Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/080149

A Note on the Bounded Solutions to $x''+c(t,x,x')+q(t)b(x)=f(t)$

Cemil Tunç[*]

Yüzüncü Yıl University, Department of Mathematics, Faculty of Sciences, 65080, Van, Turkey

Received: 9 Jul. 2013, Revised: 22 Nov. 2013, Accepted: 23 Nov. 2013
Published online: 1 Jan. 2014

Abstract: Kroopnick (2010) discussed the bounded solutions of certain non-autonomous differential equations of the second order by using the integral test. In this paper, instead of the integral test, we study the results of Kroopnick [9] by using the Liapunov's function approach. We compare the established results with that of Kroopnick [9]. We also give some additional results on the equi-bounded solutions and uniformly-bounded solutions, and an example is given for the illustrations.

Keywords: Non-autonomous, differential equation, second order, bounded solutions

1 Introduction

In 2010, Kroopnick [9] first considered the nonlinear differential equation of the second order

$$x''+q(t)b(x)=f(t), \tag{1}$$

where $q(.)$ is a continuously differentiable function for $t\geq 0$, $q(t)>q_0>0$ for some constant q_0, $b(.)$ and $q(.)$, $f(.)$ are continuous on $\Re$ and $[0,\infty)$, respectively.

Kroopnick [9] proved a result on the bounded solutions of Eq. (1) with appropriate conditions on $q(t)$, $b(x)$ and $f(t)$.

It is worth mentioning that equations in the form of Eq. (1) are quite important in applied mathematics. Consider such examples as the harmonic oscillator $x''+k^2x=0$ (see Kroopnick [[9], p. 829]), the theory of nonlinear oscillations and conservative systems $x''+f(x)=0$ (see Kroopnick [[9], p. 829]) or Duffing's equation $x''+ax+bx^3=K\sin(\Omega t)$ (see Kroopnick [[9], p. 829]). Eq.(1) characterizes all of the mentioned applications. It is worth noting, too, that the linear homogeneous equation $x''+p(t)x'+a(t)x=0$ used in the study of electrical and mechanical systems (see Kroopnick [[9], p. 829]) may be transformed into Eq. (1) using the transformation $x=y\exp(-1/2\int_0^t p(s)ds$ (see Kroopnick [[9], p. 829]), which should give the reader an idea of the robustness of this equation. For two outstanding surveys of applications see Kroopnick [[9], p. 829].

The Hill equation $x''+a(t)x=0$ is significant in investigation of stability and instability of geodesic on Riemannian manifolds where Jacobi fields can be expressed in form of the Hill equation system (see Gallot et. al. [4]). This fact has been used by some physicists to study dynamics in Hamiltonian systems (see Pettini and Valdettaro [11]).

It should be noted that Kroopnick [9] first proved the following theorem by using the integral test.

Theorem 1.*(Kroopnick [9, Theorem 1]). Consider the differential equation*

$$x''+q(t)b(x)=f(t).$$

Suppose $b(.)$ is continuous on $(-\infty,+\infty)$ and $B(x)=\int^x b(u)du\to\infty$ as $|x|\to\infty$. Moreover, suppose $q(.)$ belongs to $C^1[0,\infty)$ such that $q(t)>q_0>0$ and $q'(.)$ does not change sign on $[0,\infty)$. If $f(.)$ is continuous and an element of $L^1[0,\infty)$, then all solutions to Eq. (1) are bounded as $t\to\infty$. Furthermore, if $xb(x)>0$ for $x\neq 0$, then the derivates are also bounded.

In the same paper, [9], Kroopnick also considered the nonlinear differential equation of the second order

$$x''+c(t,x,x')+q(t)b(x)=f(t), \tag{2}$$

[*] Corresponding author e-mail: cemtunc@yahoo.com

where $q(.)$ is a continuously differentiable function for $t \geq 0$, $q(t) > q_0 > 0$ for some constant q_0 and $c(.)$, $b(.)$ and $f(.)$ are continuous on $[0,\infty) \times \Re^2$, $\Re$ and $[0,\infty)$, respectively.

Kroopnick [9] proved the second result on the bounded solutions of Eq. (2) with appropriate conditions on $c(t,x,x')$, $q(t)$, $b(x)$ and $f(t)$.

In [9], by using the integral test, Kroopnick proved the following theorem.

Theorem 2._(Kroopnick [9, Theorem 3]). Given the differential equation_

$$x'' + c(t,x,x') + q(t)b(x) = f(t).$$

Suppose that $c(t,x,y)$ is continuous on $[0,\infty) \times \Re^2$ such that $c(t,x,y)y \geq 0$, $b(.)$ is continuous on $(-\infty,+\infty)$ with $B(x) = \int^x b(u)du \to \infty$ as $|x| \to \infty$, $q(.)$ belongs to $C^1[0,\infty)$ such that $q(t) > q_0 > 0$ and $q'(.)$ does not change sign on $[0,\infty)$. If $f(.)$ is continuous and an element of $L^1[0,\infty)$, then all solutions to Eq. (2) are bounded as $t \to \infty$.

The motivation for this paper comes from the paper of Kroopnick [9] and the papers mentioned above. By defining certain Liapunov functions, sufficient conditions for the bounded solutions are obtained. On the other hand, to see some recent works on the qualitative behaviors of the solutions and certain important roles of linear and nonlinear differential equations of second order in many scientific areas, we refer the readers to the books of Ahmad and Rama Mohana Rao [1], Braun [2], Davis [3], Sanchez ([12], [13]), and Wylie [19] and the papers of Kroopnick ([6]-[8]), Tenenbaum and Pollard [14], Tunç [15], C. Tunç and E. Tunç [17] and the references cited in these sources.

Before stating our main results, we give two basic results on equi-bounded and uniformly-bounded solutions of a general non-autonomous system.

Consider a system of differential equations

$$\frac{dx}{dt} = F(t,x), \tag{3}$$

where x is an $n-$ vector. Suppose that $F(t,x)$ is continuous in (t,x) on $I \times D$, where I denotes the interval $0 \leq t < \infty$ and D is a connected open set in $\Re^n$. It is also assumed without loss of generality that $F(t,0) = 0$ and D is a domain such that $\|x\| < H$, $H > 0$.

Theorem 3._(Yoshizawa [18].) Suppose that there exists a Liapunov function $V(t,x)$ defined on $I \times \Re^n$ which satisfies the following conditions;_

(i) $a(\|x\|) \leq V(t,x)$, where $a(r) \in CI$, $a(r) \to \infty$ as $r \to \infty$ (CI denotes the families of continuous increasing functions),

(ii) $\dot{V}(t,x) \leq 0$.

Then, the solutions of (3) are equi-bounded.

Theorem 4._(Yoshizawa [18].) Suppose that there exists a Liapunov function $V(t,x)$ defined on $0 \leq t < \infty$, $\|x\| \geq R$, where R may be large, which satisfies the following conditions;_

(i) $a(\|x\|) \leq V(t,x) \leq b(\|x\|)$, where $a(r) \in CI$, $a(r) \to \infty$ as $r \to \infty$ and $b(r) \in CI$ (CI denotes the families of continuous increasing functions),

(ii) $\dot{V}(t,x) \leq 0$.

Then, the solutions of (3) are uniformly-bounded.

2 Main results

Our first main result is the following theorem.

Theorem 5._We assume that $c(t,x,y)$ is continuous on $[0,\infty) \times \Re^2$ such that $c(t,x,y)y \geq 0$, $b(.)$ is continuous on $(-\infty,+\infty)$ with $B(x) = \int^x b(u)du \to \infty$ as $|x| \to \infty$, $q(.) \in C^1[0,\infty)$ such that $q(t) > q_0 \geq 1$ (or $q(t)$ is positive and bounded away from 1) and $q'(.)$ does not change sign on $[0,\infty)$. If $f(.)$ is continuous and an element of $L^1[0,\infty)$, then all solutions to Eq. (2) are bounded as $t \to \infty$._

Proof(Proof of Theorem 5). Instead of Eq. (2), we consider it as a system

$$x' = y,$$
$$y' = -c(t,x,y) - q(t)b(x) + f(t). \tag{4}$$

We now consider two cases.

Case 1. Let $q'(.) \geq 0$.

For this case, we define a Liapunov function as

$$V(t,x,y) = \sqrt{2\int_0^x b(s)ds + \frac{1}{q(t)}y^2}$$

for $x^2 + y^2 \geq R^2$.

First, we find that $V(t,x,y) > 0$ for all $(x,y) \neq (0,0)$. Secondly, since $B(x) \to \infty$ as $|x| \to \infty$, $V(t,x,y) \leq K$ implies $|x| \leq K_1$ and $|y| \leq K_2$, where the constants K_1 and K_2 depend upon the constant K. Thus, we only need to show that $V(t,x,y)$ is bounded along every trajectory of (4) as $t \to \infty$.

Along a trajectory of (4), the time derivative of the Liapunov function $V(t,x,y)$ results in

$$\frac{d}{dt}V(t,x,y) = \frac{1}{\sqrt{2\int_0^x b(s)ds + \frac{1}{q(t)}y^2}}$$
$$\times \{2b(x)y - \frac{q'(t)}{q^2(t)}y^2 + \frac{2}{q(t)}y(-c(t,x,y) - q(t)b(x) + f(t))\}$$

$$\Rightarrow \sqrt{2\int_0^x b(s)ds + \frac{1}{q(t)}y^2}\ \frac{d}{dt}V(t,x,y) = -\frac{q'(t)}{q^2(t)}y^2 - \frac{2}{q(t)}c(t,x,y)y + \frac{2f(t)}{q(t)}\ y$$
$$\Rightarrow V(t,x,y)\frac{d}{dt}V(t,x,y) \leq \frac{2|f(t)|}{q(t)}\ |y| \leq \frac{2|f(t)|}{\sqrt{q(t)}}\ |y|$$
$$\Rightarrow V(t,x,y)\frac{d}{dt}V(t,x,y) \leq 2\{\frac{|y|}{\sqrt{q(t)}} + \sqrt{2B(x)}\}\ |f(t)|,$$

where

$$B(x) = \int_0^x b(s)ds.$$

It should be noted that $V(t,x,y) \neq 0$ (except possibly when $x(t)$, $y(t)$ and $f(t)$ vanish simultaneously). Also, by the Cauchy's inequality, we have

$$\left(\frac{|y|}{\sqrt{q(t)}} + \sqrt{2B(x)}\right) \leq \sqrt{2}\, V(t,x,y).$$

Hence, we obtain

$$\frac{d}{dt}V(t,x,y) \leq 2\sqrt{2}\,|f(t)|.$$

Integrating this inequality from 0 to t, we get

$$V(t,x(t),y(t)) \leq V(0,x(0),y(0)) + 2\sqrt{2}\int_0^t |f(s)|ds.$$

Since, by the assumption $f \in L^1[0,\infty)$, the last integral inequality converges when $t \to \infty$. Hence, we can conclude that $V(t,x(t),y(t))$ is bounded for all $t \geq 0$. This shows that for the case $q'(.) \geq 0$ every solution of Eq. (2), together with its derivative, is bounded as $t \to \infty$.

Case 2. Let $q'(.) \leq 0$.

For this case, we define a Liapunov function as

$$V_1(t,x,y) = \sqrt{2q(t)\int_0^x b(s)ds + y^2}$$

for $x^2 + y^2 \geq R^2$.

It follows that $V_1(t,x,y) > 0$ for all $(x,y) \neq (0,0)$. On the other hand, since $B(x) \to \infty$ as $|x| \to \infty$, $V_1(t,x,y) \leq C$ implies $|x| \leq C_1$ and $|y| \leq C_2$, where the constants C_1 and C_2 depend upon the constant C. Thus, we only need to show that $V_1(t,x,y)$ is bounded along every trajectory of (4) as $t \to \infty$.

Along a trajectory of (4) the time derivative of the Liapunov function $V_1(t,x,y)$ gives

$$\frac{d}{dt}V_1(t,x,y) = \frac{1}{\sqrt{2q(t)\int_0^x b(s)ds + y^2}} \times \{2q'(t)\int_0^x b(s)ds + 2q(t)b(x)y + 2y(-c(t,x,y) - q(t)b(x) + f(t))\}$$

$$\Rightarrow \sqrt{2q(t)\int_0^x b(s)ds + y^2}\,\frac{d}{dt}V_1(t,x,y) = 2q'(t)\int_0^x b(s)ds - 2c(t,x,y)y + 2f(t)y$$

$$\Rightarrow \sqrt{2q(t)\int_0^x b(s)ds + y^2}\,\frac{d}{dt}V_1(t,x,y) \leq 2q'(t)\int_0^x b(s)ds - 2c(t,x,y)y + 2|f(t)|\ |y| \leq 2|f(t)|\ |y| \leq 2(|y| + \sqrt{2q(t)B(x)}\,)\,|f(t)|,$$

where

$$B(x) = \int_0^x b(s)ds.$$

$$\Rightarrow V_1(t,x,y)\frac{d}{dt}V_1(t,x,y) \leq 2(|y| + \sqrt{2q(t)B(x)}\,)\,|f(t)|.$$

It should be noted that $V_1(t,x,y) \neq 0$ (except possibly when $x(t)$, $y(t)$ and $f(t)$ vanish simultaneously). Also, by the Cauchy's inequality, we have

$$(|y| + \sqrt{2q(t)B(x)}\,) \leq \sqrt{2}V_1(t,x,y).$$

Hence, we obtain

$$\frac{d}{dt}V_1(t,x,y) \leq 2\sqrt{2}\,|f(t)|.$$

Integrating this inequality and using the assumption $f \in L^1[0,\infty)$, we can conclude for the case $q'(.) \leq 0$ that every solution of Eq. (2), together with its derivative, is bounded as $t \to \infty$.

Our second main result is the following theorem.

Theorem 6. *We assume that all the assumptions of Theorem 5 hold. Then all solutions to Eq. (2) are uniformly bounded.*

Proof(Proof of Theorem 6). We now consider two cases.

Case 1. Let $q'(.) \geq 0$.

For this case, we define a Liapunov function as

$$V_2(t,x,y) = \sqrt{2\int_0^x b(s)ds + \frac{1}{q(t)}y^2 - 2\int_0^t |f(s)|ds}$$

for $x^2 + y^2 \geq R^2$.

We note that $V_2(t,x,y)$ satisfies the condition (i) of Theorem 4 for $x^2 + y^2 \geq R^2$:

$$\sqrt{2\int_0^x b(s)ds+\frac{1}{q(t)}y^2}-2\int_0^t|f(s)|ds\le V_2(t,x,y)\le\sqrt{2\int_0^x b(s)ds+\frac{1}{q_0}y^2}.$$

Along a trajectory of (4), the time derivative of the Liapunov function $V_2(t,x,y)$ results in

$$\frac{d}{dt}V_2(t,x,y)=\frac{1}{\sqrt{2\int_0^x b(s)ds+\frac{1}{q(t)}y^2}}$$
$$\times\{-\frac{q'(t)}{q^2(t)}y^2+2b(x)y+\frac{2}{q(t)}y(-c(t,x,y)-q(t)b(x)+f(t))\}$$
$$-2|f(t)|$$

$$\Rightarrow\sqrt{2\int_0^x b(s)ds+\frac{1}{q(t)}y^2}\frac{d}{dt}V_2(t,x,y)=$$
$$-\frac{q'(t)}{q^2(t)}y^2-\frac{2}{q(t)}c(t,x,y)y$$
$$+\frac{2f(t)}{q(t)}y-\sqrt{8\int_0^x b(s)ds+\frac{4}{q(t)}y^2}\ |f(t)|$$

$$\Rightarrow\sqrt{2\int_0^x b(s)ds+\frac{1}{q(t)}y^2}\frac{d}{dt}V_2(t,x,y)\le$$
$$-\frac{q'(t)}{q^2(t)}y^2-\frac{2}{q(t)}c(t,x,y)y$$
$$+\frac{2f(t)}{q(t)}y-\frac{2|f(t)|}{\sqrt{q(t)}}\ |y|$$

$$\Rightarrow\sqrt{2\int_0^x b(s)ds+\frac{1}{q(t)}y^2}\frac{d}{dt}V_2(t,x,y)\le$$
$$-\frac{q'(t)}{q^2(t)}y^2-\frac{2}{q(t)}c(t,x,y)y$$
$$+\frac{|f(t)|}{\sqrt{q(t)}}\ |y|-\frac{|f(t)|}{\sqrt{q(t)}}\ |y|$$

$$\Rightarrow\sqrt{2\int_0^x b(s)ds+\frac{1}{q(t)}y^2}\frac{d}{dt}V_2(t,x,y)\leqslant-\frac{q'(t)}{q^2(t)}y^2-\frac{2}{q(t)}c(t,x,y)y$$

$$\Rightarrow\frac{d}{dt}V_2(t,x,y)\le 0.$$

This shows that for the case $q'(.)\ge 0$ all solutions of Eq. (2) is uniformly bounded.

Case 2. Let $q'(.)\le 0$.

For this case, we define a Liapunov function as

$$V_3(t,x,y)=\sqrt{2q(t)\int_0^x b(s)ds+y^2}-2\int_0^t|f(s)|ds$$

for $x^2+y^2\ge R^2$.

It follows that $V_3(t,x,y)>0$ for all $(x,y)\ne(0,0)$ for $x^2+y^2\ge R^2$. On the other hand, since $B(x)\to\infty$ as $|x|\to\infty$, $V_3(t,x,y)\le D$ implies $|x|\le D_1$ and $|y|\le D_2$, where the constants D_1 and D_2 depend upon the constant D. Thus, we only need to show that $V_3(t,x,y)$ is bounded along every trajectory of (4) as $t\to\infty$.

Along a trajectory of (4) the time derivative of the Liapunov function $V_3(t,x,y)$ results in

$$\frac{d}{dt}V_3(t,x,y)=\frac{1}{\sqrt{2q(t)\int_0^x b(s)ds+y^2}}$$
$$\times\{2q'(t)\int_0^x b(s)ds+2q(t)b(x)y+2y(-c(t,x,y)-q(t)b(x)+f(t))\}$$
$$-2|f(t)|$$

$$\Rightarrow\sqrt{2q(t)\int_0^x b(s)ds+y^2}\frac{d}{dt}V_3(t,x,y)=$$
$$2q'(t)\int_0^x b(s)ds-2c(t,x,y)y+2f(t)y$$
$$-\sqrt{8q(t)\int_0^x b(s)ds+4y^2}\ |f(t)|$$

$$\Rightarrow V_3(t,x,y)\frac{d}{dt}V_3(t,x,y)\le$$
$$2q'(t)\int_0^x b(s)ds-2c(t,x,y)y+2|f(t)|\ |y|-2|f(t)|\ |y|$$
$$=2q'(t)\int_0^x b(s)ds-2c(t,x,y)y$$

$$\Rightarrow\frac{d}{dt}V_3(t,x,y)\le 0.$$

This shows that for the case $q'(.)\le 0$ all solutions of Eq. (2) are uniformly bounded

The proof of Theorem 6 is now completed.

*Remark.*Theorem 6 gives an additional result to that of Kroopnick [9, Theorem 3]). The assumptions of Theorem 6 also guarantee the equi-boundedness of the all solutions of Eq. (2).

*Example 1.*We consider the non-autonomous differential equation of second order:

$$x'' + ax' + b(t+1)x = 0, \tag{5}$$

where a and b are positive constants.

Eq. (5) can be written in the system form:

$$x' = y,$$
$$y' = -ay - b(t+1)x$$

so that

$$c(t,x,y)y = ay^2 \geq 0, t \geq 0$$

$$q(t) = t+1,$$

$$q(t) > 1, \;\; q_0 = 1,$$

$$q'(t) = 1 > 0,$$

that is, $q'(.)$ does not change sign on $[0,\infty)$,

$$b(x) = bx,$$

$$B(x) = \int_0^x b(u)du = b\int_0^x udu = \frac{1}{2}x^2 \to \infty \quad \text{as } |x| \to \infty.$$

On the other hand, for the case $q'(.) \geq 0$, it follows that

$$V(t,x,y) = \sqrt{2\int_0^x b(s)ds + \frac{1}{q(t)}y^2} = \sqrt{2\int_0^x bsds + \frac{1}{t+1}y^2} = \sqrt{bx^2 + \frac{1}{t+1}y^2}.$$

It can be shown that V=$V(t,x,y)$ satisfies the condition (i) of Theorem 4 for $x^2+y^2 \geq R^2$, and

$$\frac{dV}{dt} = \frac{1}{2}(bx^2 + \frac{1}{t+1}y^2)^{-\frac{1}{2}}\{2bxy - \frac{1}{(t+1)^2}y^2 + \frac{2}{t+1}y(-ay - b(t+1)x\}$$
$$= -\frac{1}{2}(bx^2 + \frac{1}{t+1}y^2)^{-\frac{1}{2}}\{\frac{1}{(t+1)^2}y^2 + \frac{2a}{t+1}y^2\}$$

$$\Rightarrow (bx^2 + \frac{1}{1+t}y^2)^{\frac{1}{2}}\frac{d}{dt}V(t,x,y) = -\frac{1}{2}\{\frac{1}{(t+1)^2}y^2 + \frac{2a}{t+1}y^2\}$$

$$\Rightarrow \frac{d}{dt}V(t,x,y) \leq 0.$$

Thus, we conclude that all solutions of the above equation for the case $q'(.) \geq 0$ are bounded as $t \to \infty$.

*Remark.*It can be easily shown that all solutions of Eq. (5) are equi-bounded and uniformly bounded.

*Example 2.*We consider the non-autonomous differential equation of second order:

$$x'' + (4 - e^{-t} + |x| + x'^2)x' + \left(4 + \frac{1}{1+t^2}\right)x^5 = \frac{1}{1+t^2}. \tag{6}$$

This equation can be written in the system form:

$$x' = y,$$
$$y' = -(4 - e^{-t} + |x| + y^2)y - \left(4 + \frac{1}{1+t^2}\right)x^5 + \frac{1}{1+t^2}$$

so that

$$c(t,x,y)y = (4 - e^{-t} + |x| + y^2)y^2 \geq 0, t \geq 0$$

$$q(t) = 4 + \frac{1}{1+t^2},$$

$$q(t) > 3 > 1, \; q_0 = 3,$$

$$q'(t) = -\frac{2t}{(1+t^2)^2} \leq 0,$$

that is, $q'(.)$ does not change sign on $[0,\infty)$,

$$b(x) = x^5,$$

$$B(x) = \int_0^x b(u)du = \int_0^x u^5du = \frac{1}{6}x^6 \to \infty \quad \text{as } |x| \to \infty,$$

$$f(t) = \frac{1}{1+t^2},$$

$$\int_0^\infty |f(t)|dt = \int_0^\infty \frac{1}{1+t^2}dt = \frac{\pi}{2} < \infty,$$

that is, $f(t)$ is an element of $L^1[0,\infty)$.

On the other hand, for the case $q'(.) \leq 0$, it follows that

$$V_4(t,x,y) = \sqrt{2q(t)\int_0^x b(s)ds + y^2} - \int_0^t |f(s)|ds = \sqrt{\frac{5+4t^2}{3+3t^2}x^6 + y^2} - \int_0^t \frac{1}{1+s^2}ds = \sqrt{\frac{5+4t^2}{3+3t^2}x^6 + y^2} - arctgt.$$

It can be shown that $V_4(t,x,y)$ satisfies the condition (i) of Theorem 4 for $x^2+y^2 \geq R^2$, and

$$\frac{d}{dt}V_4(t,x,y) = -2(\frac{5+4t^2}{3+3t^2}x^6 + y^2)^{-\frac{1}{2}}$$
$$\{\frac{t}{3(1+t^2)^2}x^5 + (4 - e^{-t} + |x| + y^2)y^2\}$$
$$-\frac{1}{1+t^2}$$

$$\Rightarrow (\frac{5+4t^2}{3+3t^2}x^6+y^2)^{\frac{1}{2}}\frac{d}{dt}V_4(t,x,y) =$$
$$-2\{\frac{t}{3(1+t^2)^2}x^5+(4-e^{-t}+|x|+y^2)y^2\}$$
$$-2(\frac{5+4t^2}{3+3t^2}x^6+y^2)^{\frac{1}{2}}\frac{1}{1+t^2}$$
$$\Rightarrow \frac{d}{dt}V_4(t,x,y) \le 0.$$

Thus, we conclude that all solutions of the above equation for the case $q'(.) \le 0$ are bounded as $t \to \infty$.

*Remark.*When $f(t) = 0$, instead of $f(t) = \frac{1}{1+t^2}$, in Eq. (6), it can be shown that all solutions of Eq. (6) are equi-bounded and uniformly bounded.

Corollary 1.*Consider the differential equation*

$$x''+q(t)b(x) = f(t).$$

Suppose $b(.)$ *is continuous on* $(-\infty,+\infty)$ *and* $B(x) = \int^x b(u)du \to \infty$ *as* $|x| \to \infty$. *Moreover, suppose* $q(.) \in C^1[0,\infty)$ *such that* $q(t) > q_0 > 0$ *and* $q'(.)$ *does not change sign on* $[0,\infty)$. *If* $f(.)$ *is continuous and an element of* $L^1[0,\infty)$, *then all solutions to Eq. (1) are bounded as* $t \to \infty$. *Furthermore, if* $xb(x) > 0$ *for* $x \ne 0$, *then the first derivates of all solutions of Eq. (1) are also bounded.*

Corollary 2.*We assume that all the assumptions of Corollary 1 hold. Then all solutions to Eq. (1) are uniformly bounded.*

*Remark.*It follows that Eq. (2) includes Eq. (1) and the assumptions of Theorem 5 and Theorem 6 reduce to that of Corollary 1 and Corollary 2 , respectively, when $c(t,x,x') = 0$, therefore, it is sufficient to give the proofs of Theorem 5 and Theorem 6.

*Remark.*Because of Remark 2, we only gave the proofs of Theorem 5 and Theorem 6.

*Remark.*Kroopnick [9] proved Theorem 1 and Theorem 2 by using the integral test as $t \to \infty$. Instead of this test, we prove the boundedness of solutions of Eq. (1) and Eq. (2) by using the Liapunov's function approach [10] when $t \to \infty$. It follows that the conditions of Theorem 5 and 6 are the same as that in Kroopnick [[9], Theorem 1, Theorem 3]) except $q_0 > 1$ or $q(t)$ is positive and bounded away from 0 instead of $q_0 > 0$. The procedure will be used in the proof of Theorem 5 and Theorem 6 is very clear and comprehensible, and it can be easily seen the boundedness of solutions of Eq. (1) and Eq. (2).

References

[1] Ahmad, S. and Rao, M. Rama Mohana, Theory of Ordinary differential Equations. With Applications in Biology and engineering , Affiliated East-West Press Pvt. Ltd., New Delhi, (1999).
[2] Braun, M., Differential equations and their applications. An introduction to applied mathematics, Fourth edition. Texts in Applied Mathematics, Springer-Verlag, New York, **11**, (1993).
[3] Davis, H. T., Introduction to nonlinear differential and integral equations, Dover Publications, Inc., New York, (1962).
[4] Gallot, S., Hulin, D. and Lafontaine, J., Riemannian geometry, Universitext, Springer-Verlag, Berlin, (2004).
[5] Kroopnick, A., Properties of solutions of differential equations of the form $x''+a(t)b(x) = 0$, Proc. Amer. Math. Soc., **34**, 319–320 (1972).
[6] Kroopnick, A., Note on bounded L^p- solutions of a generalized Lié nard equation, Pacific J. Math., **94**, 171–175 (1981).
[7] Kroopnick, A., On the boundedness and oscillation of solutions to $(m(t)x')'+a(t)b(x) = 0$, Internat. J. Math. Math. Sci., **10**, 47–50 (1987).
[8] Kroopnick, A., General boundedness theorems to some second order nonlinear differential equation with integrable forcing term, Internat. J. Math. Math. Sci., **18**, 823–824 (1995).
[9] Kroopnick, A., Bounded solutions to $x''+q(t)b(x) = f(t)$, Internat. J. Math. Ed. Sci. Tech., **41**, 829–836 (2010).
[10] Liapunov, A. M., Stability of Motion, Academic Press, London, (1966).
[11] Pettini, M. and Valdettaro, R., On the Riemannian description of chaotic instability in Hamiltonian dynamics, Chaos, **5**, 646–652 (1995).
[12] Sanchez, D. A., Ordinary differentialequations and stability theory. An introduction, Reprint of the 1968 original. Dover Publications, Inc., New York, (1979).
[13] Sánchez, D. A., Ordinary differential equations, A brief eclectic tour. Classroom Resource Materials Series. Mathematical Association of America, Washington, DC, (2002).
[14] Tenenbaum, M. and Pollard, H., OrdinaryDifferential Equations, Dover Publications, New York, (1985).
[15] Tunç, C., A note on boundedness ofsolutions to a class of non-autonomous differential equations of second order, Appl. Anal. Discrete Math., **4**, 361–372 (2010).
[16] Tunç, C.,Stability and boundedness for a kind of non-autonomous differential equations with constant delay, Appl. Math. Inf. Sci., **7**, 355–361 (2013).
[17] Tunç, C. and Tunç, E., On the asymptotic behavior of solutions of certain second-order differential equations, J. Franklin Inst., **344**, 391–398 (2007).
[18] Yoshizawa, T., Stability Theory by Liapunov's Second Method, Publications of the Mathematical Society of Japan, The Mathematical Society of Japan, Tokyo, **1**, (1966).
[19] Wylie, C. R., Advanced engineering mathematics, Fourth edition. McGraw-Hill Book Co., New York-Dü sseldorf-Johannesburg, (1975).

Cemil Tunç was born in Yeşilöz Köyü (Kalbulas), Horasan-Erzurum, Turkey, in 1958. He received the Ph. D. degree in Applied Mathematics from Erciyes University, Kayseri, in 1993. His research interests include qualitative behaviors of solutions of differential equations. At present he is Professor of Mathematics at Yüzüncü Yıl University, Van-Turkey.