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A novel four-step iterative scheme for approximating the fixed point with a supportive application

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Abstract: Our goal of this manuscript is to introduce a novel iterative scheme for approximate fixed point within a fewer number of steps under weak contraction condition in Banach spaces (BSs). Moreover, we present a problem raised from the modeling of electrical circuits as an application of our proposed procedure. Also, to demonstrate the performance and validity of our iterative scheme, we presented numerical examples and figures.

Keywords: Strong convergence results, new iterative scheme, mean value theorem, numerical experiments.

1 Introduction and preparatory iterations

In our life scientists used numerical analysis to approximate solutions of such important problems this is the reason why the numerical analysis is one of the most important subjects in science. One of the important branches of numerical analysis is the iterative methods for approximating fixed points (FPs) and studying their strong convergence. The iterative methods depend on two main pillars, the first is the number of iterations and the second is specific to time, when the number of iterations is small in a short time, the method is successful, effective and better than its counterpart in approximation.

From this standpoint, the approximation methods branched into many disciplines, including convex feasibility problems, convex optimization problems, monotone variational inequalities, image restoration problems, and SLBR model of computer virus over the internet and more see [1,2,3,4,5,6,7].

Functional equations arises from many problems in engineering and applied sciences. Such equations can be transferred to FP theorems in an easy manner. Moreover, we use the FP theory to prove the existence and uniqueness of solutions of such integral and differential equations, for example the solution of the integral equation

$$\eta(t) = \eta_0 + \int_{t_0}^{t} \mho(e, \eta(e))de,$$

where $\mho : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function and η belong to the set of all continuous real-valued functions $C(I)$, I is an interval in $\mathbb{R}$ is equivalent to the FP of the mapping $U : C(I) \to C(I)$ which defined by

$$U\eta(t) = \eta_0 + \int_{t_0}^{t} \mho(e, \eta(e))de. \quad (1)$$

So, in this direction, height umber of mathematicians create FP of the mapping (1) and studied a generalized exciting results for it under mild stipulations in various spaces see [8,9,10,11,12,13].

Now we will summarize the iterative methods used in the following manner:

Let $(\mathfrak{R}, \|.\|)$ be a BS, $\mathfrak{I}, \Upsilon : C \to \mathfrak{R}$ for an arbitrary set C such that $\Upsilon(C) \subseteq \mathfrak{I}(C)$. The sequence $\{\mathfrak{I}\kappa_n\}_{n=0}^{\infty}$ is called:

– Jungck-Mann iteration process [14] if it generated by

$$\mathfrak{I}\kappa_{n+1} = (1-\rho_n)\mathfrak{I}\kappa_n + \rho_n\Upsilon\kappa_n,\ n \geq 0, \quad (2)$$

where $\{\rho_n\}_{n=0}^{\infty} \in [0,1]$.

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–Jungck-Ishikawa iteration process [15] if it generated by

$$\begin{cases} \Im\kappa_{n+1} = (1-\rho_n)\Im\kappa_n + \rho_n \Upsilon\eta_n, \\ \Im\eta_n = (1-\alpha_n)\Im\kappa_n + \alpha_n \Upsilon\kappa_n, \end{cases} \quad n \geq 0, \qquad (3)$$

where $\{\rho_n\}_{n=0}^{\infty}$ and $\{\alpha_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$.

–Jungck-Noor iteration process [16] if it generated by

$$\begin{cases} \Im\kappa_{n+1} = (1-\rho_n)\Im\kappa_n + \rho_n \Upsilon\eta_n, \\ \Im\eta_n = (1-\alpha_n)\Im\kappa_n + \alpha_n \Upsilon\nu_n, \quad n \geq 0, \\ \Im\nu_n = (1-\beta_n)\Im\kappa_n + \beta_n \Upsilon\kappa_n, \end{cases} \qquad (4)$$

where $\{\rho_n\}_{n=0}^{\infty}, \{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$.

It is obvious that, if we set $C = \Re$ and $\Im = I_{\Re}$, (where $I_{\Re}$ is the identity mapp) in (2), (3) and (4), we get Mann [17], Ishikawa [18] and Noor [19] iteration process respectively. Also the iterative scheme (4) reduces to (3) or (2), if we consider $\alpha_n = 0$ or $\alpha_n = \beta_n = 0$ respectively.

Further, Das and Debata [20] introduced the below scheme for accelerate the convergence rate of FP

$$\begin{cases} \kappa_1 = \kappa \in C, \\ \eta_n = \alpha_n' \Im\kappa_n + \beta_n' \kappa_n, \\ \kappa_{n+1} = \alpha_n \Upsilon\eta_n + \beta_n \kappa_n, \end{cases} \qquad (5)$$

where $\alpha_n + \beta_n = 1 = \alpha_n' + \beta_n'$.

Khan [21], generalized the iteration (5) to the three-step and estimating common FPs of two nonexpansive mappings under the name of strong convergence. It built as follows:

$$\begin{cases} \kappa_1 = \kappa \in C, \\ \kappa_{n+1} = \alpha_n \kappa_n + (1-\alpha_n)\eta_n, \\ \eta_n = \beta_n \kappa_n + (1-\beta_n)\Im\mu_n, \\ \mu_n = \gamma_n \kappa_n + (1-\gamma_n)\Upsilon\kappa_n. \end{cases} \qquad (6)$$

The definition below is very important for obtaining the strong convergence.

Definition 1.[22] *Let $(\Re, \|.\|)$ be a normed space, $\Upsilon : C \to \Re$ be a non-self operator for an arbitrary set C, then for all $\kappa, \mu \in C$, there is a sub-linear, monotone increasing function $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\theta(0) = 0$ and $\ell \in [0,1)$ so that*

$$\|\Upsilon\kappa - \Upsilon\mu\| \leq \theta(\|\kappa - \Upsilon\kappa\|) + \ell\|\kappa - \mu\|. \qquad (7)$$

According to the iterations (4) and (6), we present in this manuscript a new four-step iteration scheme to approximate FP in BSs. Also we present some numerical examples to show the efficiency and effectiveness of our new scheme.

2 Strong convergence for a new iterative scheme

This part is devoted to present a new generalized four-step iterative procedure to discuss the strong convergence of a coincidence point under a suitable contractive condition. Let C be a non-empty closed convex subset of a BS $\Re$ and $\Im, \Upsilon : C \to \Re$ with $\Upsilon(C) \subseteq \Im(C)$. We introduce the algorithm by the sequence $\{\kappa_n\}$ as follows:

$$\begin{cases} \kappa_0 \in C, \\ \Im\mu_n = (1-\gamma_n)\Im\kappa_n + \gamma_n \Upsilon\kappa_n, \\ \Im\nu_n = (1-\beta_n)\Im\kappa_n + \beta_n \Im\mu_n, \\ \Im\eta_n = (1-\alpha_n)\Upsilon(\Im\nu_n) + \alpha_n \Upsilon(\Im\mu_n), \\ \Im\kappa_{n+1} = (1-\rho_n)\Upsilon(\Im\nu_n) + \rho_n \Upsilon(\Im\eta_n). \end{cases} \qquad (8)$$

where

(i) $\{\gamma_n\}, \{\beta_n\}, \{\alpha_n\}$, and $\{\rho_n\}$ are sequences in $[\gamma, 1-\gamma], [\beta, 1-\beta], [\alpha, 1-\alpha]$, and $[\rho, 1-\rho]$ respectively.

(ii) for $\alpha, \beta, \gamma, \rho \in (0, \frac{1}{2})$, we have $U = (1-\rho) + \alpha(1-\beta) + \beta + \ell\rho(1-\alpha)(1-\beta) < 1$.

Now, we introduce our convergence theorem.

Theorem 1. *Let $\Im, \Upsilon : C \to \Re$ be non-self mappings on a BS $(\Re, \|.\|)$ so that $\Upsilon(C) \subseteq \Im(C)$. Consider q is a common FP of $\Im$ and Υ i.e., $\Upsilon q = \Im q = q$ and Υ satisfy the stipulation (7). For $\kappa_0 \in C$, let $\{\Im\kappa_n\}_{n=0}^{\infty}$ be a sequence defined by (8), where $\{\gamma_n\}, \{\beta_n\}, \{\alpha_n\}$, and $\{\rho_n\}$ are sequences satisfying (i) and (ii). Then $\{\Im\kappa_n\}_{n=0}^{\infty}$ converges strongly to the point q.*

Proof. Let q is a unique common FP of $\Im$ and Υ. We shall show that $\{\Im\kappa_n\}_{n=0}^{\infty}$ converges strongly to q. By condition (7), we get

$$\begin{aligned} \|\Im\kappa_{n+1} - q\| &= \|(1-\rho_n)\Upsilon(\Im\nu_n) + \rho_n\Upsilon(\Im\eta_n) - q\| \\ &\leq (1-\rho_n)\|\Upsilon(\Im\nu_n) - q\| + \rho_n\|\Upsilon(\Im\eta_n) - q\| \\ &= (1-\rho_n)\|\Upsilon(\Im\nu_n) - \Upsilon q\| + \rho_n\|\Upsilon(\Im\eta_n) - \Upsilon q\| \\ &\leq (1-\rho_n)[\theta(\|q - \Upsilon q\|) + \ell\|q - \Im\nu_n\|] \\ &\quad + \rho_n[\theta(\|q - \Upsilon q\|) + \ell\|q - \Im\eta_n\|] \\ &= \ell(1-\rho_n)\|\Im\nu_n - q\| + \ell\rho_n\|\Im\eta_n - q\|. \end{aligned} \qquad (9)$$

Now by (8),

$$\begin{aligned} \|\Im\eta_n - q\| &\leq \|(1-\alpha_n)(\Upsilon(\Im\nu_n) - q)\| + \|\alpha_n(\Upsilon(\Im\mu_n) - q)\| \\ &= (1-\alpha_n)\|\Upsilon(\Im\nu_n) - q\| + \alpha_n\|\Upsilon(\Im\mu_n) - q\| \\ &\leq (1-\alpha_n)[\theta(\|q - \Upsilon q\|) + \ell\|q - \Im\nu_n\|] \\ &\quad + \alpha_n[\theta(\|q - \Upsilon q\|) + \ell\|q - \Im\mu_n\|] \\ &= (1-\alpha_n)\ell\|\Im\nu_n - q\| + \alpha_n\ell\|\Im\mu_n - q\|. \end{aligned} \qquad (10)$$

Also by (8),

$$\|\Im\nu_n - q\| \leq (1-\beta_n)\|\Im\kappa_n - q\| + \beta_n\|\Im\mu_n - q\|. \qquad (11)$$

Applying (11) in (10) and (9), we have

$$\begin{aligned} &\|\Im\kappa_{n+1} - q\| \\ &\leq (1-\rho_n)\ell[(1-\beta_n)\|\Im\kappa_n - q\| + \beta_n\|\Im\mu_n - q\|] \\ &\quad + \rho_n\ell\|\Im\eta_n - q\| \\ &= \ell(1-\rho_n)(1-\beta_n)\|\Im\kappa_n - q\| \\ &\quad + \ell\beta_n(1-\rho_n)\|\Im\mu_n - q\| + \ell\rho_n\|\Im\eta_n - q\|, \end{aligned} \qquad (12)$$

and

$$\begin{aligned}
&\|\Im\eta_n - q\| \\
&\leq (1-\alpha_n)\ell \begin{bmatrix} (1-\beta_n)\|\Im\kappa_n - q\| \\ +\beta_n\|\Im\mu_n - q\| \end{bmatrix} \\
&\quad +\alpha_n\ell\|\Im\mu_n - q\| \\
&= \ell(1-\alpha_n)(1-\beta_n)\|\Im\kappa_n - q\| \\
&\quad +\ell(\beta_n(1-\alpha_n)+\alpha_n)\|\Im\mu_n - q\|. \qquad (13)
\end{aligned}$$

From (13) in (12), one can write

$$\begin{aligned}
&\|\Im\kappa_{n+1} - q\| \\
&\leq \ell(1-\rho_n)(1-\beta_n)\|\Im\kappa_n - q\| \\
&\quad +\ell\beta_n(1-\rho_n)\|\Im\mu_n - q\| \\
&\quad +\ell\rho_n \begin{bmatrix} \ell(1-\alpha_n)(1-\beta_n)\|\Im\kappa_n - q\| \\ +\ell\begin{pmatrix} \beta_n(1-\alpha_n) \\ +\alpha_n \end{pmatrix}\|\Im\mu_n - q\| \end{bmatrix} \\
&= \ell(1-\beta_n)\begin{bmatrix} (1-\rho_n) \\ +\ell\rho_n(1-\alpha_n) \end{bmatrix}\|\Im\kappa_n - q\| \\
&\quad +\ell\begin{pmatrix} \beta_n(1-\rho_n)+ \\ \beta_n(1-\alpha_n)+\alpha_n \end{pmatrix}\|\Im\mu_n - q\|. \qquad (14)
\end{aligned}$$

Again by (8) and (14), we get

$$\begin{aligned}
\|\Im\mu_n - q\| &\leq (1-\gamma_n)\|\Im\kappa_n - q\| + \gamma_n\|\Upsilon\kappa_n - q\| \\
&= (1-\gamma_n)\|\Im\kappa_n - q\| + \gamma_n\|\Upsilon\kappa_n - \Upsilon q\| \\
&\leq (1-\gamma_n)\|\Im\kappa_n - q\| + \ell\gamma_n\|\Im\kappa_n - q\| \\
&= (1-\gamma_n(1-\ell))\|\Im\kappa_n - q\| \qquad (15)
\end{aligned}$$

It follows from (15) in (14) that

$$\begin{aligned}
&\|\Im\kappa_{n+1} - q\| \\
&\leq \begin{pmatrix} \ell(1-\beta_n)\begin{bmatrix} (1-\rho_n) \\ +\ell\rho_n(1-\alpha_n) \end{bmatrix} \\ +\ell\begin{pmatrix} \beta_n(1-\rho_n) \\ +\beta_n(1-\alpha_n) \\ +\alpha_n \end{pmatrix}(1-\gamma_n(1-\ell)) \end{pmatrix}\|\Im\kappa_n - q\| \\
&\leq \begin{pmatrix} \ell(1-\beta_n)[(1-\rho_n)+\ell\rho_n(1-\alpha_n)] \\ +\ell(\beta_n(1-\rho_n)+\beta_n(1-\alpha_n)+\alpha_n) \end{pmatrix}\|\Im\kappa_n - q\| \\
&= \ell\begin{bmatrix} (1-\rho_n)+\alpha_n(1-\beta_n) \\ +\beta_n+\ell\rho_n(1-\alpha_n)(1-\beta_n) \end{bmatrix}\|\Im\kappa_n - q\| \\
&\leq [(1-\rho)+\alpha(1-\beta)+\beta+\ell\rho(1-\alpha)(1-\beta)]\|\Im\kappa_n - q\| \\
&= U\|\Im\kappa_n - q\| \\
&\leq U^n\|\Im\kappa_0 - q\| \to 0 \text{ as } n\to\infty. \qquad (16)
\end{aligned}$$

It follows from (16) that $\lim_{n\to\infty}\|\Im\kappa_{n+1} - q\| = 0$, i.e., the sequence $\{\Im\kappa_n\}_{n=0}^{\infty}$ converges strongly to q.

3 Some numerical experiments

In this part, we give some numerical examples to illustrate the performance of our algorithm.

Example1: Consider a fixed point problem taken from [23] wherein the Banach space $\Re = \mathbb{R}$ through the usual norm on $\mathbb{R}$ real numbers space. A mapping $\Upsilon : C \to C$ is defined by

$$\Upsilon(\kappa) = (5\kappa^2 - 2\kappa + 48)^{\frac{1}{3}}$$

where $C = \{\kappa : 0 \leq \kappa \leq 50\}$. Moreover, $\mathfrak{F} = I$, $\alpha = \beta = \frac{5}{2}$, $\gamma = \rho = \frac{1}{4}$ and $\alpha_n = \beta_n = \frac{1}{2}$, $\gamma_n = \rho_n = \frac{3}{4} - \frac{1}{4+n}$ and $\kappa_0 = 50$. In this experiment, we have provide the comparison of Jungck-Noor iteration process (JN_n) [16] and Khan [21], generalized the iteration (KN_n) (5) and for our proposed algorithm (HN_n). Numerical results are shown in Figures 1 and 2 and Table 1.

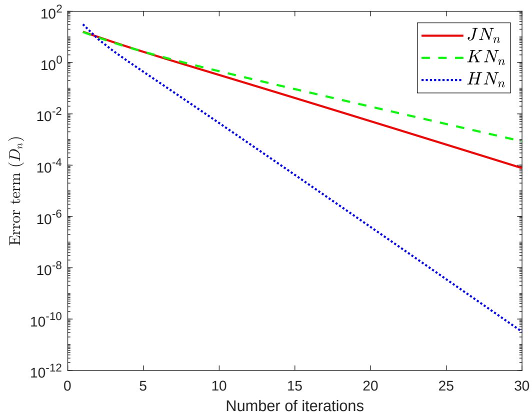

Fig. 1: Numerical behaviour of iteration sequence JN_n, KN_n and HN_n.

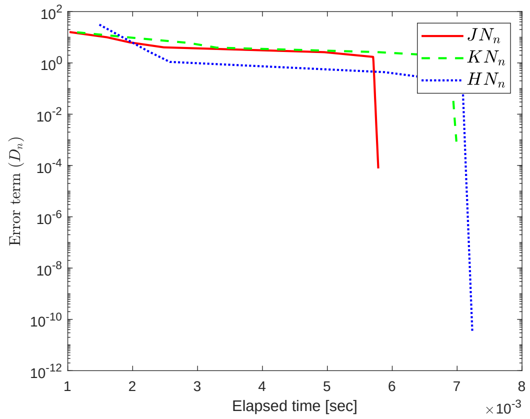

Fig. 2: Numerical behaviour of iteration sequence JN_n, KN_n and HN_n.

Table 1: Numerical comparative results of Example (1).

Iterations			Time in seconds		
JN_n	KN_n	HN_n	JN_n	KN_n	HN_n
34.0347359041649	34.4110729294678	18.9384404980078	0.001021	0.001143	0.001486
24.0453844618987	24.9070280633800	10.6717111752996	0.001604	0.002009	0.001905
17.7306126638334	18.8862718735572	7.82224976770748	0.001954	0.002861	0.002239
13.6867684277511	14.9475382881217	6.72595594523469	0.002478	0.003294	0.002571
11.0650079950452	12.3009616202608	6.29002992928956	0.0049532	0.005759	0.005881
9.34838437801081	10.4837410242492	6.11567153201474	0.0057053	0.006888	0.007079
8.21686097970758	9.21478183517224	6.04601678908814	0.0057106	0.006895	0.007089
7.46817757224950	8.31739693007599	6.01826218673631	0.0057141	0.006899	0.007095
6.97198920187550	7.67691607215996	6.00723192617061	0.0057174	0.006903	0.007101
6.64304460350676	7.21678262619221	6.00285851848182	0.0057206	0.006907	0.007107
6.42506947032072	6.88467454390853	6.00112802970075	0.0057207	0.006910	0.007114
6.28074202756228	6.64417616069234	6.00044451109818	0.0057267	0.006914	0.007120
6.18526614244217	6.46959953781067	6.00017494532316	0.0057297	0.006918	0.007125
6.12216532287317	6.34264919698940	6.00006877687214	0.0057327	0.006922	0.007131
6.08049860966701	6.25020617514437	6.00002701192236	0.0057356	0.006926	0.007137
6.05300814609534	6.18281827618756	6.00001059951659	0.0057390	0.006930	0.007144
6.03488466021744	6.13365179656770	6.00000415596658	0.0057424	0.006936	0.007151
6.02294488676860	6.09775366241140	6.00000162834350	0.0057454	0.006940	0.007157
6.01508400641595	6.07152702657616	6.00000063758216	0.0057485	0.006944	0.007164
6.00991162130003	6.05235602801519	6.00000024949794	0.0057515	0.006948	0.007170
6.00651007342166	6.03833597616918	6.00000009757980	0.0057547	0.006951	0.007176
6.00427419760786	6.02807864987641	6.00000003814467	0.0057581	0.006955	0.007183
6.00280519901692	6.02057142391450	6.00000001490410	0.0057617	0.006960	0.007189
6.00184045271503	6.01507513148867	6.00000000582090	0.0057650	0.006963	0.007196
6.00120711300908	6.01104989031300	6.00000000227248	0.0057684	0.006967	0.007202
6.00079148526016	6.00810116477533	6.00000000088685	0.0057717	0.006971	0.007208
6.00051882132880	6.00435691174675	6.00000000034598	0.0057749	0.006975	0.007214
6.00034000128762	6.00594050022444	6.00000000013493	0.0057782	0.006980	0.007221
6.00022276041118	6.00319601983588	6.00000000005261	0.0057817	0.006984	0.007227
6.00014591384677	6.00234482488425	6.00000000002050	0.0057849	0.006988	0.0072341

Example2: Consider a fixed point problem taken from [23] wherein the Banach space $\mathfrak{R} = \mathbb{R}$ through the usual norm on $\mathbb{R}$. A mapping $\Upsilon : C \to C$ is defined by

$$\Upsilon(\kappa) = (5\kappa^2 - 2\kappa + 48)^{\frac{1}{3}}$$

where $C = \{\kappa : 0 \leq \kappa \leq 50\}$. Moreover, $\mathfrak{F} = I$, $\alpha = \beta = \frac{5}{2}$, $\gamma = \rho = \frac{1}{4}$ and $\alpha_n = \beta_n = \frac{1}{2}$, $\gamma_n = \rho_n = \frac{3}{4} - \frac{1}{4+n}$ and $\kappa_0 = 50$. In this experiment, we have provide the comparison of Jungck-Noor iteration process (JN_n) [16] and Khan [21], generalized the iteration (KN_n) (5) and for our proposed algorithm (HN_n). Numerical results are shown in Figures 3 and 4 and Table 2.

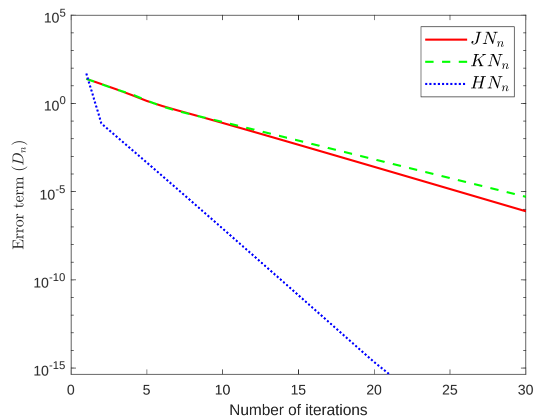

Fig. 3: Numerical behaviour of iteration sequence JN_n, KN_n and HN_n.

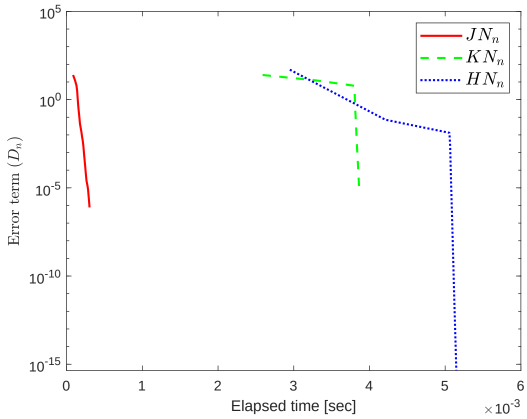

Fig. 4: Numerical behaviour of iteration sequence JN_n, KN_n and HN_n.

4 An important application

In this part, we discuss the unique solution of the initial value problem (ivp) below

$$\eta'(t) = \mho(t,\eta(t)),\ \eta(t_0) = \eta_0. \tag{17}$$

Also, to illustrate the good convergence of our iterative scheme, we shall present some numerical examples.

The lemma below is very important in the existence results.

Lemma 1.[24] *The ivp has a solution η iff*

$$\eta(t) = \eta_0 + \int_{t_0}^{t} \mho(e,\eta(e))de.$$

Assume that the superior norm $\|.\|_\infty$ on $C(I)$ defined by $\|r\|_\infty = \sup_{t\in I}|r(t)|$.

Now, we state and prove our main theorem in this part.

Theorem 2.*Consider the ivp (17) under two hypotheses below:*

(i) the function $\mho : I\times\mathbb{R}\to\mathbb{R}$ is continuous,

(ii) there is a constant $\rho\in(0,1)$ such that

$$\|\mho(t,\eta_1)-\mho(t,\eta_2)\| \le \rho\,|\eta_1-\eta_2|,\ \forall\eta_1,\eta_2\in\mathbb{R},\ t\in I. \tag{18}$$

Then the ivp has a unique solution $\eta\in C(I)$.

*Proof.*Assume that $\delta\ge 0$ so that $0\le\delta\rho<1$. Define a mapping Υ on $C(I)$ by

$$\Upsilon\eta(t) = \eta_0 + \int_{t_0}^{t} \mho(e,\eta(e))de. \tag{19}$$

It is obvious that, the unique fixed point of the mapping Υ (19) is equivalent to a unique solution of the ivp. Now, for all $\wp_1,\wp_2\in\mathbb{R}$ and $C_0 = [t_0, t_0+\delta]$, we have

$$\begin{aligned}\|\Upsilon\wp_1-\Upsilon\wp_2\| &= \sup_{t\in C_0}\left|\int_{t_0}^{t}\mho(e,\wp_1(e))de - \int_{t_0}^{t}\mho(e,\wp_2(e))de\right| \\ &\le \sup_{t\in C_0}\int_{t_0}^{t}|\mho(e,\wp_1(e))-\mho(e,\wp_2(e))|\,de \\ &\le \sup_{t\in C_0}\ell\,|\wp_1(t)-\wp_2(t)|\int_{t_0}^{t}de \\ &\le \delta\rho\,\|\wp_1-\wp_1\|.\end{aligned}$$

Hence, the stipulation (7) is fulfilled with $\ell=\delta\rho$ and $\theta=0$. Therefore the result holds.

As an application of first-order differential equations arises in the modeling of electrical circuits.The differential equation for the RL circuit in Figure 1 below was shown to be

$$L\frac{di}{dt} + Ri = E,$$

at the initial condition $i=0$ at $t=0$.

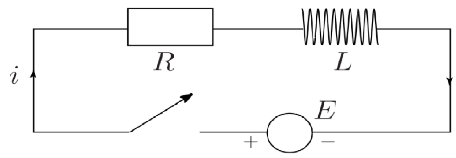

Figure 1

If $i(t_0)=i_0$, then we obtain the ivp below

$$i'(t) = \frac{1}{L}(E-Ri),\ i(t_0)=i_0. \tag{20}$$

Let $\mho(t,i(t)) = \frac{1}{L}(E-Ri)$, then we can prove that $\mho$ fulfill the stipulation (18), so according to Theorem 2, the ivp (20) has a unique solution.

We know that the exact solution of (20) is

$$i(t) = \frac{E}{R} + (i_0 - \frac{E}{R})e^{-\frac{R}{L}(t-t_0)}. \tag{21}$$

Example3: A series RL circuit with $R=40\Omega$ and $L=50H$ has a constant voltage $E=100V$ applied at $t=0$ by the closing of a switch. Find the relation between the current i and the time t?

It follows from (18) that $i(t) = 2.5(1-e^{-0.8t})$, also at $t=0$, $i_0=0$.

Table 2: Numerical comparative results of Example (2).

Iterations			Time in seconds		
JN_n	KN_n	HN_n	JN_n	KN_n	HN_n
25.1086707621037	25.1064528825808	0.211409752130777	0.000005	0.002143	0.002481
12.5994507894276	12.8485550187259	0.283693247130497	0.000110	0.002005	0.002909
6.35118074597571	6.71924043390885	0.296836425971939	0.000110	0.002861	0.002233
3.36058436520358	3.40598799025514	0.299250018207766	0.000110	0.003293	0.002572
1.98299257836461	1.95440846039890	0.299689979220810	0.000130	0.003751	0.002889
1.26183207672385	1.27835108900664	0.299769521236279	0.000139	0.003886	0.002072
0.853593995441549	0.902703256715970	0.299783802892244	0.000144	0.003894	0.002087
0.617098548212626	0.673820141791979	0.299786352741020	0.000149	0.003899	0.005092
0.480527329189438	0.531195251593426	0.299786805887070	0.000154	0.003902	0.005105
0.402282171733520	0.442419193900076	0.299786886106902	0.000159	0.003904	0.005102
0.357734434165153	0.387453392774469	0.299786900261640	0.000165	0.003910	0.005115
0.332480189437920	0.353575059103850	0.299786902752222	0.000170	0.003914	0.005121
0.318204539521391	0.332757183378794	0.299786903189382	0.000175	0.003918	0.005126
0.310150811116510	0.319987762926825	0.299786903265951	0.000185	0.003928	0.005134
0.305613723488004	0.312162594198763	0.299786903279337	0.000195	0.003921	0.005137
0.303060514052061	0.307369264735864	0.299786903281673	0.000203	0.003937	0.005142
0.301624953871998	0.304433352085335	0.299786903282080	0.000213	0.003937	0.005157
0.300818375182631	0.302634923680678	0.299786903282151	0.000220	0.003941	0.005155
0.300365469478084	0.301533048714839	0.299786903282163	0.000225	0.003942	0.005163
0.300111291542791	0.300857769835742	0.299786903282166	0.000230	0.003942	0.005171
0.299968710266453	0.300443813108807	0.299786903282166	0.000235	0.003952	0.005172
0.299888763264821	0.300189979455609	0.299786903282166	0.000241	0.003955	0.005183
0.299843953387920	0.300034288564422	0.299786903282166	0.000246	0.003964	0.005184
0.299818846582224	0.299938769074743	0.299786903282166	0.000251	0.003964	0.005196
0.299804783918478	0.299880151266729	0.299786903282166	0.000257	0.003963	0.005203
0.299796909601833	0.299844170565673	0.299786903282166	0.000262	0.003973	0.005208
0.299790034771977	0.299822080049421	0.299786903282166	0.000274	0.003975	0.005215
0.299792501650538	0.299808514658860	0.299786903282166	0.000283	0.003980	0.005221
0.299788654535853	0.299800182773454	0.299786903282166	0.000288	0.003954	0.005225
0.299787882459126	0.299795064381028	0.299786903282166	0.000293	0.003988	0.005234

Let

$$\Upsilon i(t) = i_0 + \int_{t_0}^{t} \mho(e,i(e))de = \int_{0}^{t} \frac{1}{L}(E - Ri)\,de$$
$$= \int_{0}^{t} -0.8\,(i(e) - 2.5)\,de,$$

and $\mho(e,i(e)) = 0.8\,(i(e) - 2.5)$, then $\mho$ fulfill the stipulation (18). So by Theorem 2, Υ has a unique fixed point.

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Conflict of interest

The authors declare that there is no conflict regarding the publication of this paper.

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