Mathematical Sciences Letters
An International Journal

http://dx.doi.org/10.12785/msl/040106

P^*-$*$-Connectedness in Ideal Bitopological Spaces

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Received: 5 Mar. 2013, Revised: 20 Jun. 2014, Accepted: 4 Aug. 2014
Published online: 1 Jan. 2015

Abstract: The aim of this paper is to use the concepts of ideal $\mathscr{I}$, bitopological space (X,τ_1,τ_2) and its associated supra topological space (X,τ_{12}) to introduce a new local function, A^*_{12}. The properties of these local function A^*_{12} and some important results related to it have obtained. The local function A^*_{12} is used to generate a family τ^*_{12} which is finer than τ_1,τ_2 and τ_{12}, τ^*_{12} is a supra topology not a topology in general. In addition, a supra topology τ^*_{12} is used to study connectedness in the ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$. Examples have introduced to illustrate the concepts in a friendly way. Finally, the relationship between the current study and the previous one has been given.

Keywords: Bitopological spaces, Ideal, Supra-topological spaces, P^*-Continuous mappings, P^*-separated set, P^*-Connected spaces, P^*-$*$-separated set, P^*-$*$-Connected spaces.
Mathematics Subject Classification (2010): 54A05, 54A10, 54E55.

1 Introduction

In 1963 Kelly [10] was introduced a bitopological space (X,τ_1,τ_2) as a richer structure than topological space. A study of bitopological space is a generalization of the study of general topological space as every bitopological space (X,τ_1,τ_2) can be regarded as a topological space (X,τ) if $\tau_1=\tau_2=\tau$.

The concept of ideals in topological spaces has been introduced and studied by Kuratowski [11] and Vaidyanathaswamy [17]. An ideal is a nonempty collection of subsets closed under heredity and finite additivity. The study of ideal bitopological spaces was initiated by Jafari and Rajesh [6].

As a generalized to topological spaces, Mashhour et al. [13] introduced supra-topological spaces by dropping only the intersection condition. Kandil et al. [9] generated a supra-fuzzy topological space (X,τ_{12}) from the fuzzy bitopological space (X,τ_1,τ_2) and studied some properties of the space (X,τ_1,τ_2) via properties of the space (X,τ_{12}).

The notion of connectedness in bitopological spaces has been studied by Pervin [14], Reily [15] and Swart [16]. In 2014 Mandira Kar and Thakur [12] have been studied the notion of connectedness in ideal bitopological spaces, but the studying of such spaces by using the supra-topological space has not been considered.

In this paper, given a bts (X,τ_1,τ_2) and its associated supra topological space (X,τ_{12}) [13]. Also, let $\mathscr{I}$ be an ideal on a space X, we introduce a new local function, $A^*_{12}: P(X) \longrightarrow P(X), A^*_{12}(A)=\{x\in X: O_x\cap A\notin \mathscr{I} \ \forall \ O_x \in \tau_{12}(x)\}$, where $\tau_{12}=\{U_1\cup U_2 : U_i\in\tau_i, i=1,2\}$ is a supra topology [13] generated by τ_1 and τ_2, (X,τ_{12}) is a supra topological space associate to the bts (X,τ_1,τ_2). The properties of the operator A^*_{12} have obtained. In addition, we show that $A^*_{12}(A)=A^*_1(A)\cap A^*_2(A)$. Moreover, we show that the operator $cl^*_{12}(A)=A\cup A^*_{12}(A)$ is a supra closure operator [8,9] and then induces a supra topology τ^*_{12} which is finer than τ_1,τ_2 and τ_{12}, τ^*_{12} is not a topology in general. Furthermore, a supra topology τ^*_{12} is used to study connectedness in the ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$, since the dealing with one family is easier than the dealing with two families. Also, the class of all supra-topological spaces is wider than the class of topological spaces. So, the study of supra-topological spaces is a generalization of the study of topological spaces. The notions of P^*-$*$-connected spaces, P^*-$*$-separated sets and P^*-$*s$-connected sets in ideal

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bitopological spaces are studied. Some examples are given to illustrate these concepts. Finally, comparisons between the current study and the previous one [2,12,14,15] are presented.

2 Preliminaries

In this section, we collect some needed definitions and theories of the material used in this paper.

Definition 2.1. [4] Let X be a non-empty set. A class τ of subsets of X is called a topology on X iff τ satisfies the following axioms.

1. $X, \phi \in \tau$.
2. An arbitrary union of the members of τ is in τ.
3. The intersection of any two sets in τ is in τ.

The members of τ are then called τ-open sets, or simply open sets. The pair (X,τ) is called a topological space. A subset A of a topological space (X,τ) is called a closed set if its complement $A^{'}$ is an open set. If τ satisfies the conditions 1 and 2 only, then τ is said to be supra-topology on X and the pair (X,τ) is called a supra-topological space [13].

Definition 2.2.[7] A non-empty collection $\mathscr{I}$ of subsets of a set X is called an ideal on X, if it satisfies the following conditions

1. $A \in \mathscr{I}$ and $B \in \mathscr{I} \Rightarrow A \cup B \in \mathscr{I}$,
2. $A \in \mathscr{I}$ and $B \subseteq A \Rightarrow B \in \mathscr{I}$.

Definition 2.3.[7] Let (X,τ) be a topological space and $\mathscr{I}$ be an ideal on X. Then,

$$A^*(\mathscr{I},\tau)\ (orA^*) := \{x \in X : O_x \cap A \notin \mathscr{I}\ \forall O_x\}$$

is called the local function of A with respect to $\mathscr{I}$ and τ, where O_x is an open set containing x.

Theorem 2.1.[7] Let (X,τ) be a topological space and $\mathscr{I}$ be an ideal on X. Then, the operator $cl^* : P(X) \to P(X)$ defined by:

$$cl^*(A) = A \cup A^* \tag{1}$$

satisfies Kuratwski's axioms and induces a topology $\tau^*(\mathscr{I})$ on X given by:

$$\tau^*(\mathscr{I}) = \{A \subseteq X :\ cl^*(A') = A'\}. \tag{2}$$

Proposition 2.1.[7] Let (X,τ) be a topological space and $\mathscr{I}$ be an ideal on X. Then, $\tau \subseteq \tau^*(\mathscr{I})$, i.e., $\tau^*(\mathscr{I})$ is finer than τ.

Lemma 2.1.[5] Let (X,τ,I) be an ideal topological space and $B \subseteq A \subseteq X$. Then, $cl^*_A(B) = cl^*(B) \cap A$.

Definition 2.4.[10] A bitopological space (bts, for short) is a triple (X,τ_1,τ_2), where τ_1,τ_2 are arbitrary topologies for a set X.

Definition 2.5.[14,15] Let (X,τ_1,τ_2) be a bts-space, $A,B \subset X$. Then, A and B are said to be P-separated sets if $\overline{A}^i \cap B = \phi, A \cap \overline{B}^j = \phi, i,j = 1,2, i \neq j$.

Definition 2.6.[14,15] A bts-space (X,τ_1,τ_2) is said to be P-connected space if X can not be expressed as a union of two non-empty disjoint τ_i-open set A and τ_j-open set B. If X can be so expressed we shall write $X = A|B$ and we call this a separation or disconnection.

We call (X,τ_1,τ_2) is P-disconnected space if it is not P-connected.

Definition 2.7.[8] A mapping $cl : P(X) \to P(X)$ is said to be a supra closure operator if it satisfies the following conditions.

1. $cl(\phi) = \phi$.
2. $A \subseteq cl(A)$.
3. $cl(A \cup B) \supseteq cl(A) \cup cl(B)$.
4. $cl(cl(A)) = cl(A)$.

Proposition 2.2.[8,9] For any bts (X,τ_1,τ_2) a mapping $cl_{12} : P(X) \to P(X)$, $cl_{12}(A) = cl^1(A) \cap cl^2(A)$, cl_{12} is a supra closure operator and induces a supra-topology $\tau_{12} = \{A \subseteq X : cl_{12}(A^{'}) = A^{'}\}$ and (X,τ_{12}) is a supra-topological space associated to a bts (X,τ_1,τ_2).

Proposition 2.3.[8] Let (X,τ_1,τ_2) be a bts. The operator $int_{12} : P(X) \longrightarrow P(X)$ defined by, $int_{12}(A) = A^{o1} \cup A^{o2}$, is a supra interior operator such that $\tau_{12} = \{A \subseteq X : int_{12}(A) = A\}$.

Definition 2.8. [1] Let (X,τ_1,τ_2) be a bts. Then, $A \subseteq X$ is called a P-open set if $A = U_1 \cup U_2, U_i \in \tau_i, (i = 1,2)$. The complement of a P-open set in X is a P-closed set in X.

Proposition 2.4.[3,8] Let (X,τ_1,τ_2) be a bts. Then, the family of all P-open sets in X, is a supra-topology. Moreover, $\tau_{12} = \{A \subseteq X : A \text{ is } P\text{-open }\}$.

Proposition 2.5.[8] Let (X,τ_1,τ_2) be a bts and $A \subseteq X$. Then, $x \in cl_{12}(A) \Leftrightarrow \forall\, O_x \in \tau_{12}, O_x \cap A \neq \phi$.

Definition 2.9. [3,8] Let $(X_1,\tau_1,\tau_2),(X_2,\theta_1,\theta_2)$ be two bts's. A function $f : (X_1,\tau_1,\tau_2) \to (X_2,\theta_1,\theta_2)$ is called P^*-continuous if the inverse image of every P-open subset of X_2 is a P-open subset of X_1.

Definition 2.10. [2] Let (X,τ_1,τ_2) be a bts, $A,B \subset X$. Then, A and B are said to be P^*-separated sets if $cl_{12}(A) \cap B = \phi, A \cap cl_{12}(B) = \phi$.

Definition 2.11. [2] A bts (X,τ_1,τ_2) is said to be P^*-connected space if X can not be expressed as the union of two non-empty disjoint P-open sets A and B. If X can be so expressed we shall write $X = A|B$ and we call this a P^*-disconnection.

We call (X,τ_1,τ_2,R) is P^*-disconnected space if it is not P^*-connected.

Definition 2.12. [12] An ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$ is called P-$*$-connected if X cannot be written as a union of a non-empty disjoint τ_i-open set and τ_j^*-open set , $i,j = 1,2, i \neq j$.

Definition 2.13. [12] Let $(X,\tau_1,\tau_2,\mathscr{I})$ be an ideal bitopological space, $A,B \subset X$. Then, A and B are said to be P-$*$-separated sets if $\tau_i^* cl(A) \cap B = \phi, A \cap \tau_j cl(B) = \phi$.

Definition 2.14. [12] A subset A of an ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$ is called P-$*s$-connected if A is not the union of two P-$*$-separated sets in $(X,\tau_1,\tau_2,\mathscr{I})$.

3 Bitopological spaces and the operator A^*_{12}

In this section, we consider (X,τ_1,τ_2) as a bts, (X,τ_{12}) its associated supra topological space and $\mathscr{I}$ be an ideal on X and introduce a new local function, A^*_{12}. The properties of the operator $*_{12}$ have obtained. By making use of this function, we generate a family τ^*_{12} which is finer than τ_1,τ_2 and τ_{12}, τ^*_{12} is a supra topology not a topology in general.

Definition 3.1. Let (X,τ_1,τ_2) be a bts, $\mathscr{I}$ be an ideal on a space X and $A\subseteq X$. Then, the operator $A^*_{12}:P(X)\to P(X)$ given by $A^*_{12}=\{x\in X: O_x\cap A\notin\mathscr{I}\ \forall\ O_x\in\tau_{12}(x)\}$ is a local function associated with $\mathscr{I}$.

Proposition 3.1. Let (X,τ_1,τ_2) be a bts. Then,

(i)If $\mathscr{I}$ is any ideal on X, then A^*_{12} is an increasing function,i.e. $A\subseteq B(\subseteq X)\Rightarrow A^*_{12}\subseteq B^*_{12}$.
(ii)If $\mathscr{I}_1$ and $\mathscr{I}_2$ are two ideals on X with $\mathscr{I}_1\subseteq\mathscr{I}_2$, then $A^{*\mathscr{I}_1}_{12}(A)\subseteq A^{*\mathscr{I}_2}_{12}(A)\ \forall A\subseteq X$.
(iii)For any ideal $\mathscr{I}$ on X and $A\subseteq X$, if $A\in\mathscr{I}$, then $A^*_{12}=\phi$.

Proof. It follows from the definition of the local function A^*_{12}.

Proposition 3.2. Let (X,τ_1,τ_2) be a bts and $\mathscr{I}$ be an ideal on X. Then, for all $A,B\subseteq X$

(i)$(A\cup B)^*_{12}\supseteq A^*_{12}\cup B^*_{12}$,
(ii)$(A^*_{12})^*_{12}\subseteq A^*_{12}=cl_{12}(A^*_{12})\subseteq cl_{12}(A)$.

Proof.
(i) Since $A,B\subseteq A\cup B$, by Proposition 3.1 (i), $A^*_{12}\subseteq(A\cup B)^*_{12}$ and $B^*_{12}\subseteq(A\cup B)^*_{12}$. It follows that $A^*_{12}\cup B^*_{12}\subseteq(A\cup B)^*_{12}$.
(ii) To prove that $(A^*_{12})^*_{12}\subseteq A^*_{12}$ let $x\in(A^*_{12})^*_{12}$. Then, $O_x\cap A^*_{12}\notin\mathscr{I},\forall O_x\in\tau_{12}(x)$. So, $O_x\cap A^*_{12}\neq\phi$ and consequently there exists $y\in O_x\cap A^*_{12}$. Then, $y\in O_x$ and $y\in A^*_{12}$. Thus, $O_y\cap A\notin\mathscr{I}$ for all $O_y\in\tau_{12}(y)$. Since, $y\in O_x$, $O_x\cap A\notin\mathscr{I}$, so $x\in A^*_{12}$ and therefore $(A^*_{12})^*_{12}\subseteq A^*_{12}$.

Clearly, $A^*_{12}\subseteq cl_{12}(A^*_{12})$, so, we prove that $cl_{12}(A^*_{12})\subseteq A^*_{12}$. Thus, let $x\in cl_{12}(A^*_{12})$. Then, $\forall O_x\in\tau_{12}(x)$; $O_x\cap A^*_{12}\neq\phi$. So, there exists $y\in O_x\cap A^*_{12}$. It follows that $y\in O_x$ and $y\in A^*_{12}$. So, for all $O_y\in\tau_{12}(y)$, $O_y\cap A\notin\mathscr{I}$. Hence, $O_x\cap A\notin\mathscr{I}$ and this yields $x\in A^*_{12}$. Finally, we have $A^*_{12}\supseteq cl_{12}(A^*_{12})$ and consequently $A^*_{12}=cl_{12}(A^*_{12})$. Now to complete the proof of part (ii), we show that $A^*_{12}\subseteq cl_{12}(A)$. So, let $x\notin cl_{12}(A)$. Then, there exists $O_x\in\tau_{12}(x)$ such that $O_x\cap A=\phi$, then $x\notin A^*_{12}$ and consequently $A^*_{12}\subseteq cl_{12}(A)$.

Remark 3.1. Let (X,τ_1,τ_2) be a bts and $\mathscr{I}$ be an ideal on X. Let (X,τ^*_1,τ^*_2) be a bts induced by $\mathscr{I}$, where

$$\tau^*_1=\{A\subseteq X: cl^*_1(X\backslash A)=X\backslash A\},$$

$$\tau^*_2=\{A\subseteq X: cl^*_2(X\backslash A)=X\backslash A\},$$

$$cl^*_i(A)=A\cup A^*_i\ (i=1,2)\text{ and}$$

$$A^*_i=\{x\in X: O_x\cap A\notin\mathscr{I}\ \forall\ O_x\in\tau_i(x)\}.$$

Also, note that $\tau_i\subseteq\tau^*_i$.

Lemma 3.1. Let (X,τ_1,τ_2) be a bts and $\mathscr{I}$ be an ideal on X. Let $A^*_{12}:P(X)\to P(X)$ be a local function. Then,

$$A^*_{12}=A^*_1\cap A^*_2\ \forall\ ,A\subseteq X.$$

Proof.
Let $x\notin A^*_1\cap A^*_2$. Then, $x\notin A^*_1$ or $x\notin A^*_2$. If $x\notin A^*_1$, then there exists $O_x\in\tau_1\subseteq\tau_{12}$ such that $O_x\cap A\in\mathscr{I}$. Hence, $x\notin A^*_{12}$. Similarly, if $x\notin A^*_2$, then there exists $O_x\in\tau_2\subseteq\tau_{12}$ such that $O_x\cap A\in\mathscr{I}$. Hence, $x\notin A^*_{12}$. So, in both cases, $A^*_{12}\subseteq A^*_1\cap A^*_2$. On the other hand, if $x\notin A^*_{12}$, then there exists $O_x\in\tau_{12}(x)$ such that $O_x\cap A\in\mathscr{I}$. Now, $O_x\in\tau_{12}(x)\Rightarrow O_x=O^1_x\cup O^2_x$ $(O^i_x\in\tau_i,i=1,2)\Rightarrow(O^1_x\cup O^2_x)\cap A\in\mathscr{I}\Rightarrow O^i_x\cap A\in\mathscr{I}$ (since $\mathscr{I}$ is an ideal). Now, $x\in O_x\Rightarrow x\in O^1_x$ or $x\in O^2_x\Rightarrow O^1_x\cap A\in\mathscr{I}$ or $O^2_x\cap A\in\mathscr{I}\Rightarrow x\notin A^*_1$ or $x\notin A^*_2\Rightarrow x\notin A^*_1\cap A^*_2$. Hence, the result.

The following theorem gives the properties of the local function A^*_{12} in terms of the local functions A^*_1 and A^*_2.

Theorem 3.1. Let (X,τ_1,τ_2) be a bts and $\mathscr{I}$ be an ideal on X. Then, the local function $A^*_{12}=A^*_1\cap A^*_2$ satisfies the following properties.

(i)$\phi^*_{12}=\phi$,
(ii)$A\subseteq B\Rightarrow A^*_{12}\subseteq B^*_{12}$,
(iii)$A^*_{12}\cup B^*_{12}\subseteq(A\cup B)^*_{12}$
(iv)$(A^*_{12})^*_{12}\subseteq A^*_{12}=cl_{12}(A^*_{12})\subseteq cl_{12}(A)$.

Proof.
(i)$\phi^*_{12}=\phi^*_1\cap\phi^*_2=\phi$.
(ii) Let $A\subseteq B$. Then, $A^*_{12}=A^*_1\cap A^*_2\subseteq B^*_1\cap B^*_2=B^*_{12}$ (by using the properties of A^*_1,A^*_2).
(iii) Follows from (ii).
(iv) $(A^*_{12})^*_{12}=(A^*_{12})^*_1\cap(A^*_{12})^*_2$
$=(A^*_1\cap A^*_2)^*_1\cap(A^*_1\cap A^*_2)^*_2$
$\subseteq(A^*_1)^*_1\cap(A^*_2)^*_1\cap(A^*_1)^*_2\cap(A^*_2)^*_2$
$\subseteq A^*_1\cap(A^*_2)^*_1\cap(A^*_1)^*_2\cap A^*_2.$
$\subseteq A^*_1\cap A^*_2=A^*_{12}.$

Hence, $(A^*_{12})^*_{12}\subseteq A^*_{12}$.
Clearly, $A^*_{12}\subseteq cl_{12}(A^*_{12})$.
On the other hand, $cl_{12}(A^*_{12})=\overline{A^*_{12}}^1\cap\overline{A^*_{12}}^2$
$=\overline{A^*_1\cap A^*_2}^1\cap\overline{A^*_1\cap A^*_2}^2$
$\subseteq\overline{A^*_1}^1\cap\overline{A^*_2}^1\cap\overline{A^*_1}^2\cap\overline{A^*_2}^2$
$=A^*_1\cap\overline{A^*_2}^1\cap\overline{A^*_1}^2\cap A^*_2$(since, $\overline{A^*_i}^i=A^*_i$, i=1,2)
$=A^*_1\cap A^*_2=A^*_{12}$. Hence, $A^*_{12}=cl_{12}(A^*_{12})$.

Finally, we show that $A^*_{12}\subseteq cl_{12}(A)$. Since, $A^*_{12}=A^*_1\cap A^*_2\subseteq\overline{A}^1\cap\overline{A}^2=cl_{12}(A)$ (since $A^*_i\subseteq\overline{A}^i$, i=1,2).

If $\mathscr{I}$ is an ideal on a space (X,τ_1,τ_2). Define a mapping $cl^*_{12} : P(X) \to P(X)$ by $cl^*_{12}(A) = A \cup A^*_{12}\ \forall A \subseteq X$. Then, we have the following theorem.

Theorem 3.2. The above map cl^*_{12} is a supra closure operator which induces the supra topology $\tau^*_{12} = \{A \subseteq X : cl^*_{12}(X\backslash A) = X\backslash A\}$.

Proof.
Let $cl^*_{12}(A) = A \cup A^*_{12}$. Then,
(SC1) $cl^*_{12}(\phi) = \phi \cup \phi^*_{12} = \phi$.
(SC2) Clearly, $A \subseteq cl^*_{12}(A)$.
Note that if $A \subseteq B$, then $cl^*_{12}(A) = A \cup A^*_{12} \subseteq B \cup B^*_{12} = cl^*_{12}(B)$, i.e. $A \subseteq B \Rightarrow cl^*_{12}(A) \subseteq cl^*_{12}(B)$.
(SC3) $cl^*_{12}(A) \cup cl^*_{12}(B) \subseteq cl^*_{12}(A \cup B)$ (follows from the above note).
(SC4) The proof follows by using the properties of $*_1$, $*_2$ and by using (SC2). Hence, cl^*_{12} is a supra closure operator.

It is easy to show that the family

$$\tau^*_{12} = \{A \subseteq X : cl^*_{12}(X\backslash A) = X\backslash A\},$$

is a supra topology on X it is not a topology in general.

Definition 3.2. Corresponding to an ideal $\mathscr{I}$ on a bts (X,τ_1,τ_2) there exists a unique supra topology τ^*_{12}(say) on X given by

$$\tau^*_{12} = \{U \subseteq X : cl^*_{12}(X\backslash U) = X\backslash U\},$$

which is finer than τ_{12} and $cl^*_{12}(A) = A \cup A^*_{12} = \tau^*_{12} - cl(A)\ \forall\, A \subseteq X$.

Theorem 3.3. Let (X,τ_1,τ_2) be a bts, $\mathscr{I}$ be an ideal on X and $A \subseteq X$. Then,

$$cl^*_{12}(A) = A \cup A^*_{12} = cl^*_1(A) \cap cl^*_2(A).$$

Proof.
Since, $cl^*_{12}(A) = A \cup A^*_{12}$, then
$cl^*_{12}(A) = A \cup (A^*_1 \cap A^*_2)$,
$= (A \cup A^*_1) \cap (A \cup A^*_2)$,
$= cl^*_1(A) \cap cl^*_2(A)$.

Note that Theorem 3.3 means that we can established the same supra topology from a bts (X,τ_1,τ_2) by using two equivalent methods. The first follows from the local function $*_{12}$ and the other by using the closure operators cl^*_1, cl^*_2 induced by the local functions $*_1$, $*_2$.

Theorem 3.4. Let (X,τ_1,τ_2) be a bts, $\mathscr{I}$ be an ideal on X. Let $(X,\tau_1{}^*,\tau_2{}^*)$ be a bts induced by $\mathscr{I}$ and the local functions $*_1$, $*_2$. Then,

$$\tau^*_{12} = \{U_1 \cup U_2 : U_i \in \tau^*_i, i = 1,2\}.$$

Proof.
Let $A \in \tau^*{}_{12}$. Then, $cl^*_{12}(X\backslash A) = X\backslash A$,
$\Rightarrow X\backslash A = cl^*_1(X\backslash A) \cap cl^*_2(X\backslash A)$,
$\Rightarrow X\backslash A = X\backslash int^*_1(A) \cap X\backslash int^*_2(A)$,
$\Rightarrow A = int^*_1(A) \cup int^*_2(A) = U_1 \cup U_2,\ U_i \in \tau^*_i$.

Conversely, let $A = U_1 \cup U_2$, $U_i \in \tau^*_i$. Then, $cl^*_{12}(X\backslash A) = cl^*_{12}(X\backslash U_1 \cap X\backslash U_2) = cl^*_1(X\backslash U_1 \cap X\backslash U_2) \cap cl^*_2(X\backslash U_1 \cap X\backslash U_2) \subseteq cl^*_1(X\backslash U_1) \cap cl^*_1(X\backslash U_2) \cap cl^*_2(X\backslash U_1) \cap cl^*_2(X\backslash U_2) = X\backslash U_1 \cap cl^*_1(X\backslash U_2) \cap cl^*_2(X\backslash U_1) \cap X\backslash U_2 = X\backslash U_1 \cap X\backslash U_2 = X\backslash A$. But, $X\backslash A \subseteq cl^*_{12}(X\backslash A)$. Hence, $cl^*_{12}(X\backslash A) = X\backslash A$ and consequently $A \in \tau^*_{12}$.

Remark 3.2. Let (X,τ_1,τ_2) be a bts, $\mathscr{I}$ be an ideal on X. Then,

(1)$(A \cup B)^*_{12} \neq (A)^*_{12} \cup (B)^*_{12}$ in general.
(2)$cl^*_{12}(A \cup B) \neq cl^*_{12}(A) \cup cl^*_{12}(B)$ in general.
(3)τ^*_{12} which induced by cl^*_{12} may be not a topology in general but it is a supra topology finer than τ_1, τ_2 and τ_{12}.
(4)$\tau_1 \cup \tau_2 \subseteq \tau_{12} \subseteq \tau^*_{12}$.

Example 3.1. Let $X = \{1,2,3,4\}$, $\mathscr{I} = \{\phi,\{2\}\}$, τ_1 and τ_2 be two topologies on X such that $\tau_1 = \{\phi, X, \{2,4\}\}$ and $\tau_2 = \{\phi, X, \{1,2,3\}\}$, then $\tau_{12} = \{\phi, X, \{2,4\}, \{1,2,3\}\}$ is a supra topology, since $\{2,4\} \cap \{1,2,3\} = \{2\} \notin \tau_{12}$.Now, let $A = \{4\}$, $B = \{3\}$, since, $\{4\}^*_{12} = \{4\}$, $\{3\}^*_{12} = \{1,3\}$ and $\{3,4\}^*_{12} = X$, then $(A \cup B)^*_{12} \neq (A)^*_{12} \cup (B)^*_{12}$ and, $cl^*_{12}(A \cup B) = X, cl^*_{12}(A) = \{4\}, cl^*_{12}(B) = \{1,3\}$. Therefore, $cl^*_{12}(A \cup B) \neq cl^*_{12}(A) \cup cl^*_{12}(B)$. Also, $\tau_1,\tau_2 \subseteq \tau_{12} \subseteq \tau^*_{12} = \{\phi, X, \{\{4\}, \{2,4\}, \{1,2,3\}, \{1,3,4\}\}$ and τ^*_{12} is a supra topology as $\{1,2,3\} \cap \{1,3,4\} = \{1,3\} \notin \tau^*_{12}$.

Definition 3.3. Let (X,τ_1,τ_2) be a bts, $\mathscr{I}$ be an ideal on X. Then, $A \subseteq X$ is called a P^*-open set if $A = U_1 \cup U_2, U_i \in \tau^*_i, (i = 1,2)(or\ A \in \tau^*_{12})$. The complement of a P^*-open set in X is a P^*-closed set in X.

Corollary 3.1. Let (X,τ_1,τ_2) be a bts, $\mathscr{I}$ be an ideal on X. Then, the family of all P^*-open sets in X, is a supra-topology. Moreover, $\tau^*_{12} = \{A \subseteq X : A \text{ is } P^*\text{-open}\}$.

4 P^*-$*$-Connectedness in ideal bitopological spaces

The aim of this section is to introduce the notion of P^*-$*$-connected spaces, P^*-$*$-separated sets, P^*-$*s$-connected sets in ideal bitopological spaces. Some examples are given to illustrate the concepts. Furthermore, the relationship between the current notion of connectedness and the previous one in [2,12,14,15] is obtained.

Definition 4.1. An ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$ is called P^*-$*$-connected if X cannot be written as union of a nonempty disjoint P-open set and P^*-open set.

Example 3.1, shows that, $(X,\tau_1,\tau_2,\mathscr{I})$ is a P^*-$*$-connected.

Remark 4.1. Every P^*-$*$-connected is P^*-connected.

Example 3.1 shows that the converse of Remark 4.1 is not true, i.e., $(X,\tau_1,\tau_2,\mathscr{I})$ is P^*-connected, but not P^*-$*$-connected (as $\exists$ a non-empty disjoint P-open set $A = \{1,2,3\}$ and $\exists$ P^*-open set $B = \{4\}$ such that $X = A \cup B$.

Remark 4.2. Every P^*-$*$-connected is P-connected.

Example 3.1 shows that the converse of Remark 4.2 is not true, i.e., $(X,\tau_1,\tau_2,\mathscr{I})$ is P-connected, but not P^*-$*$-connected.

Definition 4.2. Let $(X,\tau_1,\tau_2,\mathscr{I})$ be an ideal bitopological space, $A,B\subset X$. Then, A and B are said to be P^*-$*$-separated sets if $cl^*_{12}(A)\cap B=\phi, A\cap cl_{12}(B)=\phi$.

Remark 4.3. Every P-$*$-separated sets are P^*-$*$-separated sets.

Example 3.1 shows that the converse of Remark 4.3 is not true, as $A=\{1,2,3\}, B=\{4\}$ are P^*-$*$-separated sets, but not P-$*$-separated sets as $(B\cap\tau_2^*cl(A)=\{4\}\cap X=\{4\}\neq\phi)$.

Remark 4.4. Every P^*-separated sets are P^*-$*$-separated sets.

Example 3.1 shows that the converse of Remark 4.4 is not true, as $A=\{1,2,3\}, B=\{4\}$ are P^*-$*$-separated sets, but not P^*-separated sets since, $\tau_{12}cl(A)\cap B=X\cap\{4\}=\{4\}\neq\phi$.

On account of Remarks 4.3 and 4.4 and [2,12] we have the following proposition which studies the relationship between the current definitions and the previous definitions [2,12,14,15].

Proposition 4.1. For a bto-space (X,τ_1,τ_2,R), we have the following implications

$$\begin{array}{ccc} P\text{-}separated\ sets & \Rightarrow & P^*\text{-}separated\ sets. \\ \Downarrow & & \Downarrow \\ P\text{-}*\text{-}separated\ sets & \Rightarrow & P^*\text{-}*\text{-}separated\ sets. \end{array}$$

Theorem 4.1. Let $(X,\tau_1,\tau_2,\mathscr{I})$ be an ideal bitopological space and $A\subseteq B\subseteq Y\subseteq X$. Then, A and B are P^*-$*$-separated sets in $Y \Leftrightarrow A,B$ are P^*-$*$-separated sets in X.

Proof. It follows from Lemma 2.1 that $cl^*_{12}(A)\cap B=A\cap cl_{12}(B)=\phi$.

Theorem 4.2. Let $f:(X,\tau_1,\tau_2,\mathscr{I})\to(Y,\eta_1,\eta_2)$ be a P^*-continuous and surjective function. If X is a P^*-$*$-connected space, then (Y,η_1,η_2,R^*) is P^*-connected space.

Proof. It is known that P^*-connectedness space is preserved by P^*-continuous and surjective function [2]. Also, every P^*-$*$-connected space is P^*-connected (see Remark 4.1). Hence, the proof has done.

Definition 4.3. A subset A of an ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$ is called P^*-$*s$-connected if A is not the union of two P^*-$*$-separated sets in $(X,\tau_1,\tau_2,\mathscr{I})$.

Remark 4.5. Every P^*-$*s$-connected set is P-$*s$-connected set.

Example 3.1 shows that the converse of Remark 4.5 is not true, as $A=\{1,3,4\}$ is P-$*s$-connected set, but not P^*-$*s$-connected as, $\exists\, B=\{4\}, C=\{1,3\}$ which are P^*-$*$-separated sets and whose union is A.

Theorem 4.3. Let $(X,\tau_1,\tau_2,\mathscr{I})$ be an ideal bitopological space. If A is a P^*-$*s$-connected set of X and H,G are P^*-$*$-separated sets of X with $A\subseteq H\cup G$, then either $A\subseteq H$ or $A\subseteq G$.

Proof.
Let $A\subseteq H\cup G$. Since, $A=(A\cap H)\cup(A\cap G)$, then $(A\cap G)\cap cl^*_{12}(A\cap H)\subseteq G\cap cl^*_{12}(H)=\phi$. By similar reasoning, we have $(A\cap H)\cap cl_{12}(A\cap G)\subseteq H\cap cl_{12}(G)=\phi$. Suppose that $A\cap H$ and $A\cap G$ are nonempty. Then, A is not P^*-$*s$-connected. This is a contradiction. Thus, either $A\cap H=\phi$ or $A\cap G=\phi$. This implies that either $A\subseteq H$ or $A\subseteq G$.

Theorem 4.4. If A is a P^*-$*s$-connected set of an ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$ and $A\subseteq B\subseteq cl^*_{12}(A)$, then B is P^*-$*s$-connected.

Proof.
Suppose B is not P^*-$*s$-connected. There exist P^*-$*$-separated sets H and G of X such that $B=H\cup G$. This implies that H and G are nonempty and $cl^*_{12}(H)\cap G=H\cap cl_{12}(G)=\phi$. By Theorem 4.3, we have either $A\subseteq H$ or $A\subseteq G$. Suppose that $A\subseteq H$. Then, $cl^*_{12}(A)\subseteq cl^*_{12}(H)$ and $G\cap cl^*_{12}(A)=\phi$ This implies that $G\subseteq B\subseteq cl^*_{12}(A)$ and $G=cl^*_{12}(A)\cap G=\phi$. Thus, G is an empty set for if G is nonempty, this is a contradiction. Suppose that $A\subseteq G$. By similar way, it follows that H is empty. This is a contradiction. Hence, B is P^*-$*s$-connected.

Corollary 4.1. If A is a P^*-$*s$-connected set in an ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$, then $cl^*_{12}(A)$ is P^*-$*s$-connected ordered.

Theorem 4.5. If $\{M_i : i\in I\}$ is a nonempty family of P^*-$*s$-connected sets of an ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$ with $\cap_{i\in I}M_i\neq\phi$. Then, $\cup_{i\in I}M_i$ is P^*-$*s$-connected.

Proof.
Suppose that $\cup_{i\in I}M_i$ is not P^*-$*s$-connected. Then, we have $\cup_{i\in I}M_i=H\cup G$, where H and G are P^*-$*$-separated sets in X. Since $\cap_{i\in I}M_i\neq\phi$ we have a point x in $\cap_{i\in I}M_i$. Since $x\in\cup_{i\in I}M_i$, either $x\in H$ or $x\in G$. Suppose that $x\in H$. Since $x\in M_i$ for each $i\in N$, then M_i and H intersect for each $i\in I$. By Theorem 4.3, $M_i\subseteq H$ or $M_i\subseteq G$. Since H and G are disjoint, $M_i\subseteq H\ \forall i\in I$ and hence $\cup_{i\in I}M_i\subseteq H$. This implies that G is empty. This is a contradiction. Suppose that $x\in G$. By similar way, we have that H is empty. This is a contradiction. Thus, $\cup_{i\in I}M_i$ is P^*-$*s$-connected.

References

[1] B. Dvalishvili, Bitopological Spaces Theory, Relations with Generalized Algebraic Structures and Applications, Elsevier (2005).

[2] S. A. El-Sheikh, Dimension Theory of Bitopological Spaces, Master Thesis, Ain Shams University, Cairo, Egypt (1987).

[3] S. A. El-Sheikh, A new approach to fuzzy bitopological Spaces, Inform. Sci. 137 (2011) 283−301.

[4] R. Engelking, General Topology, Warszwawa (1977).

[5] E. Hatir, A. Keskin, T. Noiri, A note on strong β-$\mathscr{I}$-sets and strongly β-$\mathscr{I}$-continuous functions. Acta Math. Hungar. 108 (1−2) (2005) 87−94.

[6] S. Jafari. and N. Rajesh, On qI-open sets in ideal bitopological spaces, University of Bacau, Faculty of Sciences, Scientific Studies and Research, Series Mathematics and Informatics. 20(2) (2010) 29–38.
[7] D. Jankovic, T.R. Hamlet, New topologies from old via ideals, The American Mathematical Monthly 97 (1990) 295−310.
[8] A. Kandil, S. A. El-Sheikh and H. Fawaz, On bitopological spaces, first Intern. conf. on Mathematics and statistics, Assuit University 3 (1990) 73−96.
[9] A. Kandil, A. Nouh and S. A. El-Sheikh, On fuzzy bitopological Spaces, Fuzzy sets and systems 74 (1995) 353−363.
[10] J. C. Kelly, Bitopological spaces, Proc. London Math. Soc. 13 (1963) 71−89.
[11] K. Kuratowski, Topology Vol. I , Academic Press, New York (1966).
[12] Mandira Kar, S. S. Thakur, Connectedness in ideal bitopological spaces, Int. Journal of Engineering Research and Applications 4(1−1) (2014) 165−167.
[13] A. S. Mashhour, A. A. Allam, A. A. Mahmoud and F. H. Khedr, On supratopological space, Indian J. Pure Appl. Math. 4(4) (1983) 502−510.
[14] W. J. Pervin, Connectedness in bitopological spaces, Indog Math. J. 29 (1967), 369−372.
[15] I. L. Reilly, On Paqirwise Connected bitopological spaces, Kyungpook Math. J. 11 (1) (1971) 25−28.
[16] J. Swart, Total disconectedness in bitopological spaces and product bitopological spaces, Indagationes Mathematicae (Proceedings) 74 (1971) 135–145.
[17] R. Vaidynathaswamy, The localization theory in set topology, Proc. Ind. Acad. of Sci. 20 (1945) 515–61

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