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New Aspects of Caputo-Fabrizio Fractional Derivative

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Abstract: In this paper, we consider classes of linear and nonlinear fractional differential equations involving the Caputo-Fabrizio fractional derivative of non-singular kernel. We transform the fractional problems to equivalent initial value problems with integer derivatives. We illustrate the obtained results by presenting two mathematical models of fractional differential equations and their equivalent initial value problems. We show that it is impossible to convert all types of linear fractional differential equations to the integer ones.The obtained results will lead to better understanding of fractional models, as the solutions of their equivalent models can be studied analytically and numerically using well-known techniques of differential equations.

Keywords: Fractional differential equations, initial value problems, Caputo-Fabrizio fractional operator.

1 Introduction

The Caputo-Fabrizio fractional derivative has added a new dimension in the study of fractional differential equations. The beauty of the new derivative is that, it has a nonsingular kernel [1]. The Caputo-Fabrizio derivative is designed with the convolution of an ordinary derivative and an exponential function but, it has the same supplementary motivating properties of heterogeneous and configuration [2,3,4] with different scales as it is in the Caputo and Riemann-Liouville fractional derivatives. Many results about the new Caputo-Fabrizio fractional derivative have been developed in the last two years. For instance; the corresponding fractional integral as well as the solutions of several linear fractional differential equations were discussed in [5]. Maximum principle theory of boundary-value problems, which plays an important role of solving many fractional diffusion equations, was discussed in [6,7,8]. The existence and uniqueness results for fractional boundary value problems were studied by many authors using different techniques [2,6,9,10]. A reduction of order formula and a fundamental set of solutions results were established in [11] for a class of linear fractional differential equations. Also, numerical techniques of Caputo-Fabrizio fractional models were examined by many authors. For instance; the numerical solutions for groundwater pollution [12], for the model of RLC circuit [13], for the model of wave movement on the surface of shallow water [14], and for the heat transfer model [15,16], were discussed. Caputo-Fabrizio fractional derivative has been implemented in many areas of mathematical modeling to model real world problems. For instance, Mass-spring-damper motion model [4,17], non-linear Fisher's-diffusion equation [7], Elasticity model [18], Liénard model for fluid transmission line [19] and the Korteweg-de Vries-Burgers functional differential equation [14,20].
The aim of this paper is, to convert Caputo-Fabrizio fractional differential equations to equivalent initial value problems with integer derivatives, without losing the non-locality of the real world phenomena.

The paper is organized as follows. In Section 2, we discuss a class of fractional differential equations of variable coefficients of order $0 < \alpha < 1$, and its equivalent initial value problem of order 2. In section 3, we extend the results for a class of higher order fractional differential equations with variable coefficients. In Section 4, we present two applications of the new results. Finally, we close up with some concluding remarks in section 5.

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2 Fractional differential equations with variable coefficients

In this section, we show that a class of linear and nonlinear fractional differential equations with variable coefficients can be transformed to initial value problems with integer derivatives. We start with the definition and main properties of the Caputo-Fabrizio fractional derivative.

Definition 1.[1] *Let $f \in H^1(a,b)$, $a < b$, $a \in (-\infty, t)$, $0 < \alpha < 1$, the Caputo-Fabrizio fractional derivative in the Caputo sense is defined by*

$$({}^{CFC}D_a^{\alpha} f)(t) = \frac{B(\alpha)}{1-\alpha}\int_a^t f'(s) e^{-\frac{\alpha}{1-\alpha}(t-s)} ds, \tag{1}$$

where $B(\alpha) > 0$ is a normalization function satisfying $B(0) = B(1) = 1$.

The corresponding fractional integral is defined by, see [5]

$$({}^{CFC}I_a^{\alpha} f)(t) = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)}\int_a^t f(s) ds, \quad 0 < \alpha < 1. \tag{2}$$

The relation between the Caputo-Fabrizio fractional derivative and the corresponding integral is given by

$$({}^{CFC}I_a^{\alpha})({}^{CFC}D_a^{\alpha}) f(t) = f(t) - f(a). \tag{3}$$

For more about the Caputo-Fabrizio fractional derivatives we refer the readers to [1,5,9,21].

Lemma 1. *Let $z \in H^1(a,b)$ and consider the linear fractional differential equation*

$$k_0(t)({}^{CFC}D_a^{\alpha} z)(t) + k_1(t) z'(t) + k_2(t) z(t) = f(t), \quad 0 < \alpha < 1, \tag{4}$$

where $k_0, k_1, k_2, f \in C[a,b]$, and ${}^{CFC}D_a^{\alpha}$ is the Caputo-Fabrizio fractional derivative in the Caputo sense. This fractional differential equation is equivalent to the initial value problem

$$c_2(t) v''(t) + c_1(t) v'(t) + c_0(t) v(t) = h(t), \tag{5}$$

$$v(a) = 0, v'(a) = e^{\mu_\alpha a} z(a), \tag{6}$$

where $v(t) = \int_a^t e^{\mu_\alpha s} z(s) ds$, $\mu_\alpha = \frac{\alpha}{1-\alpha}$, and

$$h(t) = (1-\alpha) e^{\mu_\alpha t} f(t) + k_0(t) B(\alpha) v'(a), \tag{7}$$

$$c_2(t) = (1-\alpha) k_1(t), \tag{8}$$

$$c_1(t) = k_0(t) B(\alpha) + (1-\alpha)(k_2(t) - \mu_\alpha k_1(t)), \tag{9}$$

$$c_0(t) = -\mu_\alpha k_0(t) B(\alpha). \tag{10}$$

Proof. We have

$$({}^{CFC}D_a^{\alpha} z)(t) = \frac{B(\alpha)}{1-\alpha}\int_a^t z'(s) e^{-\frac{\alpha}{1-\alpha}(t-s)} ds = \frac{B(\alpha)}{1-\alpha} e^{-\mu_\alpha t}\int_a^t e^{\mu_\alpha s} z'(s) ds. \tag{11}$$

Integrating by parts yields

$$({}^{CFC}D_a^{\alpha} z)(t) = \frac{B(\alpha)}{1-\alpha}[z(t) - e^{-\mu_\alpha(t-a)} z(a) - \mu_\alpha e^{-\mu_\alpha t}\int_a^t e^{\mu_\alpha s} z(s) ds]. \tag{12}$$

We have $v(t) = \int_a^t e^{\mu_\alpha s} z(s) ds$, and since $z \in H^1(a,b)$, then z is continuous on (a,b). Applying the Fundamental Theorem of Calculus, it holds that $v'(t) = e^{\mu_\alpha t} z(t)$, $a < t < b$. We have

$$z(t) = e^{-\mu_\alpha t} v'(t), \; z'(t) = e^{-\mu_\alpha t}(-\mu_\alpha v'(t) + v''(t)), \; v(a) = 0, \text{and } v'(a) = e^{\mu_\alpha a} z(a).$$

Therefore,

$$\begin{aligned}({}^{CFC}D_a^{\alpha} z)(t) &= \frac{B(\alpha)}{1-\alpha}[e^{-\mu_\alpha t} v'(t) - e^{-\mu_\alpha t} v'(a) - \mu_\alpha e^{-\mu_\alpha t} v(t)], \\ &= e^{-\mu_\alpha t}\frac{B(\alpha)}{1-\alpha}[v'(t) - v'(a) - \mu_\alpha v(t)]. \end{aligned} \tag{13}$$

By substituting in equation (4) we get

$$e^{-\mu_\alpha t}\left(k_0(t)\frac{B(\alpha)}{1-\alpha}[v'(t)-v'(a)-\mu_\alpha v(t)]+k_1(t)[-\mu_\alpha v^{'}(t)+v^{''}(t)]+k_2(t)v'(t)\right)=f(t).$$

Multiplying the above equation by $(1-\alpha)e^{\mu_\alpha t}$ and arranging the terms yields

$$(1-\alpha)k_1(t)v^{''}(t)+\left(k_0(t)B(\alpha)+(1-\alpha)(k_2(t)-\mu_\alpha k_1(t))\right)v^{'}(t)-\mu_\alpha k_0(t)B(\alpha)v(t)$$
$$=(1-\alpha)e^{\mu_\alpha t}f(t)+k_0(t)B(\alpha)v'(a),$$

which proves the result.

*Remark.*In equation (4) if the coefficients k_0,k_1 and k_2 are all constants, then the fractional differential equation reduces to the second order initial value problem with constant coefficients

$$c_2v''+c_1v'(t)+c_0v(t)=h(t),\ v(a)=0,v'(a)=e^{\mu_\alpha a}z(a).$$

This special case has been discussed in [17].

*Remark.*In equation (4) if we choose $k_1(t)=0$, then the fractional differential equation reduces to the first order initial value problem

$$[(1-\alpha)k_2(t)+k_0(t)B(\alpha)]v'(t)-\mu_\alpha B(\alpha)k_0(t)v(t)=e^{\mu_\alpha t}(1-\alpha)f(t)+k_0(t)B(\alpha)v'(a),$$
$$v(a)=0.$$

*Remark.*We remark here that, since $v(t)=\int_a^t e^{\mu_s}z(s)ds$, and $z\in H^1(a,b)$, then $v\in H^2(a,b)$.

*Remark.*Using equation (12) one can define the Caputo-Fabrizio fractional derivative for a more wider space, such as $L^1(a,b)$, see [18] for more details. However, the restriction in the space we have considered, which is $H^1(a,b)$, comes from the appearance of the term z' in Eq. (4). This constrain doesn't allow us to consider a more wider space.

Following analogous steps in the proof of Lemma 1 we have

Lemma 2.*Let $z\in H^1(a,b)$ and consider the nonlinear fractional differential equation*

$$k_0(t)({}^{CFC}D_a^\alpha z)(t)+k_1(t)z^{'}(t)+k_2(t)z(t)=f(t,z(t)),\quad 0<\alpha<1, \tag{14}$$

where $k_0,k_1,k_2,f\in C[a,b]$, and ${}^{CFC}D_a^\alpha$ is the Caputo-Fabrizio fractional derivative in the Caputo sense. This fractional differential equation is equivalent to the initial value problem

$$c_2(t)v''(t)+c_1(t)v'(t)+c_0(t)v(t)=(1-\alpha)e^{\mu_\alpha t}f(t,e^{-\mu_\alpha t}v'(t))+k_0(t)B(\alpha)v'(a),$$
$$v(a)=0,v'(a)=e^{\mu_\alpha a}z(a),$$

where $v(t),\mu_\alpha,c_2(t),c_1(t),c_0(t)$ are as defined in Lemma 1.

3 Higher order fractional differential equations

We extend the results obtained in Section 2 to a class of higher order fractional differential equations. We start with the following result.

Lemma 3.*Let $f\in H^2(a,b)$ and $0<\alpha<1$, then it holds that,*

$$({}^{CFC}D_a^{\alpha+1}z)(t)=\frac{B(\alpha)}{1-\alpha}[z^{'}(t)-e^{-\mu_\alpha(t-a)}z'(a)]-\mu_\alpha({}^{CFC}D_a^\alpha z)(t),\quad \mu_\alpha=\frac{\alpha}{1-\alpha}. \tag{15}$$

*Proof.*We have

$$({}^{CFC}D_a^{\alpha+1}z)(t)=\frac{B(\alpha)}{1-\alpha}\int_a^t z''(s)e^{-\frac{\alpha}{1-\alpha}(t-s)}ds,$$
$$=\frac{B(\alpha)}{1-\alpha}e^{-\mu_\alpha t}\int_a^t e^{\mu_\alpha s}z''(s)ds,\ \ \mu_\alpha=\frac{\alpha}{1-\alpha}.$$

Integration by parts yields

$$\begin{aligned}({}^{CFC}D_a^{\alpha+1}z)(t) &= \frac{B(\alpha)}{1-\alpha}e^{-\mu_\alpha t}[e^{\mu_\alpha s}z^{'}(s)|_a^t - \mu_\alpha\int_a^t e^{\mu_\alpha s}z^{'}(s)ds],\\ &= \frac{B(\alpha)}{1-\alpha}[z^{'}(t) - e^{-\mu_\alpha(t-a)}z^{'}(a)] - \mu_\alpha({}^{CFC}D_a^{\alpha}z)(t),\end{aligned}$$

which proves the result.

Lemma 4. *Let $z \in H^2(a,b)$ and consider the linear fractional differential equation*

$$k_0(t)({}^{CFC}D_a^{\alpha+1}z)(t) + k_1(t)({}^{CFC}D_a^{\alpha}z)(t) + k_2(t)z(t) = f(t), \quad 0<\alpha<1, \tag{16}$$

where $k_0, k_1, k_2, f \in C[a,b]$, and ${}^{CFC}D_a^{\alpha}$ is the Caputo-Fabrizio fractional derivative in the Caputo sense. This fractional differential equation is equivalent to the initial value problem

$$\begin{aligned}d_2(t)v''(t) + d_1(t)v'(t) + d_0(t)v(t) &= g(t),\\ v(a) = 0, v'(a) = e^{\mu_\alpha a}z(a),\end{aligned} \tag{17}$$

where $v(t) = \int_a^t e^{\mu_\alpha s}z(s)ds$, $\mu_\alpha = \frac{\alpha}{1-\alpha}$, *and*

$$g(t) = \frac{(1-\alpha)}{B(\alpha)}e^{\mu_\alpha t}f(t) + B(\alpha)[k_0(t)v''(a) - (2\mu_\alpha k_0(t) - k_1(t))v'(a)], \tag{18}$$

$$d_2(t) = k_0(t), \tag{19}$$

$$d_1(t) = k_1 - 2\mu_\alpha k_0 + k_2(1-\alpha), \tag{20}$$

$$d_0(t) = \mu_\alpha(\mu_\alpha k_0 - k_1). \tag{21}$$

Proof. We have $v'(t) = e^{\mu_\alpha t}z(t)$, $z(t) = e^{-\mu_\alpha t}v'(t)$, $z'(t) = e^{-\mu_\alpha t}(v''(t) - \mu_\alpha v'(t))$, and from Eq. (13) it holds that

$$({}^{CFC}D_a^{\alpha}z)(t) = \frac{e^{-\mu_\alpha t}B(\alpha)}{1-\alpha}[v'(t) - v'(a) - \mu_\alpha v(t)]. \tag{22}$$

Substituting in Eq. (15) yields

$$({}^{CFC}D_0^{\alpha+1}z)(t) = \frac{e^{-\mu_\alpha t}B(\alpha)}{1-\alpha}[v^{''}(t) - 2\mu_\alpha v^{'}(t) + \mu_\alpha^2 v(t) - v''(a) + 2\mu_\alpha v'(a)]. \tag{23}$$

By substituting the above results in Eq. (16) we have

$$\begin{aligned}k_0(t)\frac{e^{-\mu_\alpha t}B(\alpha)}{1-\alpha}[v^{''}(t) - 2\mu_\alpha v^{'}(t) + \mu_\alpha^2 v(t) - v''(a) + 2\mu_\alpha v'(a)] + k_1(t)\frac{e^{-\mu_\alpha t}B(\alpha)}{1-\alpha}[v'(t) - v'(a) - \mu_\alpha v(t)]\\ + k_2(t)e^{-\mu_\alpha t}v'(t) = f(t).\end{aligned}$$

The last equation yields

$$\begin{aligned}&k_0(t)v^{''}(t) + [k_1 - 2\mu_\alpha k_0 + k_2(1-\alpha)]v^{'}(t) + [k_0\mu_\alpha^2 - k_1\mu_\alpha]v(t)\\ &= \frac{(1-\alpha)}{B(\alpha)}e^{\mu_\alpha t}f(t) + B(\alpha)[k_0(t)v''(a) - (2\mu_\alpha k_0(t) - k_1(t))v'(a)],\end{aligned}$$

which proves the result.

Lemma 5. *Let $z \in H^{n+1}(a,b)$ and $0<\alpha<1$, then it holds that*

$$\begin{aligned}({}^{CFC}D_a^{\alpha+n}z)(t) &= \frac{B(\alpha)}{1-\alpha}\left(\sum_{k=0}^{n-1}(-1)^k\mu_\alpha^k z^{(n-k)}(t) - e^{-\mu_\alpha(t-a)}\sum_{k=0}^{n-1}(-1)^k\mu_\alpha^k z^{(n-k)}(a)\right)\\ &+ (-1)^n\mu_\alpha^n({}^{CFC}D_a^{\alpha})z(t), \quad \text{where} \quad n\ge 1, \quad \mu_\alpha = \frac{\alpha}{1-\alpha}.\end{aligned} \tag{24}$$

*Proof.*We apply mathematical induction to prove the Lemma. For $n=1$ Eq. (24) yields

$$({}^{CFC}D_a^{\alpha+1}z)(t)=\frac{B(\alpha)}{1-\alpha}\big(z'(t)-e^{-\mu_\alpha(t-a)}z'(a)\big)-\mu_\alpha({}^{CFC}D_a^{\alpha}z)(t). \tag{25}$$

This result is true by virtue of Lemma 3. Assume (24) holds true for $k=n$, and want to show that it is true for $k=n+1$, i.e;

$$\begin{aligned}({}^{CFC}D_a^{\alpha+(n+1)}z)(t)&=\frac{B(\alpha)}{1-\alpha}\left(\sum_{k=0}^{n}(-1)^k\mu_\alpha^k z^{(n+1-k)}(t)-e^{-\mu_\alpha(t-a)}\sum_{k=0}^{n}(-1)^k\mu_\alpha^k z^{(n+1-k)}(a)\right)\\&\quad+(-1)^{n+1}\mu_\alpha^{n+1}({}^{CFC}D_a^{\alpha})z(t).\end{aligned} \tag{26}$$

Now integrating by parts of $({}^{CFC}D_a^{\alpha+(n+1)}z)(t)$ yields

$$\begin{aligned}({}^{CFC}D_a^{\alpha+(n+1)}z)(t)&=\frac{B(\alpha)}{1-\alpha}e^{-\mu_\alpha t}\int_a^t e^{\mu_\alpha s}z^{(n+2)}(s)ds,\\&=\frac{B(\alpha)}{1-\alpha}e^{-\mu_\alpha t}\left(e^{\mu_\alpha s}z^{(n+1)}(s)|_a^t-\mu_\alpha\int_a^t e^{\mu_\alpha s}z^{(n+1)}(s)ds\right),\\&=\frac{B(\alpha)}{1-\alpha}e^{-\mu_\alpha t}\left(e^{\mu_\alpha t}z^{(n+1)}(t)-e^{\mu_\alpha a}z^{(n+1)}(a)\right)-\mu_\alpha({}^{CFC}D_a^{\alpha+n}z)(t),\\&=\frac{B(\alpha)}{1-\alpha}\big(z^{(n+1)}(t)-z^{(n+1)}(a)\big)-\mu_\alpha({}^{CFC}D_a^{\alpha+n}z)(t).\end{aligned}$$

Applying the induction hypothesis we have

$$\begin{aligned}({}^{CFC}D_a^{\alpha+(n+1)}z)(t)&=\frac{B(\alpha)}{1-\alpha}\left(z^{(n+1)}(t)-z^{(n+1)}(a)\right)-\mu_\alpha\left(\frac{B(\alpha)}{1-\alpha}(\sum_{k=0}^{n-1}(-1)^k\mu_\alpha^k z^{(n-k)}(t)\right.\\&\qquad\left.-e^{-\mu_\alpha(t-a)}\sum_{k=0}^{n-1}(-1)^k\mu_\alpha^k z^{(n-k)}(a))+(-1)^n\mu_\alpha^n({}^{CFC}D_a^{\alpha})z(t)\right),\\&=\frac{B(\alpha)}{1-\alpha}\left(z^{(n+1)}(t)-\mu_\alpha\sum_{k=0}^{n-1}(-1)^k\mu_\alpha^k z^{(n-k)}(t)-z^{(n+1)}(a)\right.\\&\qquad\left.+\mu_\alpha e^{-\mu_\alpha(t-a)}\sum_{k=0}^{n-1}(-1)^k\mu_\alpha^k z^{(n-k)}(a)\right)-(-1)^n\mu_\alpha^n({}^{CFC}D_a^{\alpha})z(t).\end{aligned} \tag{27}$$

We have

$$\begin{aligned}z^{(n+1)}(t)-\mu_\alpha\sum_{k=0}^{n-1}(-1)^k\mu_\alpha^k z^{(n-k)}(t)&=z^{(n+1)}(t)+\sum_{k=0}^{n-1}(-1)^{k+1}\mu_\alpha^{k+1}z^{(n-k)}(t)\\&=z^{(n+1)}(t)+\sum_{k=1}^{n}(-1)^k\mu_\alpha^k z^{(n+1-k)}(t)\\&=\sum_{k=0}^{n}(-1)^k\mu_\alpha^k z^{(n+1-k)}(t),\end{aligned}$$

and

$$z^{(n+1)}(a)-\mu_\alpha\sum_{k=0}^{n-1}(-1)^k\mu_\alpha^k z^{(n-k)}(a)=\sum_{k=0}^{n}(-1)^k\mu_\alpha^k z^{(n+1-k)}(a).$$

Thus,

$$\begin{aligned}({}^{CFC}D_a^{\alpha+(n+1)}z)(t)&=\frac{B(\alpha)}{1-\alpha}\left(\sum_{k=0}^{n}(-1)^k\mu_\alpha^k z^{(n+1-k)}(t)-e^{-\mu_\alpha(t-a)}\sum_{k=0}^{n}(-1)^k\mu_\alpha^k z^{(n+1-k)}(a)\right)\\&\quad+(-1)^{n+1}\mu_\alpha^{n+1}({}^{CFC}D_a^{\alpha})z(t),\end{aligned}$$

which proves the result for $k=n+1$.

Lemma 6. *Let $z \in H^n(a,b)$ and $0<\alpha<1$, and consider the linear fractional differential equation*

$$c_0(t)({}^{CFC}D_a^\alpha z)(t)+c_1(t)z'(t)+c_2(t)z''(t)+\cdots+c_n z^{(n)}(t)+c_{n+1}(t)z(t)=g(t), \tag{28}$$

where $c_0,c_1,\cdots,c_{n+1},g \in C[a,b]$, and ${}^{CFC}D_a^\alpha$ is the Caputo-Fabrizio fractional derivative in the Caputo sense. This fractional differential equation is equivalent to a linear initial value problem of order $(n+1)$.

Proof. We have

$$\begin{aligned}({}^{CFC}D_a^\alpha z)(t) &= \frac{B(\alpha)}{1-\alpha}e^{-\mu_\alpha t}\int_a^t z'(s)e^{\mu_\alpha s}ds, \quad where \quad 0<\alpha<1, \quad \mu_\alpha=\frac{\alpha}{1-\alpha}\\ &= \frac{B(\alpha)}{1-\alpha}[z(t)-e^{-\mu_\alpha t}z(a)-\mu_\alpha e^{-\mu_\alpha t}\int_a^t e^{\mu_\alpha s}z(s)ds]\end{aligned}$$

Let $v(t)=\int_a^t e^{\mu_\alpha s}z(s)ds$, then, $v'(t)=e^{\mu_\alpha t}z(t)$, and $z(t)=e^{-\mu_\alpha t}v'(t)$. Substituting in Eq. (28) yields

$$c_0(t)\frac{B(\alpha)}{1-\alpha}e^{-\mu_\alpha t}[v'(t)-e^{-\mu_\alpha a}v'(a)-v(t)]+\sum_{k=1}^n c_k(t)(e^{-\mu_\alpha t}v'(t))^{(k)}+c_{n+1}(t)e^{-\mu_\alpha t}v'(t)=g(t).$$

Since

$$\begin{aligned}\left(e^{-\mu_\alpha t}v'(t)\right)^{(k)} &= \sum_{j=0}^k \binom{k}{j}(e^{-\mu_\alpha t})^{(j)}(v')^{(k-j)}(t)\\ &= e^{-\mu_\alpha t}\sum_{j=0}^k \binom{k}{j}(-\mu_\alpha)^{(j)}(v)^{(k-j+1)}(t),\end{aligned}$$

we have

$$\begin{aligned}c_0(t)\frac{B(\alpha)}{1-\alpha}[e^{-\mu_\alpha t}v'(t)-e^{-\mu_\alpha(t+a)}v'(a)-\mu_\alpha e^{-\mu_\alpha t}v(t)] + \sum_{k=1}^n c_k(t)e^{-\mu_\alpha t}\sum_{j=0}^k\binom{k}{j}(-\mu_\alpha)^{(j)}(v)^{(k-j+1)}(t)\\ + c_{n+1}(t)e^{-\mu_\alpha t}v'(t)=g(t),\end{aligned}$$

or,

$$\begin{aligned}c_0(t)\frac{B(\alpha)}{1-\alpha}[v'(t)-\mu_\alpha v(t)]+\sum_{k=1}^n c_k(t)\sum_{j=0}^k\binom{k}{j}(-\mu_\alpha)^{(j)}(v)^{(k-j+1)}(t)+c_{n+1}(t)v'(t)\\ = g(t)e^{\mu_\alpha t}+c_0(t)\frac{B(\alpha)}{1-\alpha}e^{-\mu_\alpha a}v'(a),\end{aligned}$$

which is an ordinary differential equation of order $(n+1)$ of the function $v(t)$. The initial conditions $v(a),v'(a),\cdots,v^{(n-1)}(a)$, follow from $v(t)=\int_a^t e^{\mu_\alpha s}z(s)ds$.

Lemma 7. *Let $z \in H^n(a,b)$ and $0<\alpha<1$, and consider the linear fractional differential equation*

$$k_0({}^{CFC}D_a^{\alpha+n}z)(t)+k_1({}^{CFC}D_a^{\alpha+n-1}z)(t)+\cdots+k_n({}^{CFC}D_a^\alpha z)(t)+k_{n+1}z(t)=h(t), \tag{29}$$

where $k_0,k_1,\cdots,k_{n+1},h \in C[a,b]$, and ${}^{CFC}D_a^\alpha$ is the Caputo-Fabrizio fractional derivative in the Caputo sense. This fractional differential equation is equivalent to a linear initial value problem of order $(n+1)$.

Proof. We show that Eq. (29) can be transformed to have the same form as in Eq. (28), and hence the result follows from Lemma 6. From Eq. (24) we have

$$\begin{aligned}({}^{CFC}D_a^{\alpha+n}z)(t) &= \frac{B(\alpha)}{1-\alpha}\sum_{m=0}^{n-1}(-1)^m\mu_\alpha^m[z^{(n-m)}(t)-e^{-\mu_\alpha(t-a)}z^{(n-m)}(a)]+(-1)^n\mu_\alpha^n({}^{CFC}D_a^\alpha z)(t)\\ &= (-1)^n\mu_\alpha^n({}^{CFC}D_a^\alpha z)(t)+\sum_{m=0}^{n-1}C_m^n,\end{aligned}$$

where

$$C_m^n = \frac{B(\alpha)}{1-\alpha}(-1)^m \mu_\alpha^m [z^{(n-m)}(t) - e^{-\mu_\alpha(t-a)} z^{(n-m)}(a)]. \tag{30}$$

Thus,

$$\begin{aligned}
h(t) &= k_0({}^{CFC}D_a^{\alpha+n}z)(t) + k_1({}^{CFC}D_a^{\alpha+n-1}z)(t) + k_2({}^{CFC}D_a^{\alpha+n-2}z)(t) + \cdots + k_n({}^{CFC}D_a^{\alpha}z)(t) \\
&\quad + k_{n+1}z(t) \\
&= \sum_{j=0}^{n} k_j({}^{CFC}D_a^{\alpha+n-j}z)(t) + k_{n+1}z(t) \\
&= \sum_{j=0}^{n} k_j\left((-1)^{n-j}\mu_\alpha^{n-j}({}^{CFC}D_a^{\alpha}z)(t) + \sum_{m=0}^{n-j-1} C_m^{n-j}\right) + k_{n+1}z(t) \\
&= \sum_{j=0}^{n} k_j(-1)^{n-j}\mu_\alpha^{n-j}({}^{CFC}D_a^{\alpha}z)(t) + \sum_{j=0}^{n}\sum_{m=0}^{n-j-1} k_j C_m^{n-j} + k_{n+1}z(t) \\
&= ({}^{CFC}D_a^{\alpha}z)(t)\sum_{j=0}^{n} k_j(-1)^{n-j}\mu_\alpha^{n-j} + k_{n+1}z(t) \\
&\quad + \sum_{j=0}^{n}\sum_{m=0}^{n-j-1} k_j\left(\frac{B(\alpha)}{1-\alpha}(-1)^m\mu_\alpha^m[z^{(n-j-m)}(t) - e^{-\mu_\alpha(t-a)}z^{(n-j-m)}(a)]\right) \\
&= ({}^{CFC}D_a^{\alpha}z)(t)\sum_{j=0}^{n} k_j(-1)^{n-j}\mu_\alpha^{n-j} + k_{n+1}z(t) \\
&\quad + \frac{B(\alpha)}{1-\alpha}\sum_{j=0}^{n}\sum_{m=0}^{n-j-1} k_j(-1)^m\mu_\alpha^m z^{(n-j-m)}(t) - e^{-\mu_\alpha(t-a)}\frac{B(\alpha)}{1-\alpha}\sum_{j=0}^{n}\sum_{m=0}^{n-j-1} k_j(-1)^m\mu_\alpha^m z^{(n-j-m)}(a).
\end{aligned}$$

The last equation yields

$$\begin{aligned}
&h(t) + e^{-\mu_\alpha(t-a)}\frac{B(\alpha)}{1-\alpha}\sum_{j=0}^{n}\sum_{m=0}^{n-j-1} k_j(-1)^m\mu_\alpha^m z^{(n-j-m)}(a) \\
&= ({}^{CFC}D_a^{\alpha}z)(t)\sum_{j=0}^{n} k_j(-1)^{n-j}\mu_\alpha^{n-j} + \frac{B(\alpha)}{1-\alpha}\sum_{j=0}^{n}\sum_{m=0}^{n-j-1} k_j(-1)^m\mu_\alpha^m z^{(n-j-m)}(t) + k_{n+1}z(t),
\end{aligned}$$

which proves the result.

Following analogous steps in the proof of Lemma 7 we have

Lemma 8. *Let $z \in H^n(a,b)$ and $0 < \alpha < 1$, and consider the nonlinear fractional differential equation*

$$c_0(t)({}^{CFC}D_a^{\alpha}z)(t) + c_1(t)z'(t) + c_2(t)z''(t) + \cdots + c_n z^{(n)}(t) + c_{n+1}(t)z(t) = g(t, z(t)), \tag{31}$$

where $c_0, c_1, \cdots, c_{n+1}, g \in C[a,b]$, and ${}^{CFC}D_a^{\alpha}$ is the Caputo-Fabrizio fractional derivative in the Caputo sense. This nonlinear fractional differential equation is equivalent to a nonlinear initial value problem of order $(n+1)$.

4 Applications

We consider two applications of fractional differential equations with Caputo-Fabrizio fractional derivative.

4.1 Fractional oscillatory system

In [22], the authors modified the classical oscillatory system to a fractional oscillatory system. They replaced the second derivative($\frac{d^2}{dt^\alpha}$) by the fractional derivative ($\frac{d^{2\alpha}}{dt^{2\alpha}}$), where $0 < \alpha < 1$, with justifications on the mechanical system. In this manuscript, we propose a modified fractional oscillatory system by replacing the derivative term $\frac{d^{2\alpha}}{dt^{2\alpha}}$ with $\frac{d^{1+\alpha}}{dt^{1+\alpha}}$, without losing the behaviour of the physical system.

In order to be consistent with the time dimensionality of a fractional oscillatory system, the authors in [22], introduced a parameter σ in the fractional operator as follows:

$$\frac{d}{dt} \to \frac{1}{\sigma^{1-\alpha}}\frac{d^\alpha}{dt^\alpha}, \frac{d^2}{dt^2} \to \frac{1}{\sigma^{2(1-\alpha)}}\frac{d^{2\alpha}}{dt^{2\alpha}}, \quad 0<\alpha\leq 1,$$

where the parameter σ represents the no-local fractional time components in the system. The proposed fractional differential equation corresponding to the mass-spring-damper mechanical system is given by

$$\frac{m}{\sigma^{2(1-\alpha)}}\frac{d^{2\alpha}x(t)}{dt^{2\alpha}} + \frac{\beta}{\sigma^{1-\alpha}}\frac{d^\alpha}{dt^\alpha}x(t) + kx(t) = 0, \quad 0<\alpha\leq 1, \tag{32}$$

where m is the mass, β is the damped coefficient, and k is the spring constant. In this manuscript, we introduce the fractional operator as follows:

$$\frac{d}{dt} \to \frac{1}{\sigma^{1-\alpha}}\frac{d^\alpha}{dt^\alpha} \to \frac{1}{\sigma^{1-\alpha}}D^\alpha, \frac{d^2}{dt^2} \to \frac{1}{\sigma^{2(1-\alpha)}}\frac{d^{1+\alpha}}{dt^{1+\alpha}} \to \frac{1}{\sigma^{2(1-\alpha)}}D^{1+\alpha}, \quad 0<\alpha\leq 1.$$

Thus, the fractional differential equation (32) can be written as

$$\frac{m}{\sigma^{2(1-\alpha)}}\frac{d^{1+\alpha}x(t)}{dt^{1+\alpha}} + \frac{\beta}{\sigma^{1-\alpha}}\frac{d^\alpha}{dt^\alpha}x(t) + kx(t) = 0, \quad 0<\alpha\leq 1, \tag{33}$$

or in a formal fractional sense

$$\frac{m}{\sigma^{2(1-\alpha)}}\,{}^{CFC}D_0^{1+\alpha}x(t) + \frac{\beta}{\sigma^{1-\alpha}}\,{}^{CFC}D_0^{\alpha}x(t) + kx(t) = 0, \quad 0<\alpha\leq 1, \tag{34}$$

which is a fractional differential equation of order $0<\alpha\leq 1$, where ${}^{CFC}D_0^{1+\alpha}$ denotes the Caputo-Fabrizio fractional operator. By virtue of Lemma 3.2, this fractional differential equation is equivalent to the second order initial value problem

$$\gamma_0 v''(t) + \gamma_1 v'(t) + \gamma_2 v(t) = g(t),$$
$$v(0)=0, v'(0)=x(0),$$

where, $v(t)=\int_0^t e^{\mu_\alpha s}x(s)ds$, $\mu_\alpha=\frac{\alpha}{1-\alpha}$, and

$$g(t) = \frac{m}{\sigma^{2(1-\alpha)}}v''(0) - (\frac{2m\mu_\alpha}{\sigma^{2(1-\alpha)}} - \frac{\beta}{\sigma^{1-\alpha}})v'(0), \tag{35}$$

$$\gamma_0(t) = \frac{m}{\sigma^{2(1-\alpha)}}, \tag{36}$$

$$\gamma_1(t) = \frac{\beta}{\sigma^{1-\alpha}} - \frac{2m\mu_\alpha}{\sigma^{2(1-\alpha)}} + k(1-\alpha), \tag{37}$$

$$\gamma_2(t) = \frac{m\mu_\alpha^2}{\sigma^{2(1-\alpha)}} - \frac{\beta\mu_\alpha}{\sigma^{1-\alpha}}. \tag{38}$$

The above problem possesses a solution in a closed form as the differential equation is linear, second order with constant coefficients.

4.2 Application to corneal topography

Okrasiński and Ṕlociniczak [23] introduced the nonhomogenous fractional Bessel differential equation of order 0, for human corneal topography as follows:

$$x^\alpha D_0^\alpha(xz'(x)) - x^2 z(x) = \frac{a}{b}x^2, \quad z'(0)=0, \quad z(\sqrt{a})=0, \quad 0<\alpha\leq 1,$$

where a and b are dimensionless, positive constants and D_0^α is the Riemann-Liouville fractional derivative. Here we consider the fractional Bessel equation with the Caputo-Fabrizio fractional derivative. We have

$$x^\alpha\,{}^{CFC}D_0^\alpha(xz'(x)) - x^2 z(x) = \frac{a}{b}x^2, \tag{39}$$

where ${}^{CFC}D_0^{\alpha}(xz'(x)) = \frac{B(\alpha)}{1-\alpha}e^{-\mu_\alpha x}\int_0^x e^{\mu_\alpha s}\left(sz'(s)\right)'ds$, and $\mu_\alpha = \frac{\alpha}{1-\alpha}$. Integration by parts twice yields

$$
\begin{aligned}
\int_0^x e^{\mu_\alpha s}\left(sz'(s)\right)'ds &= e^{\mu_\alpha s}sz'(s)|_0^x - \mu_\alpha \int_0^x e^{\mu_\alpha s}sz'(s)ds, \\
&= xe^{\mu_\alpha x}z'(x) - \mu_\alpha\left(e^{\mu_\alpha s}sz(s)|_0^x - \int_0^x \left(e^{\mu_\alpha s}s\right)'z(s)ds\right) \\
&= xe^{\mu_\alpha x}z'(x) - \mu_\alpha xe^{\mu_\alpha x}z(x) + \mu_\alpha \int_0^x \left(e^{\mu_\alpha s}s\right)'z(s)ds. \qquad (40)
\end{aligned}
$$

Let $w(x) = \int_0^x \left(e^{\mu_\alpha s}s\right)'z(s)ds$, then it holds that

$$
w'(x) = e^{\mu_\alpha x}(1+\mu_\alpha x)z(x), \; w(0) = 0, \; w'(0) = z(0),
$$

and

$$
z'(x) = \frac{e^{-\mu_\alpha x}}{(1+\mu_\alpha x)^2}\Big((1+\mu_\alpha x)w''(x) - \mu_\alpha(2+\mu_\alpha)w'(x)\Big). \qquad (41)
$$

Using Equations (40) and (41) we have

$$
{}^{CFC}D_0^{\alpha}(xz'(x)) = \frac{B(\alpha)}{1-\alpha}\frac{e^{-\mu_\alpha x}}{1+\mu_\alpha x}\Big(xw''(x) - \mu_\alpha x(1+\frac{2+\mu_\alpha}{1+\mu_\alpha x})w'(x) + \mu_\alpha(1+\mu_\alpha x)w(x)\Big). \qquad (42)
$$

Substituting Eq. (42) in Eq. (39) yileds

$$
xw''(x) - \left(\mu_\alpha x(1+\frac{2+\mu_\alpha}{1+\mu_\alpha x}) + x^{2-\alpha}\right)w'(x) + \mu_\alpha(1+\mu_\alpha x)w(x) = \frac{a}{b}\frac{1-\alpha}{B(\alpha)}x^{2-\alpha}(1+\mu_\alpha x)e^{\mu_\alpha x},
$$

a second order linear differential equation of w.

5 Conclusion

The Caputo-Fabrizio fractional derivative has attracted the attention of many researchers because of its appearance in various applications and its kernel is nonlocal and nonsingular of convolution type. We have developed a simple and efficient technique to convert fractional differential equations to initial value problems with integer derivatives. The degree of the resulting differential equation depends on the order of the fractional derivative α. It is observed that, for $0 < \alpha < 1$ the linear and nonlinear fractional differential equations of order $(\alpha + n)$ can be transformed to an initial value problems of integer order $(n+1)$. The technique works for several classes of linear and nonlinear fractional equations and can be extended to other types of equations. We indicate here that, the transformation is valid for a restricted space $H^1(a,b)$, while the Caputo-Fabrizio derivative can be defined for a more general space. Also, it is not possible to convert all types of linear fractional equations to integer ones. For instant the linear fractional equation

$$
k_0(t)({}^{CFC}D_a^{\alpha}z)(t) + k_1(t){}^{CFC}D_a^{\beta}z(t) + k_2(t)z(t) = f(t), \quad 0 < \alpha < 1, 1 < \beta < 2, \qquad (43)
$$

where $\beta \neq \alpha + 1$, can't be converted to initial value problem with integer order using the current technique.

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