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A Mathematical Model with Memory for Propagation of Computer Virus Under Human Intervention

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Abstract: In this paper, we propose a fractional order model for the propagation behavior of computer virus under human intervention to study the spread of computer virus across the internet. Numerical simulations are used to show the behavior of the solutions of the proposed fractional order system.

Keywords: Fractional calculus, computer virus, numerical solution, predictor-corrector method.

1 Introduction

Computer viruses and network worms, are defined as malicious codes that can replicate themselves and spread among computers [1]. The spread of computer viruses still causes enormous financial losses that large organizations suffer for computer security problems [2]. The most devastating computer virus to date is ”My Doom”, which caused over $38 billion in damages [3]. So, individuals and organizations are troubled by computer viruses [4]. Throughout the past two decades, computer viruses were inherently limited by the fact that human mediation was required for them to propagate [5]. But, in modern life, human intervention plays a significant role in preventing the breakout of computer viruses [6]. Myriad of different computer viruses have been made and developed by human programmers to damage the computer systems, erasing data or stealing information. Such viruses may attack computers through many ways like downloading files via internet, running an infected program, opening infected e-mail attachments, and using infected USB devices [7]. Mathematical modeling of the spread process of computer virus is an effective approach to understand the behavior of computer viruses and how to prevent infection [8]. It helps decision makers to put their strategies to control the spread of computer viruses. In this paper, we present a fractional order SIR (Susceptible-Infected-Removed) model to discuss the effects of human intervention in spread of viruses over network. This model is borrowed from epidemiological SIR model which is used to study the dynamics of infectious diseases [9].
The rest of the paper is organized as follows. A brief review of the fractional calculus theory is given in Section 2. In Section 3, fractional order epidemic models are discussed. In Section 4, we present the equilibrium points and their stability, while in Section 5, the existence of the solution is discussed. Section 6 is devoted to the numerical results. Finally, this paper is summarized by a conclusion.

2 Fractional Calculus

Recently, many mathematicians and applied researchers have tried to model real processes using the fractional calculus [10,11]. Fractional calculus and fractional-order differential equations date back to near the foundations of calculus, and they have been used in engineering fields for several decades [12,13,14] Many applications of fractional calculus amount

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to replacing the time derivative in a given evolution equation by a derivative of fractional order [15]. The concept of fractional calculus has tremendous potential to change the way we see the model, and control the nature around us. The major reason of using is that fractional differential equations are naturally related to systems with memory which exists in most biological and systems [16,17,18] Also, they are closely related to fractals, which are abundant in biological systems [19,20,21,25]. Moreover, factional order differential equations are, at least, as stable as their integer order counterpart [21, 22,23,24,25,26]. In other words, the fractional order derivative can capture the history of the variable that is. It is difficult to be done by means of the integer order derivatives. The physical meaning of the fractional order is considered in [27] to be the index of memory. In the models with memory, a memory process usually consists of two stages [28]:

- Short stage with permanent retention.
- The other is governed by a simple model of fractional order derivative.

We use Caputo derivative as it is attractive when physical models are presented because of clarity the physical interpretation of the prescribed data [29]. Also Caputo derivative is useful because the initial conditions for the fractional-order models with the Caputo derivatives can be the same as for the integer-order differential equations [30].

Definition 1. The fractional integral of order α of a function $f : R^+ \to R$ is given by

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \ \ \alpha > 0, x > 0,$$

$$J^0 f(x) = f(x)$$

which is an integral with memory. Hence we have

$$J^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}, \ \ \alpha > 0, \ \gamma > -1, \ t > 0.$$

Definition 2. The Caputo fractional derivative and Riemann–Liouville derivative of order α where $\alpha \in (m-1, m)$ of a continuous function $f : R^+ \to R$ is given respectively by

$$D^{\alpha} f(x) = D^m \left(J^{m-\alpha} f(x)\right),$$

$$D_*^{\alpha} f(x) = J^{m-\alpha} \left(D^m f(x)\right),$$

where

$$m-1 < \alpha \leq m, \ m \in N.$$

The definition of fractional derivative involves an integration which is non-local operator (as it is defined on an interval) so fractional derivative is a non-local operator. In other words, calculating time-fractional derivative of a function $f(t)$ at some time $t = t_1$ requires all the previous history, i.e. all $f(t)$ from $t = 0$ to $t = t_1$.

Definition 3. For $\alpha > 0$, the Grunwald-Letnikov αth order fractional derivative of function $f(t)$ with respect to t and the terminal value a is given [34]

$${}_{a}^{GL}D_t^{\alpha} f(t) = \lim_{h \to 0} h^{-\alpha} \sum_{j=0}^{n} (-1)^j \binom{\alpha}{j} f(t-jh)$$

,

where $nh = t - a$

Some problems appeared when discrete time fractional order derivative is used as follows [35]:

- Fractionalizing the discrete-time systems using classic tools have resulted in finite dimension integer-order systems that are difficult to manipulate.
- Integer order discrete-time systems that is used to approximate continuous-time fractional systems which have long memory, are known for their short memory.

3 Model Derivation

We propose the following fractional order models which are based on the integer order models given in [6]. In this model, all internal computers are further categorized into three populations: susceptible computers $S(t)$, that is, virus-free computers having no immunity; infected computers $I(t)$; recovered computers $R(t)$, that is, virus-free computers having immunity [6]. So, the e-epidemic fractional order model is given as follows

$$\begin{aligned} D^{\alpha}S(t) &= \omega - \mu_1 SI - \omega S + \gamma_2 I - \beta SI + \mu_2 R, \\ D^{\alpha}I(t) &= \beta SI - \gamma_2 I - \omega I - \gamma_1 I, \\ D^{\alpha}R(t) &= \gamma_1 I - \mu_1 SI - \omega R - \mu_2 R, \end{aligned} \tag{1}$$

where $0 < \alpha \leq 1$, is the index of memory, ω is defined as the probability that at any time an internal computer is disconnected from the Internet while βSI is responsible of increasing the percentage of infected computers. γ_1 is the probability of an infected computer becomes recovered at any time or becomes susceptible with probability γ_2. The parameter μ_2 is the probability of a recovered computer becomes susceptible at any time. $\mu_1 I$ is the probability of a susceptible computer becomes recovered. The initial conditions are:

$$S(0) \geq 0, I(0) \geq 0 \text{ and } R(0) \geq 0.$$

The basic reproduction R_0 is defined as the average number of susceptible computers that are infected by a single infected computer during its life span. From the above model, basic reproduction number has been driven in [6] as

$$R_0 = \frac{\beta}{\gamma_1 + \gamma_2 + \omega}. \tag{2}$$

Based on [6] and since $S + I + R \equiv 1$, system (1) can be simplified to the following SIS (susceptible-Infected) fractional model:

$$\begin{aligned} D^{\alpha}S(t) &= \omega - \mu_1 SI - \omega S + \gamma_2 I - \beta SI + \mu_2(1 - S - I), \\ D^{\alpha}I(t) &= \beta SI - \gamma_2 I - \omega I - \gamma_1 I, \end{aligned} \tag{3}$$

with initial conditions $S(0) \geq 0, I(0) \geq 0$.
A sufficient condition for the local asymptotic stability of the equilibrium point $E(S_{eq}, I_{eq})$ is that the eigenvalues λ_i of the Jacobian matrix of E satisfy the condition $|\arg \lambda_i| > \alpha\frac{\pi}{2}$: $(|\arg \lambda_1| > \alpha\frac{\pi}{2}, |\arg \lambda_2| > \alpha\frac{\pi}{2})$ [19]-[36]. This confirms that fractional-order differential equations are, at least, as stable as their integer order counterpart.

4 The Virus-Free Equilibrium Point and Its Stability

To evaluate the equilibrium points, let

$$D^{\alpha}S(t) = 0,$$

$$D^{\alpha}I(t) = 0.$$

System (3) always has a virus-free equilibrium $E^0(1,0)$.
Theorem 1. The equilibrium point E^0 is globally asymptotically stable with respect to feasible region $\Omega = \{(S,I) : S \geq 0, I \geq 0, S + I \leq 1\}$ if $R_0 \leq 1$.
The proof: in [6].
When $R_0 > 1$, the system (3) has a unique viral-equilibrium $E^*(S^*, I^*)$, where

$$S^* = \frac{\gamma_1 + \gamma_2 + \omega}{\beta} = \frac{1}{R_0},$$

$$I^* = \frac{(\mu_2 + \omega)(R_0 - 1)}{\mu_1 + (\omega + \gamma_1 + \mu_2) R_0} > 0. \tag{4}$$

5 Existence of Uniformly Stable Solution

Let

$$x_1(t) = S(t) \quad x_2(t) = I(t)$$

,

$$f_1(x_1(t), x_2(t)) = \omega - \mu_1(x_1(t), x_2(t)) - \delta(x_1(t)) + \gamma_2(x_2(t)) - \beta(x_1(t) x_2(t)) - \mu_2(1 - x_1(t) - x_2(t)),$$

$$f_2(x_1(t), x_2(t)) = \beta((x_1(t)\ x_2(t)) - \gamma_2(x_2(t)) - \omega(x_2(t)) - \gamma_1(x_2(t)). \quad (5)$$

Let D= $\{x_1, x_2 \in R : |x_i(t)| \leq \alpha, \ t \in [0,T], i = 1,2\}$.
Then on D we have

$$\left|\frac{\partial}{\partial x_1} f_1(x_1, x_2)\right| \leq k_1, \qquad \left|\frac{\partial}{\partial x_2} f_1(x_1, x_2)\right| \leq k_2,$$

$$\left|\frac{\partial}{\partial x_1} f_2(x_1, x_2)\right| \leq k_3, \qquad \left|\frac{\partial}{\partial x_2} f_2(x_1, x_2)\right| \leq k_4,$$

where k_1, k_2, k_3, k_4 are positive constants.
This implies that each of the two functions f_1, f_2 satisfies the Lipschitz condition with respect to the two arguments x_1 and x_2 then each of the two functions f_1, f_2 is absolutely continuous with respect to the two arguments x_1 and x_2.
Consider the following initial value problem which represents the fractional-order SIR model (6) and (7)

$$D^{\alpha} x_1(t) = f_1(x_1, x_2), \ t > 0 \ , x_1(0) = x_{0^1}, \quad (6)$$

$$D^{\alpha} x_2(t) = f_2(x_1, x_2), \ t > 0 \ , x_2(0) = x_{0^2}. \quad (7)$$

Definition 5.1. By a solution of the fractional-order SIR model (6) and (7), we mean a column vector$(x_1(t)\ x_2(t))^{\tau}$, x_1 and $x_2 \in C[0, T]$, T $< \infty$ where $C[0, T]$ is the class of continuous functions defined on the interval [0, T] and s denotes the transpose of the matrix.

Theorem 5.1
The fractional-order SIR model (6) and (7)has a unique uniformly Lyapunov stable solution.
Proof. Write the model (6) and (7) in the matrix form

$D^{\alpha} X(t) = F(x(t)), \quad t > 0$ and $x(0) = x_0$,
where

$$x(t) = (x_1(t)\ x_2(t))^{\tau}$$

and

$$F(x(t)) = (f_1(x_1(t), x_2(t))\ f_2(x_1(t)\ x_2(t)))^{\tau}.$$

6 Numerical results

In this section, the predictor corrector method is applied to get the numerical solutions of system (3) [36]. We will propose two cases for the model (3) with various of values of parameters (γ_2). In the first case, $\beta = 0.3, \omega = 0.1, \mu_1 = 0.2, \mu_2 = 0.4, \gamma_1 = 0.1, \gamma_2 = 0.2$ and with initial conditions: $S(0) = 0.5$, $I(0) = 0.4$. In this case, $R_0 = 0.75 < 1$, then the virus-free equilibrium is globally stable and the virus is eliminated. In the second case, $\beta = 0.3, \delta = 0.1, \mu_1 = 0.2, \mu_2 = 0.4, \gamma_1 = 0.1, \gamma_2 = 0.05$ with initial conditions: $S(0)$ = 0.5, $I(0)$=*0.4,* then $R_0 = 1.2 > 1$ which implies that the virus still persists and the viral equilibrium is globally stable.

7 Conclusion

In this paper, we introduced a study of propagation of computer virus under the impact of human intervention. We modified the ODE model proposed in [6] into a system of fractional-order (SIR) model. The possibility that an infected computer becomes susceptible as well as the possibility that a susceptible computer becomes recovered is considered here in the proposed model. The results show that the solution continuously depends on the time-fractional derivative. When $\alpha \to 1$ the solution of the fractional models reduce to the standard solution of the integer order models. According to the results of the simulation experiments figures (1-6) it is observed that R_0 is increasing with β, and it is decreasing with γ_1, γ_2 and ω respectively Figures 1 and 4.. This implies that prevention is more important than cure, and higher disconnecting rate from the Internet contributes to the suppression of virus diffusion.

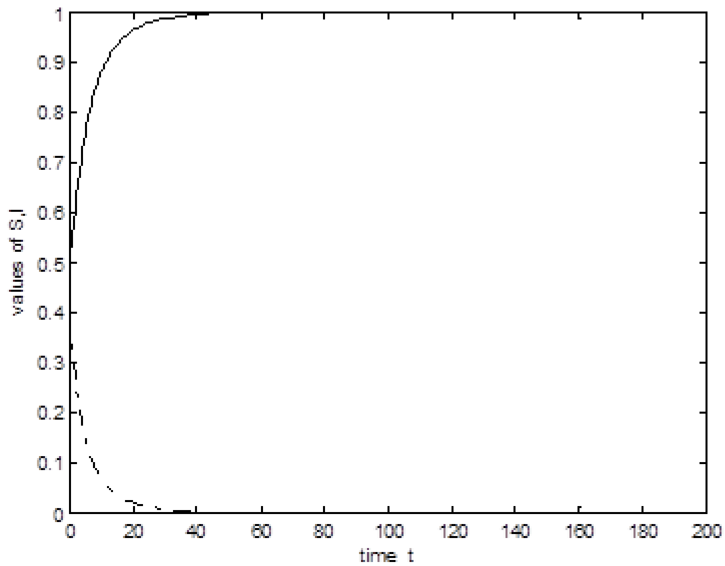

Fig. 1: Evolutions of $S(t)$; $I(t)$, and for the system (3) for $\alpha = 1$ for case (1) with value of $\gamma_2 = 0.2$.

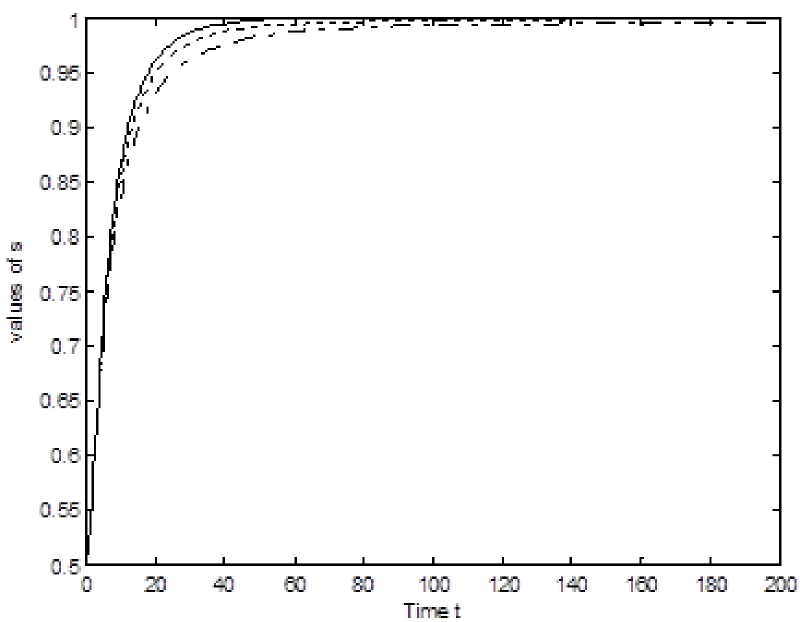

Fig. 2: The density of susceptible computer for $\alpha = 1$ (solid line), $\alpha = 0.98$ (dashed line), and $\alpha = 0.95$ (Dashed-dotted).

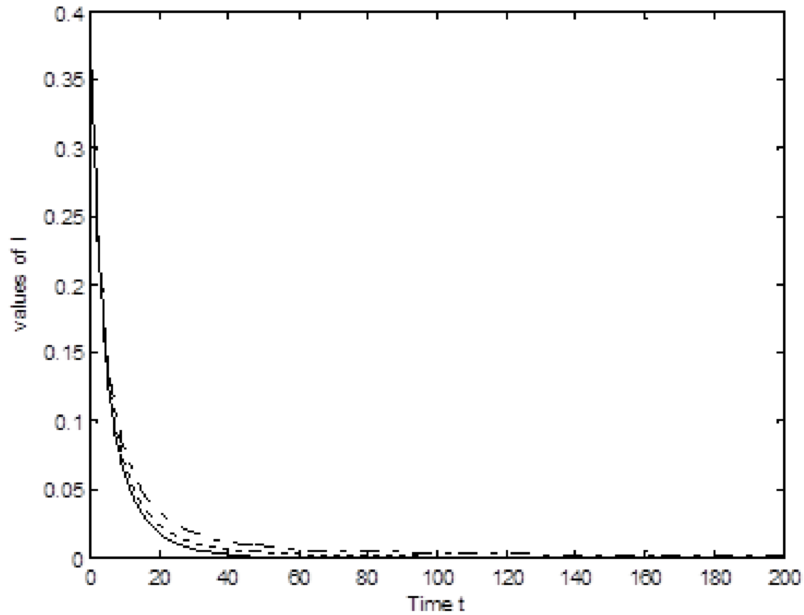

Fig. 3: The density of infected computer for α = 1 (solid line), α = 0.98 (dashed line), and α = 0.95 (Dashed-dotted).

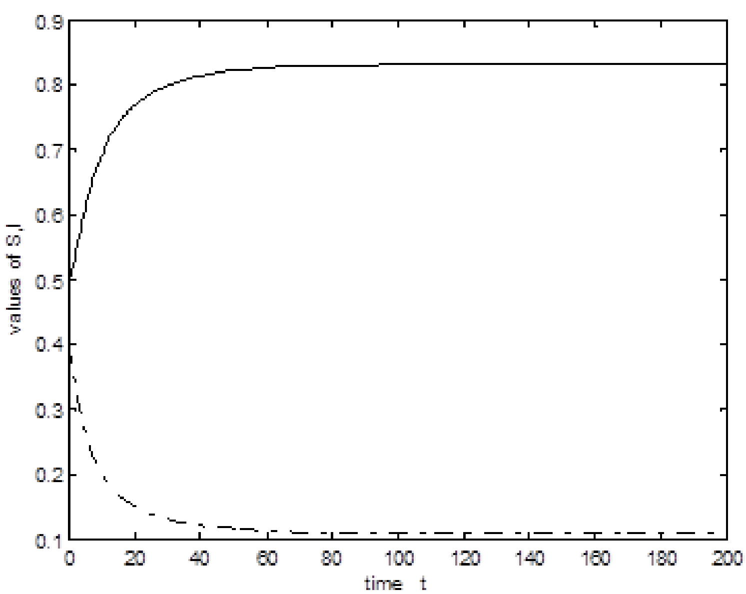

Fig. 4: Evolutions of S (t) (solid line)*;* I (t) (dashed dotted line), for case (2) at α = 1 and value of $\gamma_2 = 0.05$

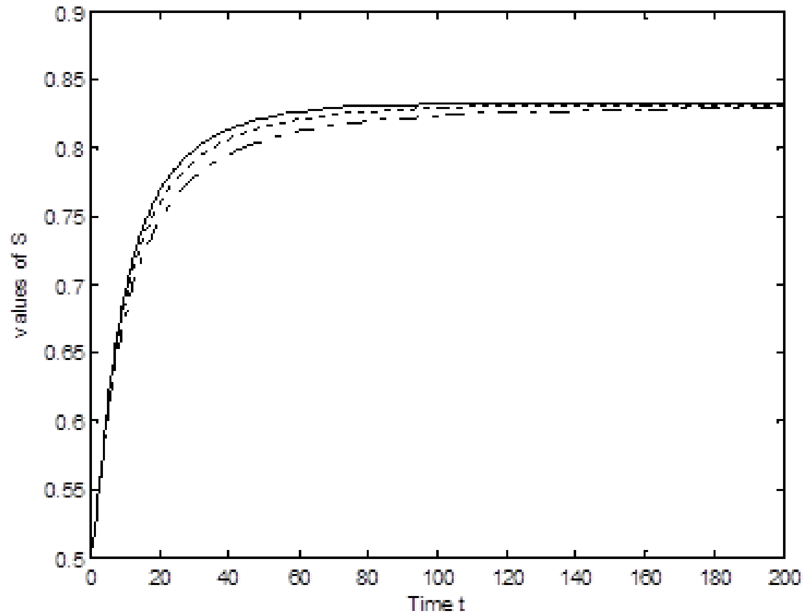

Fig. 5: The density of susceptible computer for α = 1 (solid line), α =0.98 (dashed line), and α = 0.95 (Dashed-dotted).

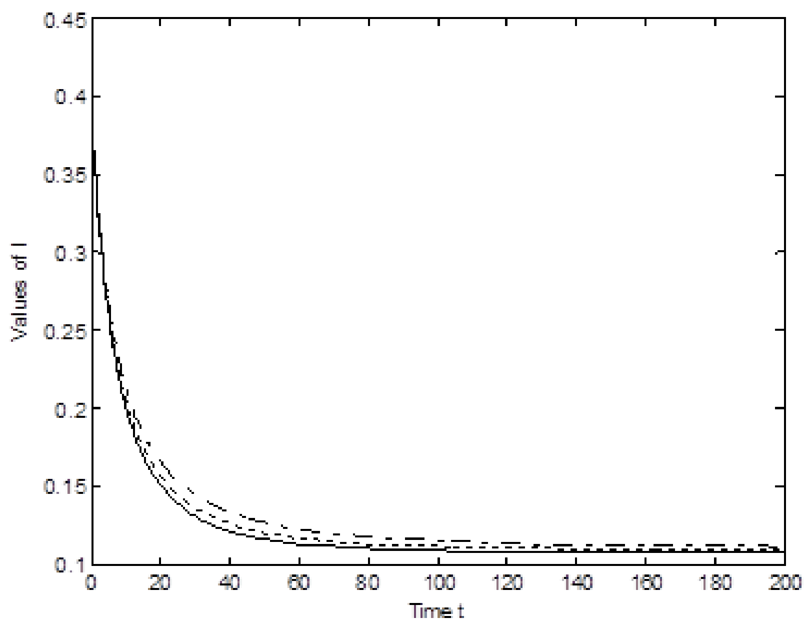

Fig. 6: The density of infected computer for α = 1 (solid line), α =0.98 (dashed line), and α = 0.95 (Dashed-dotted).

References

[1] C. Zhang, T. Feng, Y. Zhao and G. Jiang, A new model for capturing the spread of computer viruses on complex-networks, *Discr. Dyn. Nat. Soc.* Volume **2013**, Article ID 956893, (2013).
[2] G. Serazzi and S. Zanero, *Computer virus propagation models, Proceedings of 11th IEEE/ACM Symposium on Modeling*, Analysis and Simulation of Computer and Telecommunication Systems (MASCOTS), 2003.
[3] 10 of the most costly computer viruses of all time http://www.investopedia.com/financial-edge/0512/10-of-the-most-costly computer-viruses-of-all-time.aspx
[4] Q. Zhu, X. Yang and J. Ren, Modeling and analysis of the spread of computer virus, *Commun Nonlin Sci. Numer. Simul.* **17**, 5117-5124 (2012).
[5] S. Shakkottai and R. Srikant, Peer to peer networks for defense against internet worms, IEEE, Journal on Selected Areas in Communications **25**, 1745-1752 (2007).
[6] C. Gan, X. Yang, W. Liu, Q. Zhu and X. Zhang. Propagation of computer virus under human intervention: a dynamical model, *Discr. Dyn. Nat. Soc.* **2012** ,Article ID 106950, (2012).
[7] B. K. Mishra and D. Saini, Mathematical models on computer viruses, *Appl. Math. Comput.* **187**, 929-936 (2007).
[8] C. Gan, X. Yang, W. Liu, Q. Zhu, J. Jin and L. He, Propagation of computer virus both across the Internet and external computers: A complex-network approach, *Commun. Nonli. Sci. Numer. Simul.* **19**, 2785-2792 (2014).
[9] J. R. C. Piqueira, A. A. de Vasconcelos, C. E. C. J. Gabriel and V. O. Araujo, Dynamic models for computer viruses, *Comput. Secur.* **27**, 355-359 (2008).
[10] Y. Ding and H. Ye, A fractional-order differential equation model of HIV infection of CD4+ T-cells, *Math. Comput. Model.* **50**, 386-392 (2009).
[11] A. M .A. El-Sayed, A. E. M. El-Mesiry and H. A. A. El-Saka, Numerical solution for multi-term fractional (arbitrary) orders differential equations, *Comput. Appl. Math.* **23**, 33-54 (2004).
[12] J. F. Gómez-Aguilar, R. Razo-Hernández and D. Granados-Lieberman, A physical interpretation of fractional calculus in observables terms: analysis of the fractional time constant and the transitory response, Rev. Mex.Fis. **60**, 32-38 (2014).
[13] M. D. Ortigueira and F. J. Coito, System initial conditions vs derivative initial conditions, *Comput. Math. Appl.* **59**, 1782-1789 (2010).
[14] R. Magin, M. D. Ortigueira, I. Podlubny and J. J. Trujillo. On the fractional signals and systems, *Signal Proc.* **91**, 350-371 (2011).
[15] D. Y. Xue, C. N. Zhao and Y. Q. Chen, Fractional order PID control of A DC-motor with elastic shaft: a case study, American Control Conference, 3182-3187 (2006).
[16] A. A. M. Arafa, S. Z. Rida and M. Khalil, The effect of anti-viral drug treatment of human immunodeficiency, *Appl. Math. Model.* 37, 2189-2196 (2013).
[17] A. M. A. El-Sayed, S. H. Behiry and W. E. Raslan, Adomian's decomposition method for solving an intermediate fractional advection dispersion equation, *Comput. Math. Appl.* **59**, 1759-1765 (2010).
[18] W. Deng, Smoothness and stability of the solutions for nonlinear fractional differential equations, *Nonlinear Anal.* **72**, 1768-1777 (2010).
[19] E. Ahmed, A. M. A. El-Sayed and H. A. A. El-Saka, Equilibrium points, stability and numerical solutions of fractional-order predator-prey and rabies models, *J. Math. Anal. Appl.* **325**, 542-553 (2007).
[20] A. A. M. Arafa, S.cZ. Rida and M. Khalil, Fractional order model of human T-cell lymphotropic virus I (HTLV-I) infection of $CD4^{+}$T-cells, *Adv. Stud. Biol.*, **3**, 347 - 353 (2011).
[21] A. E. M. El-Misiery and E. Ahmed, On a fractional model for earthquakes, *Appl. Comput. Math.* **178**, 207-211 (2006).
[22] A. M. A. El-Sayed, S. Z. Rida and A. A. M. Arafa, On the Solutions of Time-fractional Bacterial Chemotaxis in a Diffusion Gradient Chamber, *Int. J. Nonlin. Sci.* **7**, 485-492 (2009).
[23] A. M. A. El-Sayed, A. E. M. El-Mesiry and H. A. A. El-Saka, Numerical solution for multi-term fractional (arbitrary) orders differential equations, *Comput. Appl. Math.* **23**, 33-54 (2004).
[24] A. M. A. El-Sayed, S. Z. Rida and A. A. M. Arafa, On the solutions of the generalized reaction-diffusion model for bacteria growth, *Acta Appl. Math.*, **110**, 1501-1511 (2010).
[25] A. M. A. El-Sayed, S. H. Behiry and W. E. Raslan, Adomian's decomposition method for solving an intermediate fractional advection dispersion equation, *Comput. Math. Appl.* **59**, 1759-1765 (2010).
[26] A. Dzielinski and D. Sierociuk, Fractional order model of beam heating process and its experimental verification,in New trends in nanotechnology and fractional calculus applications, Eds. D. Baleanu, Z. B. Guvenc and J. A. T. Machado, Springer Netherlands, 287-294 (2010).
[27] M. Du, Z. Wang and H. Hu, Measuring memory with the order of fractional derivative, *Sci. Rep.* **3**, 221-227 (2013).
[28] K. Moaddy, A. G. Radwan, K. N. Salama, S. Momani and I. Hashim, The fractional-order modeling and synchronization of electrically coupled neuron systems, *Comput. Math. Appl.* **64**, 3329-3339 (2012).
[29] K. Diethelm, The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type, Lecture Notes in Mathematics, 2004, Springer-Verlag, Berlin, 2010.
[30] D. Sierociuk, I. Podlubny and I. Petras, Experimental evidence of variable-order behavior of ladders and nested ladders, *IEEE Trans. Contr. Syst. Tech.* **21**, 459-466 (2013).
[31] C. M. Pinto and J. A. T. Machado, Fractional model for malaria transmission under control strategies, *Comput. Math. Appl.* **66**, 908–916 (2013).

[32] Mathematical Problems in Engineering C. M. A. Pinto and J. A. Tenreiro Machado Fractional Dynamics of Computer Virus Propagation, Volume 2014 (2014), Article ID 476502, 7 pages.
[33] N. Varalta, A. V. Gomes and R. F.Camargo, A prelude to the fractional calculus applied to tumor dynamic, TEMA (Sao Carlos) **15**, 211-221 (2014).
[34] M. Salah, O. Abdelouahab and N. E. Hamri, The Grunwald-Letnikov fractional-order derivative with fixed memory length, *Mediterr. J. Math.*, 1-16 (2015).
[35] M. D. Ortigueira, F. J. V. Coito and J. J. Trujillo, Discrete-time differential systems, *Signal Proc.* **107**, 198-217 (2015).
[36] H. A. A. El-Saka, The fractional-order SIS epidemic model with variable population size, *J. Egypt. Math. Soc.* **22**,50–54 (2014).
[37] D. Rodabazlgh and J. R. Wesson, On the efficient use of predictor-corrector methods in the numerical solution of differential equations, Technical Report, NASA TN D-2946, 1965.