary properties inherent and memory in different processes. and materials. Therefore, the derivatives models described using fractional derivatives are better suited to epidemics models. For numerous applications involving epidemic models, models built with fractional-order derivatives have been proved to outperform models built with integer-order derivatives in terms of real-world fit ([1], [4] and [5]). In addition, due to biological systems have time lags, we presented a Salmonella fractional-order model with some delay terms in this paper to depict the dynamic of Salmonella with memory effect, which includes the effect of previous values of the variables. This model's stability will be investigated.

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Fractional calculus gained a lot of importance due to its attractive applications in many scientific fields such as biology ([1], [4], [6]), system control ([3], [7], [8]), viscoelasticity [9], thermoelasticity ([10], [11]), hydrology [12], finance [13] and fractional dynamics [14].

Delay fractional differential equations have recently attracted the attention of many researchers ([15], [16], [17], [18], [19], [20] and [21]). In order to characterize the solution behavior of delay fractional differential equations, accurate approximation and efficient numerical techniques are essential.

The main contribution of this work is to numerically investigate the following fractional-order Salmonella epidemic model with time delay memory τ:

$$\begin{aligned}
{}_0^cD_t^\alpha S(t) &= \mu^\alpha N + r^\alpha R(t) - (\beta_c^\alpha I_c(t) + \beta_s^\alpha I_s(t-\tau) + \beta_{lt}^\alpha I_{lt}(t) + \mu^\alpha)S(t),\\
{}_0^cD_t^\alpha I_c(t) &= f^\alpha(\beta_c^\alpha I_c(t) + \beta_s^\alpha I_s(t-\tau) + \beta_{lt}^\alpha I_{lt}(t))S(t) - (e^\alpha + m^\alpha + \mu^\alpha)I_c(t),\\
{}_0^cD_t^\alpha I_s(t) &= (1-f^\alpha)(\beta_c^\alpha I_c(t) + \beta_s^\alpha I_s(t-\tau) + \beta_{lt}^\alpha I_{lt}(t))S(t) + e^\alpha I_c(t) - (h^\alpha + \mu^\alpha)I_s(t),\\
{}_0^cD_t^\alpha I_{lt}(t) &= f_{lt}^\alpha h^\alpha I_s(t) - (h_{lt}^\alpha + \mu^\alpha)I_{lt}(t),\\
{}_0^cD_t^\alpha R(t) &= (1-f_{lt}^\alpha)h^\alpha I_s(t) + h_{lt}^\alpha I_{lt}(t) - (r^\alpha + \mu^\alpha)R(t),
\end{aligned} \tag{1}$$

subject to the initial conditions:

$$S(0) = s_0 \geq 0, I_c(0) = i_{c0} \geq 0, I_{lt}(0) = i_{lt0} \geq 0, \text{ and } R(0) = r_0 \geq 0.$$

and

$$I_s(t) = i_{s0} \geq 0,\ t \in [-\tau, 0].$$

This fractional order delay Salmonella model is numerically approximated using an efficient implicit technique for numerical treatments.

All of the parameters in this system are linked to the fractional-order α. For the sake of clarity in the notations, we will remove the symbol α from above of the parameters to simplify the notations. Table (1) describes all variables and parameters in the suggested system (1), along with their definitions.

The flow of individuals from and into the various groups is described in System (1). Therefore, once infected, a susceptible individual moves from the susceptible compartment to the infectious compartment, where it becomes infectious. The infected animals enter the reconstructed compartment. Individuals who have recovered from the disease have temporary immunity and are classified.

Salmonella infection mathematical modeling was important in understanding disease transmission. In the case $\alpha = 1$, Y. Y. Lo [22] proposed the above system. Due to the impact of the epidemic is not instantaneous, we prepare in fractional-order with time delay case, so incorporating memory is very important for explaining and understanding the flow of salmonella epidemic. More details about the proposed system's biological impulse and its assumptions can be found in ([22]).

1.1 Preliminaries

Many definitions of fractional derivatives have been introduced in the literature (see e.g., [3], [23], [24]). The time fractional derivatives are typically defined using the operators Riemann-Liouville, Grunwald-Letnikov, and Caputo. Caputo's definition of the fractional derivative is now one of the most commonly used fractional derivatives in engineering and applied science because it deals with any initial value problem appropriately.

Definition 1.*The Caputo fractional derivatives of order α, $\alpha \in R^+$, are defined as follows:*

$$({}_0^cD_t^\alpha f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(x)}{(t-x)^{1-n+\alpha}} dx, \quad t > 0, \tag{2}$$

where $f(x) \in C^n[0,\infty[$, $n = [\alpha] + 1$.

In particular, when $0 < \alpha \leq 1$, we have:

$$({}_0^cD_t^\alpha f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(x)}{(t-x)^{\alpha}} dx, \quad t > 0, \tag{3}$$

It is well known that the Caputo fractional derivative has the following properties:

–${}_0^cD_t^\alpha k = 0$, k is constant.

Table 1: All symbols in the system (1) and their definition

Symbol	Definition
t	Time, $t \geq 0$.
α	Order of the fractional derivative.
${}_0^c D_t^\alpha$	Caputo fractional derivative.
$S(t)$	Individuals that do not have the bacterial infection (susceptible).
$I_c(t)$	Individuals that have the bacterial infection (clinically infected).
$I_s(t)$	Individuals that recovered from the infection and have temporary immunity (subclinically infected).
$I_{lt}(t)$	Individuals without clinical signs reported to shed Salmonella persistently or intermittently.
$R(t)$	Number of recovered individuals.
N	Size population, $N = S(t) + I_c(t) + I_s(t) + I_{lt}(t) + R(t)$.
μ	Replacement and exit rate (denotes the mortality rate in every compartment).
β_c	Transmission coefficient for clinical animals.
β_s	Transmission coefficient for subclinical animals.
β_{lt}	Transmission coeffcient for long-term shedders.
f	Proportion of infected animals that develop clinical case.
f_{lt}	Proportion of subclinical cases that become long-term shedders.
e	Rate of clinical cases that become subclinical.
h	Recovery rate for subclinical case.
h_{lt}	Recovery rate for long-term shedders.
m	Disease-induced induce mortality rate.
r	Immunity loss rate.

–Caputo's fractional differentiation is a linear operation. i.e.,

$$ {}_0^c D_t^\alpha (\lambda f(t) + \gamma g(t)) = \lambda \, {}_0^c D_t^\alpha f(t) + \gamma \, {}_0^c D_t^\alpha g(t). $$

The article is structured as follows: Section 2 introduces the basic reproduction number and the stability analysis. Section 3 employs the weighted average finite difference method to approximate the proposed model's solutions. Section 5 reports numerical simulations for the suggested model to demonstrate the efficiency and applicability of the method used. Finally, in Section 6, a brief conclusion is provided.

2 Basic reproduction number and stability analysis

2.1 Basic reproduction number

The basic reproduction number, R_0, is defined as the average number of secondary infections generated when a single infective individual is introduced into a host population where the remaining of the population is susceptible ([2], [22]). This parameter is used to determine the local stability of the disease-free equilibrium. If $R_0 < 1$, a locally asymptotically stable equilibrium exists. From a biological standpoint, this means that the infected individual, on average, produces less than one new infected over the course of the infectious period. Therefore, the infection will be stopped, and the model will eventually reach a locally stable disease-free equilibrium. The disease-free equilibrium, on the other hand, is locally unstable when $R_0 > 1$. Each infected individual will spread the disease to at least one susceptible individual, and thus the infection will persist.

The the basic reproduction number R_0 for system (1) is:

$$ R_0 = \frac{f\beta_c s_0}{e+\mu+m} + \frac{(1-f)\beta_s s_0}{h+\mu} + \frac{fe\beta_s s_0}{(e+\mu+m)(h+\mu)} + \frac{(1-f)f_{lt}h\beta_{lt}s_0}{(h+\mu)(h_{lt}+\mu)} + \frac{fef_{lt}h\beta_{lt}s_0}{(e+\mu+m)(h+\mu)(h_{lt}+\mu)}. \tag{4} $$

2.2 Stability analysis

Theorem 1.*[25] The system (1) is locally asymptotically stable if and only if the eigenvalues λ_i of the Jacobian matrix $J = \frac{\partial g}{\partial t}$ satisfy $|arg(\lambda_i)| > \frac{\alpha\pi}{2}$.*

2.2.1 Stability of the disease free equilibrium point

System (1) has a disease-free equilibrium point $\xi_1 = (N,0,0,0,0)$.

The linearized system corresponding to (1) at ξ_1 is provided by:

$$\begin{aligned}
{}_0^cD_t^\alpha S(t) &= \mu N + rR(t) - \beta_s N I_s(t-\tau) - \mu S(t),\\
{}_0^cD_t^\alpha I_c(t) &= fN\beta_s I_s(t-\tau) - (e+m+\mu)I_c(t),\\
{}_0^cD_t^\alpha I_s(t) &= (1-f)N\beta_s I_s(t-\tau) + eI_c(t) - (h+\mu)I_s(t),\\
{}_0^cD_t^\alpha I_{lt}(t) &= f_{lt}hI_s(t) - (h_{lt}+\mu)I_{lt}(t),\\
{}_0^cD_t^\alpha R(t) &= (1-f_{lt})hI_s(t) + h_{lt}I_{lt}(t) - (r+\mu)R(t).
\end{aligned} \tag{5}$$

The characteristic polynomial associated with the linearized system is

$$det\left[\begin{pmatrix}
\lambda+\mu & 0 & \beta_s N e^{-\lambda\tau} & 0 & -r\\
0 & \lambda+(e+m+\mu) & -f\beta_s N e^{-\lambda\tau} & 0 & 0\\
0 & -e & \lambda-(1-f)\beta_s N e^{-\lambda\tau}+(h+\mu) & 0 & 0\\
0 & 0 & -f_{lt}h & \lambda+(h_{lt}+\mu) & 0\\
0 & 0 & -(1-f_{lt})h & -h_{lt} & \lambda+(r+\mu)
\end{pmatrix}\right] = 0,$$

where λ is known as the eigenvalues.

This leads to

$$(\lambda+\mu)(\lambda+r+\mu)(\lambda+h_{lt}+\mu)[(\lambda+e+m+\mu)(\lambda+h+\mu-(1-f)\beta_s N e^{-\lambda\tau}) - ef\beta_s N e^{-\lambda\tau}] = 0.$$

1- for $\tau = 0$: the eigenvalues are given by

$$\lambda_1 = -\mu,$$
$$\lambda_2 = -\mu - r,$$
$$\lambda_3 = -\mu - h_{lt},$$
$$\lambda_4 = \frac{(1-f)\beta_s N - h - m - e - 2\mu + \sqrt{(AN^2+BN+C)}}{2},$$
$$\lambda_5 = \frac{(1-f)\beta_s N - h - m - e - 2\mu - \sqrt{(AN^2+BN+C)}}{2},$$

where

$$A = (\beta_s^2 f^2 - 2\beta_s^2 f + \beta_s^2),$$
$$B = 2\beta_s ef + 2\beta_s e + 2\beta_s fh - 2\beta_s fm - 2\beta_s h + 2\beta_s m,$$
$$C = e^2 - 2eh + 2em + h^2 - 2hm + m^2,$$

It is clear that, if $\lambda_4, \lambda_5 \leq 0$ or λ_4 and λ_5 has negative real part then the free equilibrium point for the proposed model is asymptotically stable

2- for $\tau > 0$: the eigenvalues are given by

$$\lambda_1 = -\mu,$$
$$\lambda_2 = -\mu - r,$$
$$\lambda_3 = -\mu - h_{lt}.$$

The remaining roots are determined as follows:

$$[(\lambda+e+m+\mu)(\lambda+h+\mu-(1-f)\beta_s N e^{-\lambda\tau}) - ef\beta_s N e^{-\lambda\tau}] = 0.$$

For any value of the delay τ, these roots have no pure imaginary roots. Therefore, all of the roots of the characteristic polynomial have negative real parts. Consequently, regardless of the delay value, the free equilibrium point is locally asymptotically stable and this leads to the disease-free equilibrium ξ_1 is locally asymptotically stable.

2.2.2 Stability of the endemic equilibrium point

The endemic equilibrium for the system (1) exists if at least one of the infected groups is not equal zero [2]. The analytic expression is very complicated and is not our aim. Numerically, considering the values of parameters from the section of the numerical simulation. The basic reproduction number is $R_0 = 2.5771 > 1$. The endemic equilibrium is $\xi_2 = (193.63, 0.65371, 7.9516, 4.1119, 0.56645)$

The linearization of system (1) around the endemic equilibrium point ξ_2 is given by

$$\begin{aligned}
{}_0^cD_t^\alpha S(t) &= 0.2997 + 0.01R(t) - 0.0116I_s(t-\tau) - 0.0011S(t),\\
{}_0^cD_t^\alpha I_c(t) &= 0.1252 + 0.0058I_s(t-\tau) - 0.2621I_c(t),\\
{}_0^cD_t^\alpha I_s(t) &= 0.1252 + 0.0058I_s(t-\tau) + 0.25I_c(t) - 0.0421I_s(t),\\
{}_0^cD_t^\alpha I_{lt}(t) &= 0.0057I_s(t) - 0.0111I_{lt}(t),\\
{}_0^cD_t^\alpha R(t) &= 0.0353I_s(t) + 0.01I_{lt}(t) - 0.0111R(t).
\end{aligned} \tag{6}$$

The corresponding characteristic equation for ξ_2 is:

$$det\left[\begin{pmatrix}
\lambda+0.0011 & 0 & 0.0116e^{-\lambda\tau} & 0 & -0.01\\
0 & \lambda+0.2621 & -0.0058e^{-\lambda\tau} & 0 & 0\\
0 & -0.25 & \lambda-0.0058e^{-\lambda\tau}+0.0421 & 0 & 0\\
0 & 0 & -0.0057 & \lambda+0.0111 & 0\\
0 & 0 & -0.0353 & -0.01 & \lambda+0.0111
\end{pmatrix}\right] = 0,$$

where λ is the eigenvalue.

So,

$$(\lambda+0.0111)(\lambda+0.0011)[(\lambda+0.2621)(\lambda-0.0058e^{-\lambda\tau}+0.0421) - 0.00145e^{-\lambda\tau}] = 0.$$

1- for $\tau = 0$: the eigenvalues are given by

$$\lambda_1 = -0.0111,$$

$$\lambda_2 = -0.0011,$$

$$\lambda_3 = -0.2683,$$

and

$$\lambda_4 = -0.0301.$$

All these eigenvalues are negative, thus the endemic equilibrium point ξ_2 is asymptotical stable.

2- for $\tau > 0$: the eigenvalues are given by

$$\lambda_1 = -0.0111,$$

$$\lambda_2 = -0.0011,$$

All the other roots are determined by the following equation

$$[(\lambda+0.2621)(\lambda-0.0058e^{-\lambda\tau}+0.0421) - 0.00145e^{-\lambda\tau}] = 0.$$

The roots of this equation are numbers with no pure imaginary part for any value of the delay τ. Therefore all the roots of the characteristic equation have negative real parts. Hence, the endemic equilibrium point is locally asymptotically stable.

3 Weighted average finite difference scheme

Due to the exact solution to the fractional delay differential equations cannot be obtained explicitly, numerical techniques are now very important for approximating the solution. Depending on the weight factor $\theta \in [0,1]$ [26], the weighted average finite difference method (WAFDM) can be either explicit or implicit. We first discretize the Caputo fractional operator to approximate the solutions of the system (1) using WAFDM. If we assume that the mesh points as $\{t_0 =$

$0, t_1, t_2, \ldots t_N = T_{final}\}$, then, using the quadrature formula, the following discrete approximation of the Caputo fractional derivative can be found [5]:

$$\begin{aligned}
({}_0^cD_t^\alpha f)(t)\big|_{t=t_n} &= \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{f'(x)}{(t_n-x)^\alpha}dx, \qquad n=1,2,...,N,\\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{i=1}^{n}\int_{(i-1)h}^{ih}[\frac{f_i-f_{i-1}}{h}](nh-x)^{-\alpha}dx\\
&= \frac{1}{(1-\alpha)\Gamma(1-\alpha)}\sum_{i=1}^{n}[\frac{f_i-f_{i-1}}{h}][(n-i+1)^{1-\alpha}-(n-i)^{1-\alpha}]h^{1-\alpha}\\
&= \frac{1}{h^\alpha(1-\alpha)\Gamma(1-\alpha)}\sum_{i=1}^{n}[f_i-f_{i-1}][(n-i+1)^{1-\alpha}-(n-i)^{1-\alpha}], \qquad (7)
\end{aligned}$$

so,

$$({}_0^cD_t^\alpha f)(t)\big|_{t=t_n} = g(\alpha,h)\sum_{i=1}^{n} w_i^\alpha (f_i - f_{i-1}), \qquad (8)$$

where

$$g := g(\alpha,h) = \frac{1}{h^\alpha(1-\alpha)\Gamma(1-\alpha)},$$

and

$$w_i := w_i^\alpha = [(n-i+1)^{1-\alpha}-(n-i)^{1-\alpha}], \qquad w_n = 1.$$

System (1) using this discrtization with $\tau = qh$ and WAFDM can be written as follows

$$\begin{aligned}
g\sum_{i=1}^{n} w_i(S_i - S_{i-1}) &= \mu N + (1-\theta)\{rR_n - (\beta_c I_{cn} + \beta_s I_{s(n-q)} + \beta_{lt} I_{ltn} + \mu)S_n\}\\
&\quad + \theta\{rR_{n-1} - (\beta_c I_{c(n-1)} + \beta_s I_{s(n-q-1)} + \beta_{lt} I_{lt(n-1)} + \mu)S_{n-1}\},\\
g\sum_{i=1}^{n} w_i(I_{ci} - I_{c(i-1)}) &= (1-\theta)\{f(\beta_c I_{cn} + \beta_s I_{s(n-q)} + \beta_{lt} I_{ltn})S_n - (e+m+\mu)I_{cn}\}\\
&\quad + \theta\{f(\beta_c I_{c(n-1)} + \beta_s I_{s(n-q-1)} + \beta_{lt} I_{lt(n-1)})S_{n-1} - (e+m+\mu)I_{c(n-1)}\},\\
g\sum_{i=1}^{n} w_i(I_{si} - I_{s(i-1)}) &= (1-\theta)\{(1-f)(\beta_c I_{cn} + \beta_s I_{s(n-q)} + \beta_{lt} I_{ltn})S_n + eI_{cn} - (h+\mu)I_{sn}\}\\
&\quad + \theta\{(1-f)(\beta_c I_{c(n-1)} + \beta_s I_{s(n-q-1)} + \beta_{lt} I_{lt(n-1)})S_{n-1} + eI_{c(n-1)} - (h+\mu)I_{s(n-1)}\},\\
g\sum_{i=1}^{n} w_i(I_{lti} - I_{lt(i-1)}) &= (1-\theta)\{f_{lt} hI_{sn} - (h_{lt}+\mu)I_{ltn}\}\\
&\quad + \theta\{f_{lt} hI_{s(n-1)} - (h_{lt}+\mu)I_{lt(n-1)}\},\\
g\sum_{i=1}^{n} w_i(R_i - R_{i-1}) &= (1-\theta)\{(1-f_{lt})hI_{sn} + h_{lt} I_{ltn} - (r+\mu)R_n\}\\
&\quad + \theta\{(1-f_{lt})hI_{s(n-1)} + h_{lt} I_{lt(n-1)} - (r+\mu)R_{n-1}\}, \qquad (9)
\end{aligned}$$

this system is a nonlinear algebraic system with $5N$ equations of $5N$ unknowns (S_n, I_{cn}, I_{sn}, I_{ltn}, R_n) that can be solved using appropriate iterative method depending on the supposed initial conditions.

We notice that this schemes is explicit for $\theta = 1$, partially implicit for $0 < \theta < 1$ and a fully implicit when $\theta = 0$. In the following section, we will demonstrate that this method is unconditionally stable in case $0 \le \theta < 1$.

3.1 Stability of WAFDM

To Investigate the stability of the weighted implicit scheme ($\theta \neq 1$),, we consider the following linear fractional delay differential equation [27]

$$({}_0^cD_t^\alpha f)(t) = \rho_0 f(t) + \rho_1 f(t-\tau), \;\; t \ge 0, \;\; 0 < \alpha \le 1, \qquad (10)$$

$$f(t)=\Psi(t),\ \ t\in[-\tau,0],\ \ f(0)=f_0,$$

such that $\rho_0<0,\ \rho_1<\rho_0$ and $\Psi(t)$ is continuous and bounded function.

Let $f(t_n)=f_n=\xi_n$ is the approximate solution of this equation then using WAFDM with the relation (8), we can rewrite Eq. (10) as follows:

$$g\sum_{i=1}^{n}w_i(\xi_i-\xi_{i-1})=(1-\theta)\{\rho_0\xi_n+\rho_1\xi_{n-q}\}+\theta\{\rho_0\xi_{n-1}+\rho_1\xi_{n-q-1}\}.$$

By doing the same steps in [27] we have

$$\xi_n(1-\frac{1-\theta}{g}\rho_0)=\sum_{i=1}^{n-1}w_i^{\alpha}(\xi_{i-1}-\xi_i)+\xi_{n-1}(1+\frac{\theta}{g}\rho_0)+\frac{1-\theta}{g}\rho_1\xi_{n-q}+\frac{\theta}{g}\rho_1\xi_{n-q-1},\ \ n>q,$$

and

$$\xi_n(1-\frac{1-\theta}{g}\rho_0)=\sum_{i=1}^{n-1}w_i^{\alpha}(\xi_{i-1}-\xi_i)+\xi_{n-1}(1+\frac{\theta}{g}\rho_0)+\frac{1-\theta}{g}\rho_1\Psi_{n-q}+\frac{\theta}{g}\rho_1\Psi_{n-q-1},\ \ n=1,2,...q-1,$$

then

$$\xi_n=\frac{1}{(1-\frac{1-\theta}{g}\rho_0)}\big[\sum_{i=1}^{n-1}w_i^{\alpha}(\xi_{i-1}-\xi_i)+\xi_{n-1}(1+\frac{\theta}{g}\rho_0)+\frac{1-\theta}{g}\rho_1\xi_{n-q}+\frac{\theta}{g}\rho_1\xi_{n-q-1}\big],\ \ n\geq 2.$$

We have $\frac{1}{(1-\frac{1-\theta}{g}\rho_0)}<1$, hence

$$\xi_1\leq\xi_0,$$
$$\xi_n\leq\xi_{n-1}\leq\xi_{n-1}\leq\ldots\leq\xi_1\leq\xi_0.$$

So the proposed implicit scheme is stable.

Notice : the truncation error of this method is of order h.

4 Numerical simulations

In this section, we used the suggested technique (9) when $\theta=0$, to obtain the numerical simulations of the model (1).

First, let $N=500$, $\mu=0.0011$, $\beta_c=0.0016$, $\beta_s=0.00006$, $\beta_{lt}=0.00006$, $f=0.5$, $f_{lt}=0.14$, $e=0.25$, $h=0.041$, $h_{lt}=0.01$, $m=0.011$, and $r=0.01$. The initial conditions are $S_0=N-1$, $I_c=1$, $I_s=0$, $I_{lt}=0$, $R=0$. Figures **??** and **??** display the numerical solutions of the system (1) using WAFDM at $t=100$ and $t=500$, respectively. It is clear that the proposed method is stable than the built-in MATLAB Code dd23.

Second, let $N=500$, $\mu=0.0011$, $e=0.25$, $\beta_c=0.0016$, $\beta_s=0.00166$, $\beta_{lt}=0.00006$, $f=0.15$, $f_{lt}=0.14$, $h=0.041$, $h_{lt}=0.01$, $r=0.01$, and $m=0.011$. The initial conditions are $S_0=N-10$, $I_c=5$, $I_s=5$, $I_{lt}=0$, $R=0$. Figure (3) shows that the summation of the unknown $(S(t)+I_c(t)+I_s(t)+I_{lt}(t)+R(t)=N$ strictly still constant during the studying time when we use the implicit method while it is not constant when we use dde23, that is, the proposed method outperforms the solver dde23.

Third, let $N=300$, $\mu=0.0011$, $e=0.25$, $\beta_c=0.0016$, $\beta_s=0.00166$, $\beta_{lt}=0.00006$, $f=0.15$, $f_{lt}=0.14$, $h=0.041$, $h_{lt}=0.01$, $r=0.01$, and $m=0.011$. The initial conditions are $S_0=N-10$, $I_c=5$, $I_s=5$, $I_{lt}=0$, $R=0$. Figure (4) shows the numerical solutions of the system (1) using WAFDM at $t_{final}=250$, $alpah=0.7$ and different values of τ.

Fourth, let $N=500$, $\mu=0.0011$, $e=0.25$, $\beta_c=0.0016$, $\beta_s=0.00166$, $\beta_{lt}=0.00006$, $f=0.15$, $f_{lt}=0.14$, $h=0.041$, $h_{lt}=0.01$, $r=0.01$, and $m=0.011$. The initial conditions are $S_0=N-10$, $I_c=5$, $I_s=5$, $I_{lt}=0$, $R=0$.

Figure (5) shows the numerical solutions of the system (1) using WAFDM at $t_{final}=1000$, $\alpha=0.9$ and different values of τ..

Fifth, let $N=500$, $\mu=0.0011$, $e=0.25$, $\beta_c=0.0016$, $\beta_s=0.00166$, $\beta_{lt}=0.00006$, $f=0.15$, $f_{lt}=0.14$, $h=0.041$, $h_{lt}=0.01$, $r=0.01$, and $m=0.011$.. The initial conditions are $S_0=N-1$, $I_c=1$, $I_s=0$, $I_{lt}=0$, $R=0$. Figures (6) and 7) show the values of R and S, respectively, which obtained by WAFDM at various values of τ and α. It is obvious that the values of $R(t)$ decrease by increasing of τ. In these figures, we compare our result with the result obtained by the solver dde23 and we see that the result of WAFDM is more reality.

Also, in Figure (8) we show the relations between $I_s(t)$ and $I_s(t-\tau)$ when $\tau=5,\ \alpha=0.9\ \ t_{final}=100$ and we see how $R(t)$ changes when $S(t)$ changes.

5 Conclusion

In this article, we presented a fractional Salmonella model with time delay. This dynamical model is better suited to describing biological phenomena than the classical model. We investigated the stability characteristics of the equilibrium points of the suggested system. The numerical simulations of the suggested model were obtained using WAFDM. Furthermore, the numerical simulations show that he implicit case has good stability properties. Some comparisons simulations with MATLAB code dde23 are provided with various time delay values. These comparisons show that the proposed method is superior to the dde23 code for the stiff systems. Finally, WAFDM can be used to solve fractional delay differential systems in a straightforward and effective manner.

Conflict of Interest

The authors declare that they have no conflict of interest.

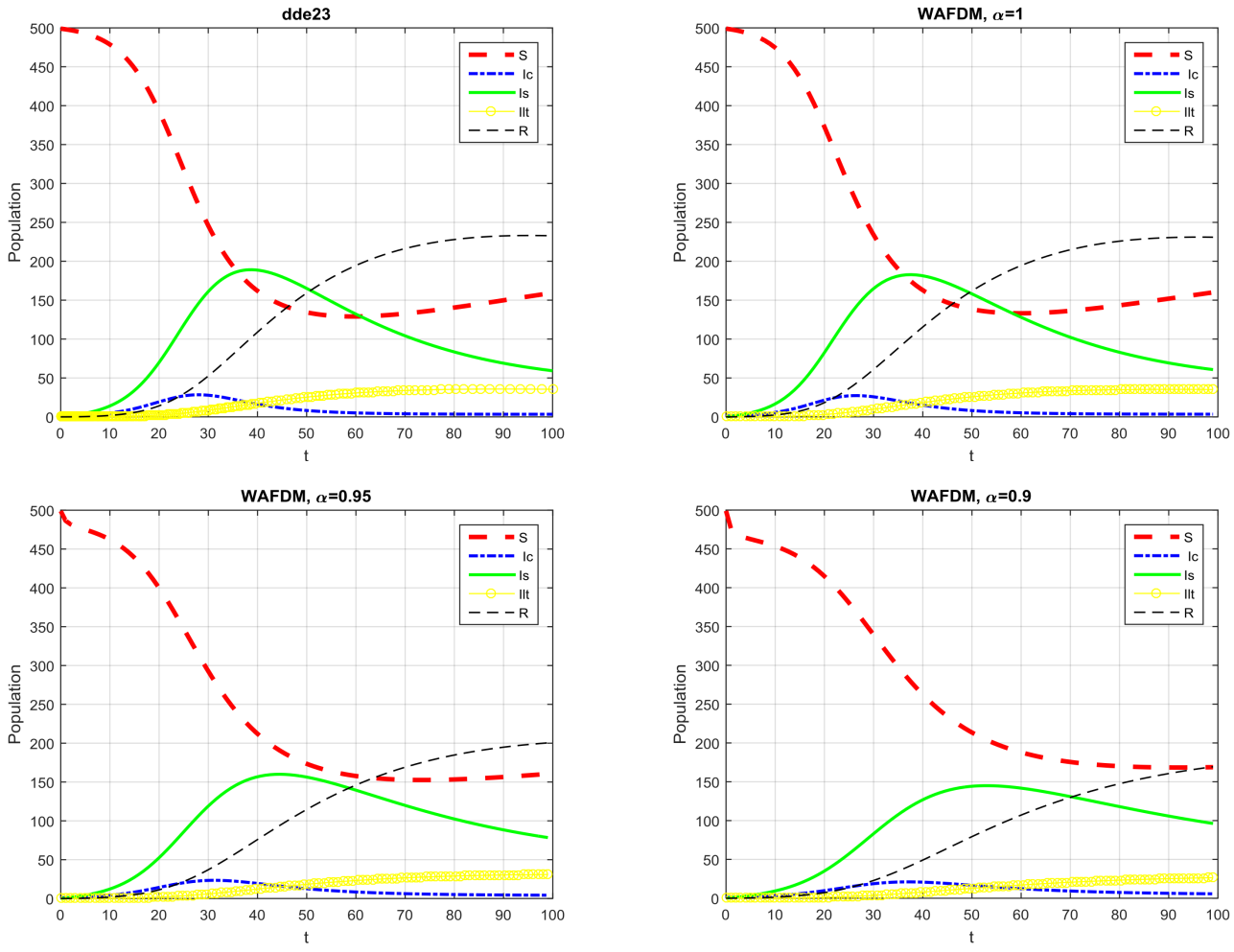

Fig. 1: Profiles obtained by dde23 and WAFDM ($\theta = 0$) for solving the fractional delay Salmonella model with different value of α when $\tau = 0$.

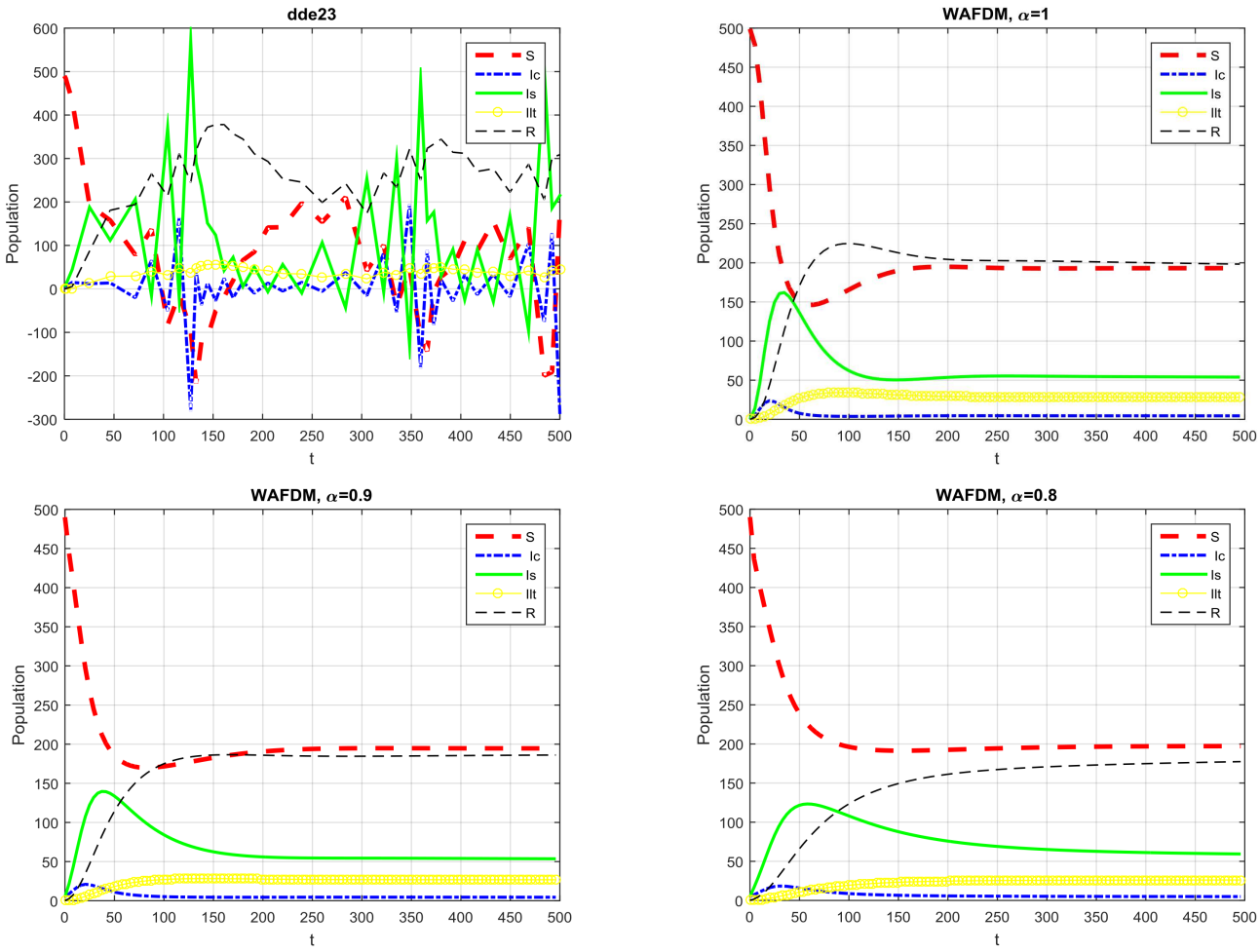

Fig. 2: Unstable solution using dde23 and stable solution using WAFDM ($\theta = 0$) for solving the fractional delay Salmonella model with different value of α when $\tau = 0$.

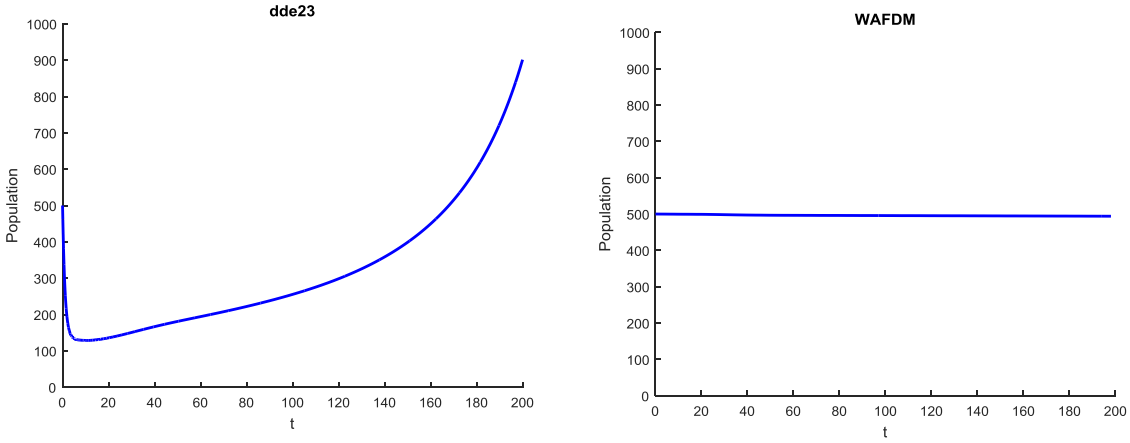

Fig. 3: Summation of the unknowns when $\tau = 10$.

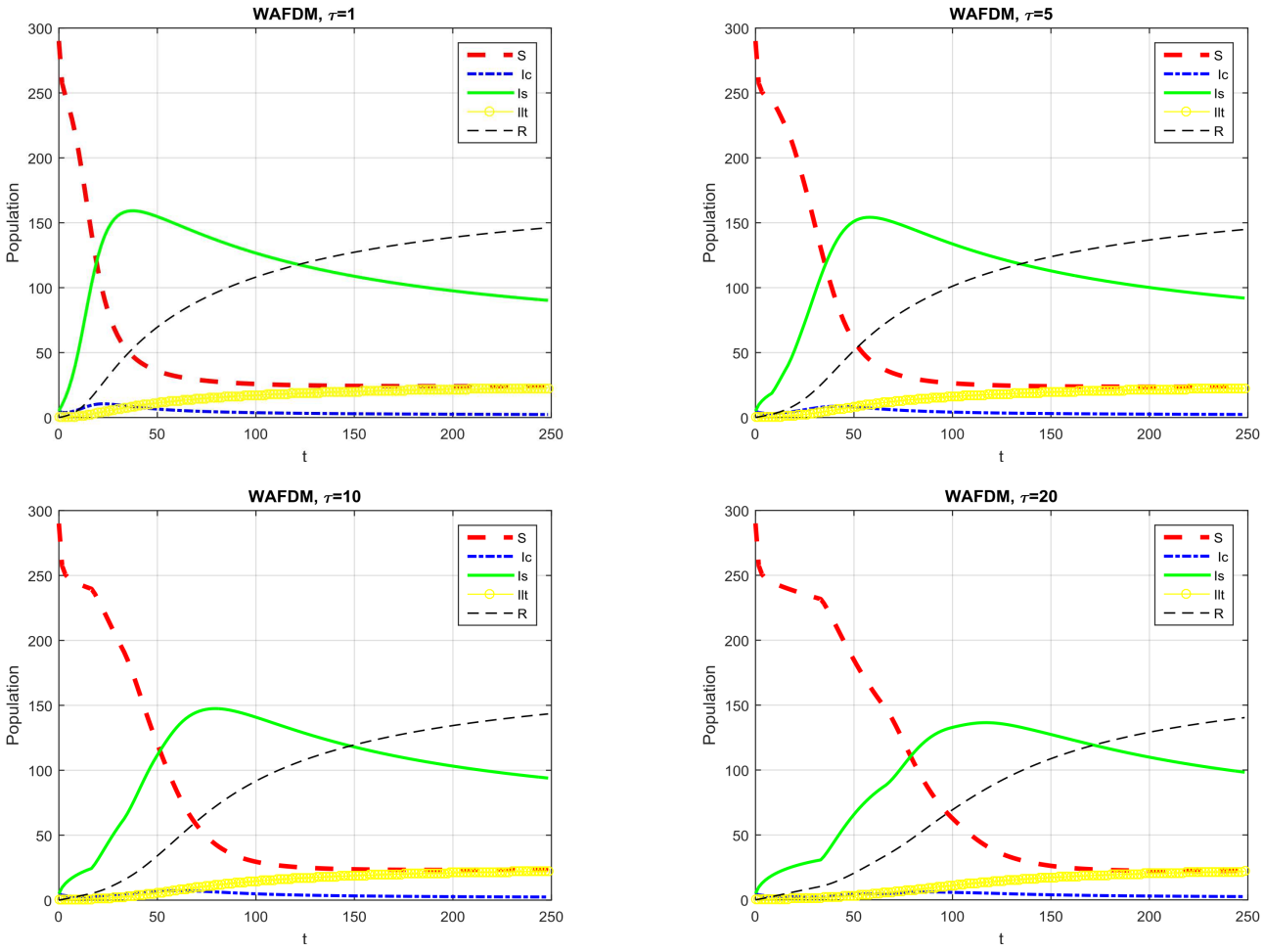

Fig. 4: Profiles obtained by WAFDM $(\theta = 0)$ for solving the fractional delay Salmonella model with different value of τ when $\alpha = 0.7$.

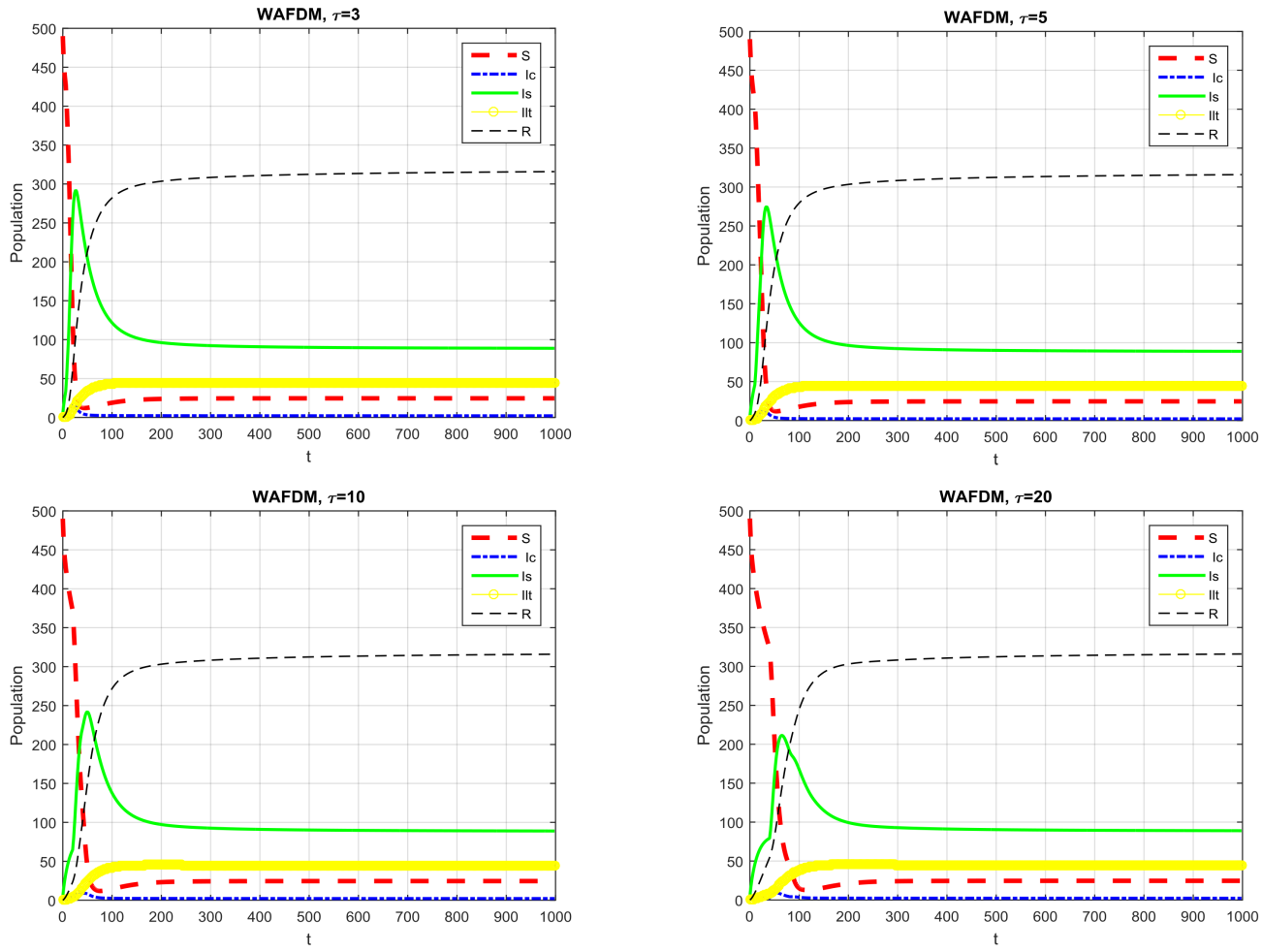

Fig. 5: Profiles obtained by WAFDM ($\theta = 0$) for solving the fractional delay Salmonella model with different value of $\tau = 0$ when $\alpha = 0.9$.

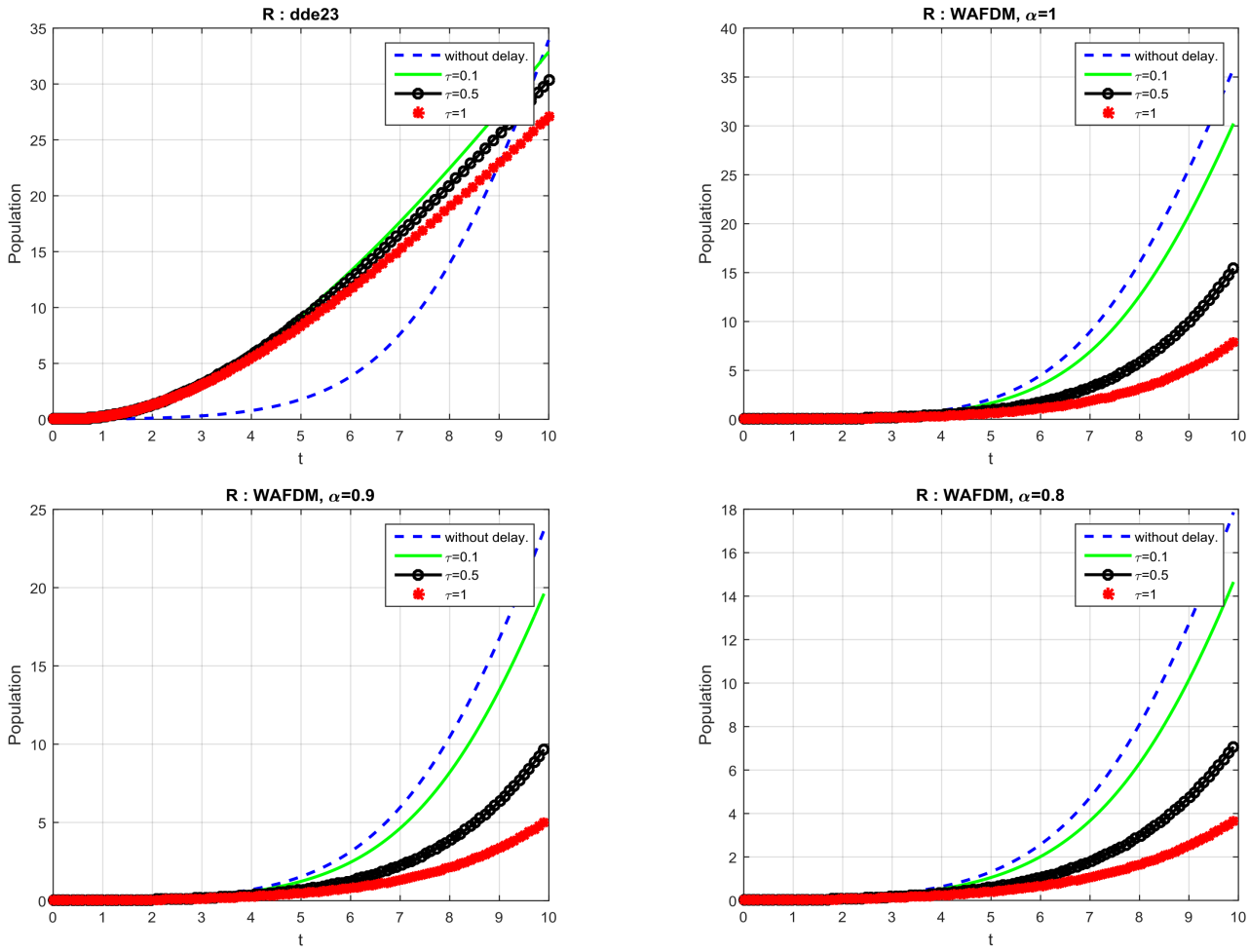

Fig. 6: Obtained values of R using dde23 and WAFDM $(\theta = 0)$ for differnt values of τ and α.

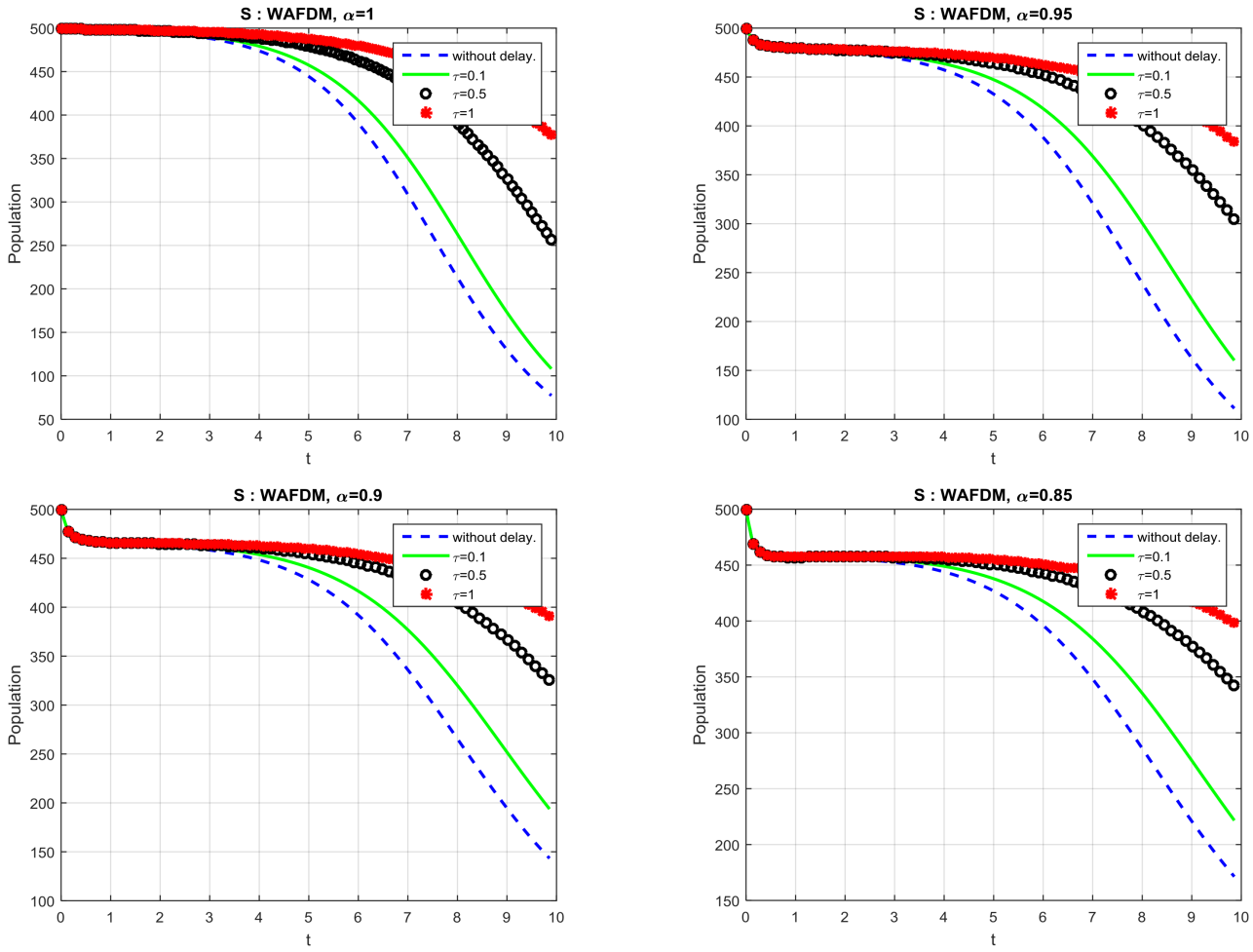

Fig. 7: Obtained values of S using WAFDM $(\theta = 0)$ for differnt values of τ and α.

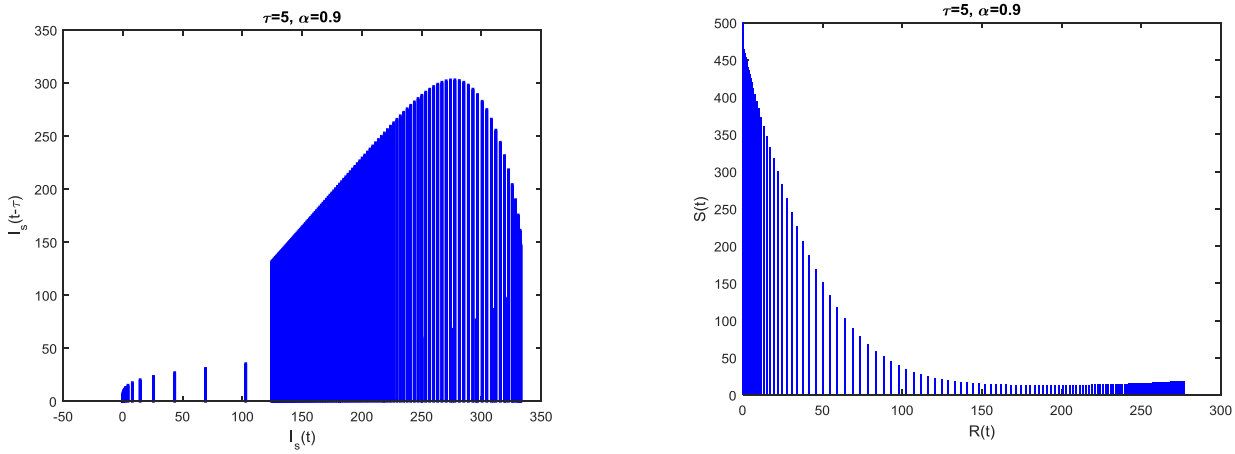

Fig. 8: The relation between I_s and $I_s(t-\tau)$ and the relation between R and S.

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