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On $(m,h_1,h_2)-$Convex Stochastic Processes using Fractional Integral Operator.

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Abstract: We consider and study a new class of convex stochastic processes, called $(m,h_1,h_2)-$convex stochastic processes. Some Ostrowski inequality for this kind of generalized convex stochastic processes are derived, using the fractional integral operator.

Keywords: $(m,h_1,h_2)-$convex stochastic processes, Ostrowski Inequality

1 Introduction

The Ostrowski's inequality was introduced by Alexander Ostrowski in [30], and with the passing of the years, generalizations on the same, involving derivatives of the function under study, have taken place.

Ostrowski's Inequality. Let $f: I\subset[0,+\infty)\to\mathbb{R}$ be a differentiable function on $int(I)$, such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'(x)|\le M$ for all $x\in[a,b]$, then the inequality:

$$\left|f(x)-\frac{1}{(b-a)}\int_a^b f(x)\,dx\right| \le M(b-a)\left[\frac{1}{4}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right] \quad (1)$$

holds for all $x\in[a,b]$.

Recently, several generalizations of the Ostrowski integral inequality are considered by many authors; for instance covering the following concepts: functions of bounded variation, Lipschitzian, monotonic, absolutely continuous and n times differentiable mappings with error estimates with some special means together with some numerical quadrature rules. For recent results and generalizations concerning Ostrowski's inequality, we refer the reader to the recent papers [1,2,3,40,45]. The convex functions play a significant role in many fields, for example, in biological system, economy, optimization and so on [12,36]. And many important inequalities are established for these class of functions. Also the evolution of the concept of convexity has had a great impact in the community of investigators. In recent years, for example, generalized concepts such as $s-$convexity (see [7]), $h-$convexity (see [37,46]) , $m-$convexity (see [4,11]), $MT-$convexity (see [28]]) and others , as well as combinations of these new concepts have been introduced. The study on convex stochastic processes began in 1974 when B. Nagy in [21], applied a characterization of measurable stochastic processes to solving a generalization of the (additive) Cauchy functional equation.

In 1980, Nikodem [24] introduced the convex stochastic processes in his article.

Later in 1995, A. Skrowronski in [43] presented some further results on convex stochastic processes. In 2014 Maden et. al. [18] introduced the convex stochastic processes in the first sense and proved Hermite-Hadamard type inequalities to these processes. In the year 2014, E. Set et. al. in [39] investigated Hermite-Hadamard type inequalities for stochastic processes in the second sense.

For other results related to stochastic processes see [5], [6], [10], [19] where further references are given.

In [37], Sarikaya M.Z, Filiz H. and Kiris M.E. established some inequalities for differentiable mappings which are connected with Ostrowski type inequality by used the

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Riemann-Liouville fractional integrals. Also, Cristescu G. in [9] wrote about weighted inequalities for Katugampola fractional integral. In the same way, Agarwal R.P., Luo M-J. and Raina R.K. in [1], showed some inequalities associated with a generalized fractional integral based in definition proposed by Raina R.K. in [34], and called fractional integral operator. In all of these papers, the concept of convexity plays a relevant role, and due to its evolution in recent years, other authors have written on it.

2 Preliminaries

This section contains definitions and properties of generalized convexity and fractional integral operators. Recall that a real-valued function f defined in a real interval J is said to be convex if for all $x,y\in J$ and for any $\in[0,1]$ the inequality

$$f(tx+(1-t)y)\leq tf(x)+(1-t)y \tag{2}$$

holds. If inequality (2) is strict when we say that f is strictly convex, and if inequality (2) is reversed the function f is said to be concave.

In [2], Alomari M. , Darus M. and Dragomir S.S. introduced the following generalized concept.

Definition 1. *Let $0<s\leq 1$. The function $f:[0,\infty)\to\mathbb{R}$ is called a $s-$convex function in second sense if*

$$f(tx+(1-t)y)\leq t^s f(x)+(1-t)^s f(y) \tag{3}$$

holds for all $x,y\in[0,\infty)$ and $t\in[0,1]$.

Dragomir S.S and Agarwal R.P, in [9], about Hadamard inequalities, introduced the following definition of $P-$convex functions.

Definition 2. *We say that a function $f:I\to\mathbb{R}$ is a $P-$convex on I or $f\in P(I)$ if f is non negative and for all $x,y\in I$ and $t\in[0,1]$ we have*

$$f(tx+(1-t)y)\leq f(x)+f(y) \tag{4}$$

Park J. in [15] introduced the concept of $MT-$convex function.

Definition 3. *A function $f:I\subset\mathbb{R}\to\mathbb{R}$ is said to be $MT-$convex function on I , if it is non negative and for all $x,y\in I$ and $t\in(0,1)$ satisfies the following inequality*

$$f(tx+(1-t)y)\leq\frac{\sqrt{t}}{2\sqrt{1-t}}f(x)+\frac{\sqrt{1-t}}{2\sqrt{t}}f(y). \tag{5}$$

Sanja Varošanec, in [23], introduced the convex functions

Definition 4. *Let $h:J\to\mathbb{R}$ be a non negative function, $h\not\equiv 0$, with $(0,1)\subset J$ and J is an interval of $\mathbb{R}$. A function $f:I\subset\mathbb{R}\to\mathbb{R}$, where I is an interval of $\mathbb{R}$, is said to be $h-$convex function if for all $x,y\in I$ and $t\in[0,1]$ the following inequality holds*

$$f(tx+(1-t)y)\leq h(t)f(x)+h(1-t)f(y). \tag{6}$$

G. Toader introduced in [44] the concept of $m-$convex function.

Definition 5. *For $f:[0,b]\to\mathbb{R}, b>0$ and $m\in(0,1]$, if*

$$f(tx+m(1-t)y)\leq tf(x)+m(1-t)f(y) \tag{7}$$

is valid for all $x,y\in[0,b]$ and $t\in[0,1]$, then we say that f is an $m-$convex function.

In [20], Shi D-P, Xi B-Y and Qi F., introduced the following definition. (See also [16]).

Definition 6.*Let $h_1,h_2:[0,1]\to\mathbb{R}$ and $m\in(0,1]$. A function $f:[0,\infty)\to\mathbb{R}$ is said to be $(m,h_1,h_2)-$convex function if the inequality*

$$f(tx+m(1-t)y)\leq h_1(t)f(x)+mh_2(t)(y)$$

holds for all $x,y\in I$ and $t\in[0,1]$.

In this paper we propose the generalization of convexity of this kind for stochastic processes.

Definition 7. *Let (Ω,F,P) be an arbitrary probability space. A function $X:\Omega\to\mathbb{R}$ is called a random variable if it is F-measurable. Let (Ω,F,P) be an arbitrary probability space and let $T\subset\mathbb{R}$ be time. A collection of random variable $X(t,w), t\in T$ with values in $\mathbb{R}$ is called a stochastic processes.*

1. *If $X(t,w)$ takes values in $S=\mathbb{R}^d$ if is called vector-valued stochastic process.*
2. *If the time T can be a discrete subset of $\mathbb{R}$, then $X(t,w)$ is called a discrete time stochastic process.*
3. *If the time T is an interval, $\mathbb{R}^+$ or $\mathbb{R}$, it is called a stochastic process with continuous time*

Throughout the paper we restrict our attention stochastic process with continuous time, i.e, index set $T=[0,+\infty)$.

Definition 8. *Let (Ω,A,P) be a probability space and $T\subset\mathbb{R}$ be an interval. We say that a stochastic process $X:T\times\Omega\to\mathbb{R}$ if*

1. *Convex if*

$$X(\lambda u+(1-\lambda)v,\cdot)\leq\lambda X(u,\cdot)+(1-\lambda)X(v,\cdot) \tag{8}$$

for all $u,v\in T$ and $\lambda\in[0,1]$.
This class of stochastic process are denoted by C.

2. $m-$convex if

$$X(tu+m(1-t)v,\cdot)\leq tX(u,\cdot)+m(1-t)X(v,\cdot) \quad (9)$$

for all $u,v\in T$ *and* $t\in[0,1], m\in(0,1]$.

3.$h-$convex if

$$X(tu+(1-t)v,\cdot)\leq h(t)X(u,\cdot)+h(1-t)X(v,\cdot)$$

for some function $h: J\rightarrow\mathbb{R}$ *non negative ,* $h\not\equiv 0$*, with* $(0,1)\subset J$ *and* J *is an interval of* $\mathbb{R}$

4.$(m,h_1,h_2)-$convex if

$$X(tu+m(1-t)v,\cdot)\leq h_1(t)X(u,\cdot)+mh_2(1-t)X(v,\cdot)$$

for some functions $h_1,h_2:[0,1]\rightarrow\mathbb{R}$ *and* $m\in(0,1]$.

Definition 9. *Let* (Ω,A,P) *be a probability space and* $T\subset\mathbb{R}$ *be an interval. We say that the stochastic process* $X:\Omega\rightarrow\mathbb{R}$ *is called*

1. *Continuous in probability in interval* T *if for all* $t_0\in T$

$$P-\lim_{t\rightarrow t_0}X(t,\cdot)=X(t_0,\cdot)$$

where $P-\lim$ *denotes the limit in probability;*

2. *Mean-square continuous in the interval* T *if for all* $t_0\in T$

$$P-\lim_{t\rightarrow t_0}E(X(t,\cdot)-X(t_0,\cdot))=0$$

where $E(X(t,\cdot))$ *denotes the expectation value of the random variable* $X(t,\cdot)$;

3. *Increasing (decreasing) if for all* $u,v\in T$ *such that* $t<s$,

$$X(u,\cdot)\leq X(v,\cdot),\quad (X(u,\cdot)\geq X(v,\cdot))(respectively)$$

4. *Monotonic if it's increasing or decreasing;*
5. *Differentiable at a point* $t\in T$ *if there is a random variable*

$$X'(t,\cdot):T\times\Omega\rightarrow\mathbb{R}$$

$$X'(t,\cdot)=P-\lim_{t\rightarrow t_0}\frac{X(t,\cdot)-X(t_0,\cdot)}{t-t_0}$$

We say that a stochastic process $X:T\times\Omega\rightarrow\mathbb{R}$ is continuous (differentiable) if it is continuous (differentiable) at every point of the interval T. [17], [42], [43], [24].

Definition 10. *Let* (Ω,A,P) *be a probability space* $T\subset\mathbb{R}$ *be an interval with* $E(X(t)^2)<\infty$ *for all* $t\in T$. *Let* $[a,b]\subset T, a=t_0<t_1<...<t_n=b$ *be a partition of* $[a,b]$ *and* $\theta_k\in[t_{k-1},t_k]$ *for* $k=1,2,...,n$. *A random variable* $Y:\Omega\rightarrow\mathbb{R}$ *is called mean-square integral of the process* $X(t,\cdot)$ *on* $[a,b]$ *if the following identity holds:*

$$\lim_{n\rightarrow\infty}E[X(\theta_k(t_k-t_{k-1})-Y)^2]=0$$

Then we can write

$$\int_a^b X(t,\cdot)dt=Y(\cdot)(a.e.).$$

Also, mean square integral operator is increasing, that is,

$$\int_a^b X(t,\cdot)dt\leq\int_a^b Z(t,\cdot)dt(a.e.)$$

where $X(t,\cdot)\leq Z(t,\cdot)$ in $[a,b]$ [41].
In throughout paper, we will consider the stochastic processes that is with continuous time and mean-square continuous.

Another important aspect for the development of this work is the following.

In [34], Raina R. K. introduced a class of functions defined formally by

$$\mathscr{F}^{\sigma}_{\rho,\lambda}(x)=\mathscr{F}^{\sigma(0),\sigma(1),..}_{\rho,\lambda}(x)=\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda)}x^k \quad (10)$$

where $\rho,\lambda>0, |x|<\mathbb{R}$ ($\mathbb{R}$ is the set of real numbers),and $\sigma=(\sigma(1),..,\sigma(k),..)$ is a bounded sequence of positive real numbers.

Note that if we take in (10) $\rho=1,\lambda=0$ and $\sigma(k)=((\alpha)_k(\beta)_k)/(\gamma)_k), k=1,2,3,...,$ where α,β and γ are parameters which can take arbitrary real or complex values (provided that $\gamma\neq 0,-1,-2,...$) , and the symbol $(a)_k$ denote the quantity

$$(a)_k=\frac{\Gamma(a+k)}{\Gamma(a)}=a(a+1)...(a+k-1),\quad k=1,2...,$$

and restrict its domain to $|x|\leq 1$ (with $x\in\mathbb{C}$), then we have the classical Hypergeometric Function, that is

$$\mathscr{F}^{\sigma}_{\rho,\lambda}(x)=F(\alpha,\beta;\gamma;x)=\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k)}{(\gamma)_k k!}x^k$$

Using (10), Agarwal, Luo and Raina in [1] defined the following left-sided and right-sided fractional integral operators respectively, as follows

$$\left(\mathscr{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi\right)(x)$$

$$=\int_a^x(x-t)^{\lambda-1}\mathscr{F}^{\sigma}_{\rho,\lambda}\left[w(x-t)^{\rho}\right]\varphi(t)dt,\quad (x>a) \quad (11)$$

and

$$\left(\mathscr{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi\right)(x)$$

$$=\int_x^b(t-x)^{\lambda-1}\mathscr{F}^{\sigma}_{\rho,\lambda}\left[w(t-x)^{\rho}\right]\varphi(t)dt,\quad (x<b), \quad (12)$$

where $\lambda,\rho>0$, $w\in R$ and φ is such that the integral on the right side exits.

It is easy to verify that $\mathscr{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi$ and $\mathscr{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi$ are bounded integral operators on $L(a,b)$, if

$$\mathfrak{M} := \mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right] < \infty.$$

In fact, for $\varphi \in L((a,b))$ we have

$$\left\| \mathscr{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi \right\|_1 \leq \mathfrak{M}\left\|\varphi\right\|_1$$

and

$$\left\| \mathscr{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi \right\|_1 \leq \mathfrak{M}\left\|\varphi\right\|_1$$

where

$$\left\|\varphi\right\|_1 = \int_a^b |\varphi(x)|\,dx$$

The importance of these operators stems indeed from their generality. Many useful fractional integral operators can be obtained by specializing the coefficient $\sigma(k)$. Here, we just point out that the classical Riemann-Liouville fractional integrals I^{α}_{a+} and I^{α}_{b-} of order α :

$$\left(I^{\alpha}_{a+}\varphi\right)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\varphi(t)dt, \quad (x>a, \alpha>0)$$

and

$$\left(I^{\alpha}_{b-}\varphi\right)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\varphi(t)dt, \quad (x<b, \alpha>0)$$

follow from (11) and (12) setting $\lambda = \alpha, \sigma(0) = 1$ and $w = 0$.

3 Main Results

Lemma 1. *Let $X : [a,b]\times\Omega \to \mathbb{R}$ be a differentiable stochastic processes on (a,b) and such that $|X(s,\cdot)| \leq M$ for all $s \in [a,b]$, and $\lambda > 0$. Then*

$$\begin{aligned}&\left(\mathscr{K}^{\sigma}_{\lambda,\rho}(b-s)+\mathscr{K}^{\sigma}_{\lambda,\rho}(s-a)\right)X(s,\cdot)\\ &\qquad -\left(\left(\mathscr{J}^{\sigma}_{\rho,\lambda,s+;w}X\right)(a,\cdot)+\left(\mathscr{J}^{\sigma}_{\rho,\lambda,s-;w}X\right)(b,\cdot)\right)\\ &= \left(\mathscr{J}^{\sigma}_{\rho,\lambda+1,s+;w}X'\right)(a,\cdot)-\left(\mathscr{J}^{\sigma}_{\rho,\lambda+1,s-;w}X'\right)(b,\cdot) \qquad (13)\end{aligned}$$

and

$$\begin{aligned}&\left(\mathscr{K}^{\sigma}_{\lambda,\rho,w}(b-s)+\mathscr{K}^{\sigma}_{\lambda,\rho,w}(s-a)\right)X(s,\cdot)\\ &\qquad -\left(\left(\mathscr{J}^{\sigma}_{\rho,\lambda,s+;w}X\right)(a,\cdot)+\left(\mathscr{J}^{\sigma}_{\rho,\lambda,s-;w}X\right)(b,\cdot)\right)\\ &= (s-a)^{\lambda+1}\int_0^1 t^{\lambda}\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w(s-a)^{\rho}t^{\rho}\right]X'(ts+(1-t)a,\cdot)\,dt\\ &-(b-s)^{\lambda+1}\int_0^1 t^{\lambda}\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-s)^{\rho}t^{\rho}\right]X'(ts+(1-t)b,\cdot)\,dt \qquad (14)\end{aligned}$$

where

$$\mathscr{K}^{\sigma}_{\lambda,\rho,w}(z-y) = (z-y)^{\lambda}\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w(z-y)^{\rho}\right].$$

Proof. Integrating by parts

$$\begin{aligned}\left(\mathscr{J}^{\sigma}_{\rho,\lambda,s-;w}X\right)(b,\cdot) &= \int_s^b (t-s)^{\lambda-1}\mathscr{F}^{\sigma}_{\rho,\lambda}\left[w(t-s)^{\rho}\right]X(t,\cdot)dt\\ &= (b-s)^{\lambda}\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-s)^{\rho}\right]X(b,\cdot)\\ &\quad -\int_s^b (t-s)^{\lambda}\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w(t-s)^{\rho}\right]X'(t,\cdot)dt\\ &= (b-s)^{\lambda}\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-s)^{\rho}\right]X(b,\cdot)\\ &\quad +\left(\mathscr{J}^{\sigma}_{\rho,\lambda+1,s-;w}X'\right)(b,\cdot)\end{aligned}$$

and, similarly

$$\begin{aligned}\left(\mathscr{J}^{\sigma}_{\rho,\lambda,s+;w}X\right)(a,\cdot) &= \int_a^s (s-t)^{\lambda-1}\mathscr{F}^{\sigma}_{\rho,\lambda}\left[w(s-t)^{\rho}\right]X(a,\cdot)dt\\ &= (s-a)^{\lambda}\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w(s-a)^{\rho}\right]X(a,\cdot)\\ &\quad -\left(\mathscr{J}^{\sigma}_{\rho,\lambda+1,s-;w}X'\right)(a,\cdot)\end{aligned}$$

So, adding both equalities

$$\begin{aligned}&\left(\mathscr{K}^{\sigma}_{\rho,\lambda,w}(b-s)+\mathscr{K}^{\sigma}_{\rho,\lambda,w}(s-a)\right)X(s,\cdot)\\ &\qquad -\left(\left(\mathscr{J}^{\sigma}_{\rho,\lambda,s+;w}X\right)(a,\cdot)+\left(\mathscr{J}^{\sigma}_{\rho,\lambda,s-;w}X\right)(b,\cdot)\right)\\ &= \left(\mathscr{J}^{\sigma}_{\rho,\lambda+1,s-;w}X'\right)(a,\cdot)-\left(\mathscr{J}^{\sigma}_{\rho,\lambda+1,s-;w}X'\right)(b,\cdot),\end{aligned}$$

where

$$\mathscr{K}^{\sigma}_{\rho,\lambda,w}(z-y) = (z-y)^{\lambda}\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w(z-y)^{\rho}\right],$$

then we just found inequality (13).

Now evaluating the following integral by parts we have

$$\begin{aligned}&\int_0^1 t^{\lambda}\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w(s-a)^{\rho}t^{\rho}\right]X'(ts+(1-t)a,\cdot)\,dt\\ &= \frac{1}{s-a}\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w(s-a)^{\rho}\right]X(s,\cdot)\\ &\quad -\frac{1}{s-a}\int_0^1 t^{\lambda-1}\mathscr{F}^{\sigma}_{\rho,\lambda}\left[w(s-a)^{\rho}t^{\rho}\right]X(ts+(1-t)a,\cdot)\,dt\\ &= \frac{1}{s-a}\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w(s-a)^{\rho}\right]X(s,\cdot)\\ &\quad -\frac{1}{(s-a)^{\lambda-1}}\int_a^s (v-a)^{\lambda-1}\mathscr{F}^{\sigma}_{\rho,\lambda}\left[w(v-a)^{\rho}\right]X(v,\cdot)\,dv\\ &= \frac{1}{s-a}\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w(s-a)^{\rho}\right]X(s,\cdot)\\ &\qquad -\frac{1}{(s-a)^{\lambda+1}}\left(\mathscr{J}^{\sigma}_{\rho,\lambda,s-;w}X'\right)(a,\cdot), \qquad (15)\end{aligned}$$

similarly

$$\begin{aligned}&\int_0^1 t^{\lambda}\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-s)^{\rho}t^{\rho}\right]X'(ts+(1-t)b,\cdot)\,dt\\ &= \frac{1}{b-s}\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w(s-a)^{\rho}\right]X(s,\cdot)\\ &\qquad -\frac{1}{(b-s)^{\lambda+1}}\left(\mathscr{J}^{\sigma}_{\rho,\lambda,s+;w}X'\right)(b,\cdot). \qquad (16)\end{aligned}$$

Multiplying (15) by $(s-a)^{\lambda+1}$ and (16) by $(b-s)^{\lambda+1}$, we have

$$(s-a)^{\lambda+1}\int_0^1 t^{\lambda}\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w(s-a)^{\rho}t^{\rho}\right]X'(ts+(1-t)a,\cdot)\,dt$$

$$=(s-a)^{\lambda}\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w(s-a)^{\rho}\right]X(s,\cdot)-\left(\mathscr{J}^{\sigma}_{\rho,\lambda,s-;w}X'\right)(a,\cdot)$$

and

$$(b-s)^{\lambda+1}\int_0^1 t^{\lambda}\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-s)^{\rho}t^{\rho}\right]X'(ts+(1-t)b,\cdot)\,dt$$

$$=(b-s)^{\lambda}\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w(s-a)^{\rho}\right]X(s,\cdot)-\left(\mathscr{J}^{\sigma}_{\rho,\lambda,s+;w}X'\right)(b,\cdot).$$

Adding the equalities we get the desired equality (1).

The proof is complete.

Theorem 1. *Let $X: I\times\Omega\to F$ be a differentiable stochastic processes and such $|X(s,\cdot)|\le M$ for all $s\in I$, and $\lambda>0$. Then we have*

$$\left|\left(\mathscr{K}^{\sigma}_{\lambda,\rho}(b-s)+\mathscr{K}^{\sigma}_{\lambda,\rho}(s-a)\right)X(s,\cdot)\right.$$
$$\left.-\left(\left(\mathscr{J}^{\sigma}_{\rho,\lambda,s+;w}X\right)(a,\cdot)+\left(\mathscr{J}^{\sigma}_{\rho,\lambda,s-;w}X\right)(b,\cdot)\right)\right|$$
$$\le M\left(\mathscr{K}^{\sigma}_{\lambda,\rho,w}(s-a)+\mathscr{K}^{\sigma}_{\lambda,\rho,w}(b-s)\right); \tag{17}$$

if in addition X' is a $(m,h_1,h_2)-$convex stochastic processes we have

$$\left|\left(\mathscr{K}^{\sigma}_{\lambda,\rho}(b-s)+\mathscr{K}^{\sigma}_{\lambda,\rho}(s-a)\right)X(s,\cdot)\right.$$
$$\left.-\left(\left(\mathscr{J}^{\sigma}_{\rho,\lambda,s+;w}X\right)(a,\cdot)+\left(\mathscr{J}^{\sigma}_{\rho,\lambda,s-;w}X\right)(b,\cdot)\right)\right|$$
$$\le\left|X'(s,\cdot)\right|\left((s-a)^{\lambda+1}I_1+(b-s)^{\lambda+1}I_3\right)$$
$$+(s-a)^{\lambda+1}m\left|X'\left(\frac{a}{m},\cdot\right)\right|I_2$$
$$+(b-s)^{\lambda+1}m\left|X'\left(\frac{b}{m},\cdot\right)\right|I_4, \tag{18}$$

where

$$I_1=\int_0^1 t^{\lambda}h_1(t)\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[|w|(s-a)^{\rho}t^{\rho}\right]dt,$$

$$I_2=\int_0^1 t^{\lambda}h_2(t)\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[|w|(s-a)^{\rho}t^{\rho}\right]dt$$

$$I_3=\int_0^1 t^{\lambda}h_1(t)\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[|w|(b-s)^{\rho}t^{\rho}\right]dt,$$

$$I_4=\int_0^1 t^{\lambda}h_2(t)\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[|w|(b-s)^{\rho}t^{\rho}\right]dt.$$

Proof. To prove (17) we use (13)

$$\left|\left(\mathscr{K}^{\sigma}_{\lambda,\rho}(b-s)+\mathscr{K}^{\sigma}_{\lambda,\rho}(s-a)\right)X(s,\cdot)\right.$$
$$\left.-\left(\left(\mathscr{J}^{\sigma}_{\rho,\lambda,s+;w}X\right)(a,\cdot)+\left(\mathscr{J}^{\sigma}_{\rho,\lambda,s-;w}X\right)(b,\cdot)\right)\right|$$
$$\le\left|\left(\mathscr{J}^{\sigma}_{\rho,\lambda+1,s+;w}X'\right)(a,\cdot)\right|+\left|\left(\mathscr{J}^{\sigma}_{\rho,\lambda+1,s-;w}X'\right)(b,\cdot)\right|$$

$$\le M\left\{\int_0^1(t-a)^{\lambda}\mathscr{F}^{\sigma}_{\rho,\lambda+1}[w(t-a)^{\rho}]\,dt\right.$$
$$\left.+\int_0^1(b-t)^{\lambda}\mathscr{F}^{\sigma}_{\rho,\lambda+1}[w(b-t)^{\rho}]\,dt\right\}.$$

Evaluating the integrals in the previous braces, we have

$$\int_0^1(t-a)^{\lambda}\mathscr{F}^{\sigma}_{\rho,\lambda+1}[w(t-x)^{\rho}]\,dt$$

$$=(s-a)^{\lambda+1}\mathscr{F}^{\sigma}_{\rho,\lambda+2}\left[w(s-a)^{\rho}\right]$$

$$=\mathscr{K}^{\sigma}_{\lambda,\rho,w}(s-a)$$

and similarly

$$\int_0^1(b-t)^{\lambda}\mathscr{F}^{\sigma}_{\rho,\lambda+1}[w(b-t)^{\rho}]\,dt$$

$$=(b-s)^{\lambda+1}\mathscr{F}^{\sigma}_{\rho,\lambda+2}\left[w(b-s)^{\rho}\right]$$

$$=\mathscr{K}^{\sigma}_{\lambda,\rho,w}(b-s)$$

making the substitution we have the desired result (17).

To prove (18) we have

$$\left|\left(\mathscr{K}^{\sigma}_{\lambda,\rho}(b-s)+\mathscr{K}^{\sigma}_{\lambda,\rho}(s-a)\right)X(s,\cdot)\right.$$
$$\left.-\left(\left(\mathscr{J}^{\sigma}_{\rho,\lambda,s+;w}X\right)(a,\cdot)+\left(\mathscr{J}^{\sigma}_{\rho,\lambda,s-;w}X\right)(b,\cdot)\right)\right|$$

$$\le(s-a)^{\lambda+1}\int_0^1 t^{\lambda}\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[|w|(s-a)^{\rho}t^{\rho}\right]\left|X'\left(ts+m(1-t)\frac{a}{m},\cdot\right)\right|dt$$

$$+(b-s)^{\lambda+1}\int_0^1 t^{\lambda}\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[|w|(b-s)^{\rho}t^{\rho}\right]\left|X'\left(ts+m(1-t)\frac{b}{m},\cdot\right)\right|dt$$

Using the $(m,h_1,h_2)-$convexity of the stochastic processes X' we have:

$$\left|\left(\mathscr{K}^{\sigma}_{\lambda,\rho}(b-s)+\mathscr{K}^{\sigma}_{\lambda,\rho}(s-a)\right)X(s,\cdot)\right.$$
$$\left.-\left(\left(\mathscr{J}^{\sigma}_{\rho,\lambda,s+;w}X\right)(a,\cdot)+\left(\mathscr{J}^{\sigma}_{\rho,\lambda,s-;w}X\right)(b,\cdot)\right)\right|$$

$$\le(s-a)^{\lambda+1}\left|X'(s,\cdot)\right|\left|\int_0^1 t^{\lambda}h_1(t)\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[|w|(s-a)^{\rho}t^{\rho}\right]dt\right.$$

$$+(s-a)^{\lambda+1}m\left|X'\left(\frac{a}{m},\cdot\right)\right|\left|\int_0^1 t^{\lambda}h_2(t)\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[|w|(s-a)^{\rho}t^{\rho}\right]dt\right.$$

$$+(b-s)^{\lambda+1}\left|X'(s,\cdot)\right|\left|\int_0^1 t^{\lambda}h_1(t)\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[|w|(b-s)^{\rho}t^{\rho}\right]dt\right.$$

$$+(b-s)^{\lambda+1}m\left|X'\left(\frac{b}{m},\cdot\right)\right|\left|\int_0^1 t^{\lambda}h_2(t)\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[|w|(b-s)^{\rho}t^{\rho}\right]dt.\right.$$

Doing

$$I_1=\int_0^1 t^{\lambda}h_1(t)\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[|w|(s-a)^{\rho}t^{\rho}\right]dt,$$

$$I_2 = \int_0^1 t^{\lambda} h_2(t) \mathscr{F}^{\sigma}_{\rho,\lambda+1} \left[|w|(s-a)^{\rho} t^{\rho}\right] dt$$

$$I_3 = \int_0^1 t^{\lambda} h_1(t) \mathscr{F}^{\sigma}_{\rho,\lambda+1} \left[|w|(b-s)^{\rho} t^{\rho}\right] dt,$$

$$I_4 = \int_0^1 t^{\lambda} h_2(t) \mathscr{F}^{\sigma}_{\rho,\lambda+1} \left[|w|(b-s)^{\rho} t^{\rho}\right] dt.$$

we have the desired inequality.

This complete the proof.

Corollary 1. *Let $X : I \times \Omega \to F$ be a differentiable stochastic processes and X' is a $P-$convex stochastic processes then we have*

$$\left|\left(\mathscr{K}^{\sigma}_{\lambda,\rho}(b-s) + \mathscr{K}^{\sigma}_{\lambda,\rho}(s-a)\right) X(s,\cdot) - \left(\left(\mathscr{J}^{\sigma}_{\rho,\lambda,s+;w} X\right)(a,\cdot) + \left(\mathscr{J}^{\sigma}_{\rho,\lambda,s-;w} X\right)(b,\cdot)\right)\right|$$
$$\leq \left|X'(s,\cdot)\right| \left(\mathscr{K}^{\sigma}_{\lambda+1,\rho}(s-a) + \mathscr{K}^{\sigma}_{\lambda+1,\rho}(b-s)\right) \quad (19)$$
$$+ \left(\left|X'(a,\cdot)\right| \mathscr{K}^{\sigma}_{\lambda+1,\rho}(s-a) + \left|X'(b,\cdot)\right| \mathscr{K}^{\sigma}_{\lambda+1,\rho}(b-s)\right)$$

Proof. If in Theorem 1 we put $m = 1, h_1(t) = h_2(t) = 1$ then we have

$$I_1 = \int_0^1 t^{\lambda} \mathscr{F}^{\sigma}_{\rho,\lambda+1} \left[|w|(s-a)^{\rho} t^{\rho}\right] dt = I_2$$

$$I_3 = \int_0^1 t^{\lambda} \mathscr{F}^{\sigma}_{\rho,\lambda+1} \left[|w|(b-s)^{\rho} t^{\rho}\right] dt = I_4$$

and so

$$I_1 = \mathscr{F}^{\sigma}_{\rho,\lambda+2} \left[|w|(s-a)^{\rho}\right] \text{ and } I_3 = \mathscr{F}^{\sigma}_{\rho,\lambda+2} \left[|w|(b-s)^{\rho}\right].$$

Replacing these values in (18) we have the desired inequality (19). The proof is complete.

Corollary 2. *Let $X : I \times \Omega \to F$ be a differentiable stochastic processes and such X' is a convex stochastic processes, and $\lambda > 0$. Then we have*

$$\left|\left(\mathscr{K}^{\sigma}_{\lambda,\rho}(b-s) + \mathscr{K}^{\sigma}_{\lambda,\rho}(s-a)\right) X(s,\cdot) - \left(\left(\mathscr{J}^{\sigma}_{\rho,\lambda,s+;w} X\right)(a,\cdot) + \left(\mathscr{J}^{\sigma}_{\rho,\lambda,s-;w} X\right)(b,\cdot)\right)\right|$$
$$\leq + \left|X'(a,\cdot)\right| \left(\mathscr{K}^{\sigma}_{\lambda+1,\rho}(s-a) - \frac{\mathscr{K}^{\sigma_1}_{\lambda+2,\rho}(s-a)}{(s-a)}\right) \quad (20)$$
$$+ \left|X'(b,\cdot)\right| \left(\mathscr{K}^{\sigma}_{\lambda+1,\rho}(b-s) - \frac{\mathscr{K}^{\sigma_1}_{\lambda+2,\rho}(b-s)}{(b-s)}\right)$$

where

$$\sigma_1(k) = (k\rho + \lambda + 1)\,\sigma(k), \text{ for all } k = 0,1,2,\ldots\ .$$

Proof. If in Theorem 1 new put $m = 1$ and $h_1(t) = t$ and $h_2(t) = (1-t)$ we get

$$I_1 = \int_0^1 t^{\lambda+1} \mathscr{F}^{\sigma}_{\rho,\lambda+1} \left[|w|(s-a)^{\rho} t^{\rho}\right] dt = \mathscr{F}^{\sigma_1}_{\rho,\lambda+3} \left[|w|(s-a)^{\rho}\right],$$

$$I_3 = \int_0^1 t^{\lambda+1} \mathscr{F}^{\sigma}_{\rho,\lambda+1} \left[|w|(b-s)^{\rho} t^{\rho}\right] dt = \mathscr{F}^{\sigma_1}_{\rho,\lambda+3} \left[|w|(b-s)^{\rho}\right],$$

$$I_2 = \int_0^1 t^{\lambda}(1-t) \mathscr{F}^{\sigma}_{\rho,\lambda+1} \left[|w|(s-a)^{\rho} t^{\rho}\right] dt = \mathscr{F}^{\sigma}_{\rho,\lambda+2} \left[|w|(s-a)^{\rho}\right] - \mathscr{F}^{\sigma_1}_{\rho,\lambda+3} \left[|w|(s-a)^{\rho}\right]$$

and

$$I_4 = \int_0^1 t^{\lambda}(1-t) \mathscr{F}^{\sigma}_{\rho,\lambda+1} \left[|w|(b-s)^{\rho} t^{\rho}\right] dt = \mathscr{F}^{\sigma}_{\rho,\lambda+2} \left[|w|(b-s)^{\rho}\right] - \mathscr{F}^{\sigma_1}_{\rho,\lambda+3} \left[|w|(b-s)^{\rho}\right]$$

where

$$\sigma_1(k) = (k\rho + \lambda + 1)\,\sigma(k), \text{ for all } k.$$

Replacing these values in (18) we have the desired inequality (20). The proof is complete.

Corollary 3. *Let $X : I \times \Omega \to F$ be a differentiable stochastic processes and such X' is a $s-$convex stochastic processes, and $\lambda > 0$. Then we have*

$$\left|\left(\mathscr{K}^{\sigma}_{\lambda,\rho}(b-s) + \mathscr{K}^{\sigma}_{\lambda,\rho}(s-a)\right) X(s,\cdot) - \left(\left(\mathscr{J}^{\sigma}_{\rho,\lambda,s+;w} X\right)(a,\cdot) + \left(\mathscr{J}^{\sigma}_{\rho,\lambda,s-;w} X\right)(b,\cdot)\right)\right|$$
$$\leq \left|X'(s,\cdot)\right| \left(\frac{\mathscr{K}^{\sigma_1}_{\rho,\lambda+s+1}(s-a)}{(s-a)^s} + \frac{\mathscr{K}^{\sigma_1}_{\rho,\lambda+s+1}(b-s)}{(b-s)^s}\right)$$
$$+ \Gamma(s+1) \left(\left|X'(a,\cdot)\right| \frac{\mathscr{K}^{\sigma}_{\rho,\lambda+s+2}(s-a)}{(s-a)^s} \quad (21)\right.$$
$$\left. + \left|X'(b,\cdot)\right| \frac{\mathscr{K}^{\sigma}_{\rho,\lambda+s+2}(s-a)}{(b-s)^s}\right)$$

where

$$\sigma_1(k) = \sigma(k) \frac{\Gamma(k\rho + \lambda + s + 1)}{\Gamma(k\rho + \lambda + 1)} \text{ for all } k.$$

Proof. Taking $m = 1, h_1(t) = t^s$ and $h_2(t) = (1-t)^s$ in Theorem 1 we have

$$I_1 = \mathscr{F}^{\sigma_1}_{\rho,\lambda+s+2} \left[|w|(s-a)^{\rho}\right],$$

$$I_3 = \mathscr{F}^{\sigma_1}_{\rho,\lambda+s+2} \left[|w|(b-s)^{\rho}\right]$$

where

$$\sigma_1(k) = \sigma(k) \frac{\Gamma(k\rho + \lambda + s + 1)}{\Gamma(k\rho + \lambda + 1)} \text{ for all } k.$$

Also,

$$I_2 = \Gamma(s+1) \mathscr{F}^{\sigma}_{\rho,\lambda+s+2} \left[|w|(s-a)^{\rho}\right],$$
$$I_4 = \Gamma(s+1) \mathscr{F}^{\sigma}_{\rho,\lambda+s+2} \left[|w|(b-s)^{\rho}\right].$$

Replacing the above values in (18) we have the desired inequality (21).

This complete the proof.

Corollary 4. *Let $X : I \times \Omega \to F$ be a differentiable stochastic processes and such X' is a $MT-$convex stochastic processes, and $\lambda > 0$. Then we have*

$$\left| \left(\mathscr{K}^{\sigma}_{\lambda,\rho}(b-s) + \mathscr{K}^{\sigma}_{\lambda,\rho}(s-a) \right) X(s,\cdot) - \left(\left(\mathscr{J}^{\sigma}_{\rho,\lambda,s+;w} X \right)(a,\cdot) + \left(\mathscr{J}^{\sigma}_{\rho,\lambda,s-;w} X \right)(b,\cdot) \right) \right|$$

$$\leq \sqrt{\pi} \left| X'(s,\cdot) \right| \left(\frac{\mathscr{K}^{\sigma_1}_{\rho,\lambda+\frac{5}{2}}(s-a)}{2\sqrt{(s-a)}} + \frac{\mathscr{K}^{\sigma_1}_{\rho,\lambda+\frac{5}{2}}(b-s)}{2\sqrt{(b-s)}} \right) \quad (22)$$

$$+ \left| X'(a,\cdot) \right| \sqrt{\pi} \frac{\mathscr{K}^{\sigma_1}_{\rho,\lambda+\frac{5}{2}}(s-a)}{4\sqrt{(s-a)}} + \left| X'(b,\cdot) \right| \sqrt{\pi} \frac{\mathscr{K}^{\sigma_1}_{\rho,\lambda+\frac{5}{2}}(b-s)}{4\sqrt{(b-s)}}$$

where

$$\sigma_1(k) = \sigma(k) \frac{\Gamma\left(k\rho + \lambda + \frac{3}{2}\right)}{\Gamma(k\rho + \lambda + 1)}.$$

Proof. Taking $m = 1, h_1(t) = \left(\sqrt{t}/2\sqrt{1-t}\right)$ and $h_2(t) = \left(\sqrt{1-t}/2\sqrt{t}\right)$ in Theorem 1 we have

$$I_1 = \int_0^1 t^{\lambda} \frac{\sqrt{t}}{2\sqrt{1-t}} \mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[|w| (s-a)^{\rho} t^{\rho} \right] dt = \frac{\Gamma\left(\frac{1}{2}\right)}{2} \mathscr{F}^{\sigma_1}_{\rho,\lambda+\frac{5}{2}}\left[|w| (s-a)^{\rho} \right],$$

$$I_3 = \int_0^1 t^{\lambda} \frac{\sqrt{t}}{2\sqrt{1-t}} \mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[|w| (b-s)^{\rho} t^{\rho} \right] dt = \frac{\Gamma\left(\frac{1}{2}\right)}{2} \mathscr{F}^{\sigma_1}_{\rho,\lambda+\frac{5}{2}}\left[|w| (b-s)^{\rho} \right],$$

$$I_2 = \int_0^1 t^{\lambda} \frac{\sqrt{1-t}}{2\sqrt{t}} \mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[|w| (s-a)^{\rho} t^{\rho} \right] dt = \frac{\Gamma\left(\frac{1}{2}\right)}{4} \mathscr{F}^{\sigma_1}_{\rho,\lambda+\frac{5}{2}}\left[|w| (s-a)^{\rho} \right]$$

and

$$I_4 = \int_0^1 t^{\lambda} \frac{\sqrt{1-t}}{2\sqrt{t}} \mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[|w| (b-s)^{\rho} t^{\rho} \right] dt = \frac{\Gamma\left(\frac{1}{2}\right)}{4} \mathscr{F}^{\sigma_1}_{\rho,\lambda+\frac{5}{2}}\left[|w| (b-s)^{\rho} \right]$$

where

$$\sigma_1(k) = \sigma(k) \frac{\Gamma\left(k\rho + \lambda + \frac{3}{2}\right)}{\Gamma(k\rho + \lambda + 1)}.$$

Replacing the above values in (18), and using the well know value $\Gamma(1/2) = \sqrt{\pi}$, we have the desired inequality (22).

The proof is complete.

Remark. Taking $\lambda = \alpha, w = 0$ and $\sigma = (1,0,0,\ldots)$ in Theorem 1 and Corollaries 1,2,3 and 4 we have the corresponding inequalities for Riemann-Liouville fractional integral:

a) For $P-$convex Stochastic Processes

$$\left| \left(\frac{(b-s)^{\alpha} + (s-a)^{\alpha}}{\Gamma(\alpha+1)} \right) X(s,\cdot) - \left(\left(\mathscr{I}^{\alpha}_{s+} X \right)(a,\cdot) + \left(\mathscr{I}^{\alpha}_{s-} X \right)(b,\cdot) \right) \right|$$

$$\leq \frac{1}{\Gamma(\alpha+2)} \left| X'(s,\cdot) \right| \left((s-a)^{\alpha+1} + (b-s)^{\alpha+1} \right) + \frac{1}{\Gamma(\alpha+2)} \left((s-a)^{\alpha+1} \left| X'(a,\cdot) \right| + (b-s)^{\alpha+1} \left| X'(b,\cdot) \right| \right)$$

b) For convex stochastic processes

$$\left| \left(\frac{(b-s)^{\alpha} + (s-a)^{\alpha}}{\Gamma(\alpha+1)} \right) X(s,\cdot) - \left(\left(\mathscr{I}^{\alpha}_{s+} X \right)(a,\cdot) + \left(\mathscr{I}^{\alpha}_{s-} X \right)(b,\cdot) \right) \right|$$

$$\leq \left| X'(s,\cdot) \right| \Gamma(\alpha+3) \left(\frac{1}{(s-a)} + \frac{1}{(b-s)} \right) + \left| X'(a,\cdot) \right| (s-a)^{\alpha+1} \left(\frac{1}{\Gamma(\alpha+2)} - \frac{1}{\Gamma(\alpha+3)} \right) + \left| X'(b,\cdot) \right| (b-s)^{\alpha+1} \left(\frac{1}{\Gamma(\alpha+2)} - \frac{1}{\Gamma(\alpha+3)} \right)$$

c) For $s-$convex stochastic processes

$$\left| \left(\frac{(b-s)^{\alpha} + (s-a)^{\alpha}}{\Gamma(\alpha+1)} \right) X(s,\cdot) - \left(\left(\mathscr{I}^{\alpha}_{s+} X \right)(a,\cdot) + \left(\mathscr{I}^{\alpha}_{s-} X \right)(b,\cdot) \right) \right|$$

$$\leq \frac{\left| X'(s,\cdot) \right|}{(\alpha+s+1)\Gamma(\alpha+1)} \left((s-a)^{\alpha+1} + (b-s)^{\alpha+1} \right) + \frac{\Gamma(s+1)}{\Gamma(\alpha+s+3)} \left(\left| X'(a,\cdot) \right| (s-a)^{\alpha+2} + \left| X'(b,\cdot) \right| (b-s)^{\alpha+2} \right)$$

d) For $MT-$convex stochastic processes

$$\left| \left(\frac{(b-s)^{\alpha} + (s-a)^{\alpha}}{\Gamma(\alpha+1)} \right) X(s,\cdot) - \left(\left(\mathscr{I}^{\alpha}_{s+} X \right)(a,\cdot) + \left(\mathscr{I}^{\alpha}_{s-} X \right)(b,\cdot) \right) \right|$$

$$\leq \frac{\sqrt{\pi}}{\Gamma(\alpha+1)\, 2\left(\alpha+\frac{3}{2}\right)} \left| X'(s,\cdot) \right| \left((s-a)^{\alpha+2} + (b-s)^{\alpha+2} \right) + \left| X'(a,\cdot) \right| \frac{\sqrt{\pi}(s-a)^{\alpha+2}}{\Gamma(\alpha+1)\, 4\left(\alpha+\frac{3}{2}\right)} + \left| X'(b,\cdot) \right| \frac{\sqrt{\pi}(b-s)^{\alpha+2}}{\Gamma(\alpha+1)\, 4\left(\alpha+\frac{3}{2}\right)}.$$

Clearly, if we put $\alpha = 1$ we have

a') for $P-$convex stochastic Processes

$$\left| ((b-a) X(s,\cdot) - \int_a^b X(t,\cdot) dt \right| \leq \frac{1}{2} \left| X'(s,\cdot) \right| \left((s-a)^2 + (b-s)^2 \right) + \frac{1}{2} \left((s-a)^2 \left| X'(a,\cdot) \right| + (b-s)^2 \left| X'(b,\cdot) \right| \right)$$

b') for convex stochastic processes

$$\left| ((b-a)) X(s,\cdot) - \int_a^b X(t,\cdot) dt \right| \leq \left| X'(s,\cdot) \right| 6 \left(\frac{1}{(s-a)} + \frac{1}{(b-s)} \right) + \left| X'(a,\cdot) \right| \frac{(s-a)^2}{3} + \left| X'(b,\cdot) \right| \frac{(b-s)^2}{3}$$

c') for s-convex function

$$\left| ((b-a)) X(s,\cdot) - \int_a^b X(t,\cdot) dt \right|$$

$$\leq \frac{\sqrt{\pi}}{5}\left|X'(s,\cdot)\right|\left((s-a)^3+(b-s)^3\right)$$

$$+\frac{\Gamma(s+1)}{\Gamma(s+4)}\left(\left|X'(a,\cdot)\right|(s-a)^3+\left|X'(b,\cdot)\right|(b-s)^3\right)$$

d') for $MT-$convex function

$$\left|((b-a))X(s,\cdot)-\int_a^b X(t,\cdot)dt\right|$$

$$\leq \frac{\sqrt{\pi}}{5}\left|X'(s,\cdot)\right|\left((s-a)^3+(b-s)^3\right)$$

$$+\left|X'(a,\cdot)\right|\frac{\sqrt{\pi}(s-a)^3}{10}+\left|X'(b,\cdot)\right|\frac{\sqrt{\pi}(b-s)^3}{10}$$

4 Conclusions

In this paper we have proved an Ostrowski inequality type for $(m,h_1,h_2)-$convex stochastic processes using the fractional integral Operator defined by Agarwal, Luo and Raina in [1], and we have obtained in Corollaries 1,2,3 y 4 and Remark 3 particularized inequalities for $P-$convex , $s-$convex, convex and $MT-$convex stochastic processes. We expect that the ideas and techniques used in this paper may inspire interested readers to explore some new applications of these newly introduced functions in various fields of pure and applied sciences.

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