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Wrapped Ishita Distribution

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Abstract: Wrapped Ishita distribution is a new circular distribution generated by wrapping Ishita random variable around the circle real line. In this paper, we obtained the explicit formulas of the probability densities functions and trigonometric moments. Also, the explicit formulas of some distribution characteristics are given.

Keywords: Circular variance, Circular standard deviation, Ishita distribution, Mean direction, Mean resultant length, Trigonometric moments, Wrapped distribution.

1 Introduction

Circular data are two-dimensional measurements used in many scientific fields, such as Geology, Physics, Medicine, and Political Science [1]. Due to the nature of these circular data linear analysis methods are not applicable [2]. Various model classes have been developed to study and analyze circular data. Different methods have been used to generate circular distributions (see e.g. [3,4]). Circular distribution can be generated by transforming a bivariate linear random variable to its directional component, or using some characterizing properties (see e.g. [5,6]). However, the most widely used method is generating a circular distribution by wrapping a linear distribution around the unit circle (see e.g. [7,8,9]). For example, let $f(x)$ be a continuous linear probability density function, we can generate the following circular probability density function $g(\theta)=\sum_{m=-\infty}^{\infty} f(\theta+2m\pi)$ for $0\leq\theta<2\pi$. Note that, $g(\theta)$ is a periodic function such that $g(\theta)=g(\theta+2m\pi)$ for $m\in\mathbb{Z}$.

The present paper is organized, as follows: Wrapped Ishita distribution, the probabilities densities functions and the characteristic function are presented in Section 2. In Section 3, we obtain the trigonometric moments (central and non-central). In Section 4, the formulas of the mean direction and mean resultant length are introduced. Also, the formulas for the measures of variations (circular variance and standard deviation) are presented in Section 5. The formulas of the skewness and kurtosis are given in Section 6. Section 7 is devoted to conclusion.

2 Wrapped Ishita Distribution

While studying the modelling of lifetime data using common lifetime distributions (i.e., Akash, Lindley, and exponential distribution), [10] noticed that these distributions were not applicable for some data. Thus, they proposed a new mixture of an exponential distribution with parameter λ and a gamma distribution with shape parameter 3 and scale parameter λ, with mixing proportion $(\lambda^3/(\lambda^3+2))$. The new distribution (i.e. Ishita distribution) was introduced by [11] as a flexible alternative for common lifetime distributions. The probability density and cumulative distribution functions for the Ishita distribution are given, respectively, by

$$f(x;\lambda)=\frac{\lambda^3}{\lambda^3+2}(\lambda+x^2)e^{-x\lambda} \qquad ;x>0,\quad \lambda>0, \tag{1}$$

and

$$F(x;\lambda)=1-\left[1+\frac{x^2\lambda^2+2x\lambda}{\lambda^3+2}e^{-x\lambda}\right] \qquad ;x>0,\quad \lambda>0. \tag{2}$$

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The characteristic function for Ishita distribution is defined as

$$\varphi_X(t) = \frac{\lambda^3}{\lambda^3+2}(\lambda^3 - t^2\lambda + 2 - 2\,it\lambda^2)(\lambda - it)^{-3} \qquad ;i = \sqrt{-1}. \tag{3}$$

Wrapped Ishita distribution is generated by wrapping Ishita distribution around the unit circle. Let X be an Ishita random variable, then the circular random variable generated by X is $\theta = X(mod\ 2\pi)$ with probability density function $g(\theta;\lambda)$ given by

$$\begin{aligned}
g(\theta;\lambda) &= \sum_{m=0}^{\infty} f(\theta+2m\pi) \\
&= \sum_{m=0}^{\infty} \frac{\lambda^3}{\lambda^3+2}[\lambda+(\theta+2m\pi)^2]e^{-(\theta+2m\pi)\lambda} \\
&= \frac{\lambda^3\, e^{-\theta\lambda}}{\lambda^3+2}\left[\sum_{m=0}^{\infty}(\lambda+\theta^2)e^{-2m\pi\lambda} + \sum_{m=0}^{\infty} 4m\pi\theta e^{-2m\pi\lambda} + \sum_{m=0}^{\infty} 4m^2\pi^2 e^{-2m\pi\lambda}\right] \\
&= \frac{\lambda^3\, e^{-\theta\lambda}}{\lambda^3+2}\left[\frac{\lambda+\theta^2}{1-e^{-2\pi\lambda}} + \frac{4\pi\theta e^{-2\pi\lambda}}{(1-e^{-2\pi\lambda})^2} + \frac{4\pi^2 e^{-2\pi\lambda}(1+e^{-2\pi\lambda})}{(1-e^{-2\pi\lambda})^3}\right].
\end{aligned}$$

The cumulative distribution function $G(\theta;\lambda)$ for $\theta \in [0,2\pi)$ and $\lambda > 0$ is given by

$$\begin{aligned}
G(\theta;\lambda) &= \sum_{m=0}^{\infty}[F(\theta+2m\pi) - F(2m\pi)] \\
&= \sum_{m=0}^{\infty}\Bigg[1 - \left(\frac{\lambda^2(\theta+2m\pi)^2 + 2\lambda(\theta+2m\pi)}{\lambda^3+2} + 1\right)e^{-(\theta+2m\pi)\lambda} \\
&\qquad - 1 + \left(\frac{\lambda^2(2m\pi)^2 + 2\lambda(2m\pi)}{\lambda^3+2} + 1\right)e^{-(2m\pi)\lambda}\Bigg] \\
&= \frac{1}{\lambda^3+2}\Bigg[(\lambda^3+2-(\lambda^3+2+\theta\lambda(\theta\lambda+2))e^{-\theta\lambda})\sum_{m=0}^{\infty}(e^{-2\pi\lambda})^m \\
&\qquad + (4\pi\lambda(1-(\theta\lambda+1)e^{-\theta\lambda}))\sum_{m=0}^{\infty} m(e^{-2\pi\lambda})^m \\
&\qquad + (4\pi^2\lambda^2(1-e^{-\theta\lambda}))\sum_{m=0}^{\infty} m^2(e^{-2\pi\lambda})^m\Bigg] \\
&= \frac{1}{\lambda^3+2}\Bigg[\frac{\lambda^3+2-(\lambda^3+2+\theta\lambda(\theta\lambda+2))e^{-\theta\lambda}}{1-e^{-2\pi\lambda}} \\
&\qquad + \frac{4\pi\lambda(1-(\theta\lambda+1)e^{-\theta\lambda})e^{-2\pi\lambda}}{(1-e^{-2\pi\lambda})^2} + \frac{4\pi^2\lambda^2(1-e^{-\theta\lambda})(1+e^{-2\pi\lambda})e^{-2\pi\lambda}}{(1-e^{-2\pi\lambda})^3}\Bigg].
\end{aligned}$$

Now, we can define the new circular distribution (i.e. wrapped Ishita distribution) as follows:

Definition 1. *A random variable $\theta \in [0,2\pi)$ is said to have a wrapped Ishita distribution with parameter $\lambda > 0$ if its probability density function is given by*

$$g(\theta;\lambda) = \frac{\lambda^3\, e^{-\theta\lambda}}{\lambda^3+2}\left[\frac{\lambda+\theta^2}{1-e^{-2\pi\lambda}} + \frac{4\pi\theta e^{-2\pi\lambda}}{(1-e^{-2\pi\lambda})^2} + \frac{4\pi^2 e^{-2\pi\lambda}(1+e^{-2\pi\lambda})}{(1-e^{-2\pi\lambda})^3}\right], \tag{4}$$

with cumulative distribution function

$$\begin{aligned}
G(\theta;\lambda) = \frac{1}{\lambda^3+2}\Bigg[&\frac{\lambda^3+2-(\lambda^3+2+\theta\lambda(\theta\lambda+2))e^{-\theta\lambda}}{1-e^{-2\pi\lambda}} \\
&+ \frac{4\pi\lambda(1-(\theta\lambda+1)e^{-\theta\lambda})e^{-2\pi\lambda}}{(1-e^{-2\pi\lambda})^2} + \frac{4\pi^2\lambda^2(1-e^{-\theta\lambda})(1+e^{-2\pi\lambda})e^{-2\pi\lambda}}{(1-e^{-2\pi\lambda})^3}\Bigg].
\end{aligned} \tag{5}$$

Figure 1 shows the cumulative distribution and probability density functions, respectively, of the wrapped Ishita distribution with different values of λ .

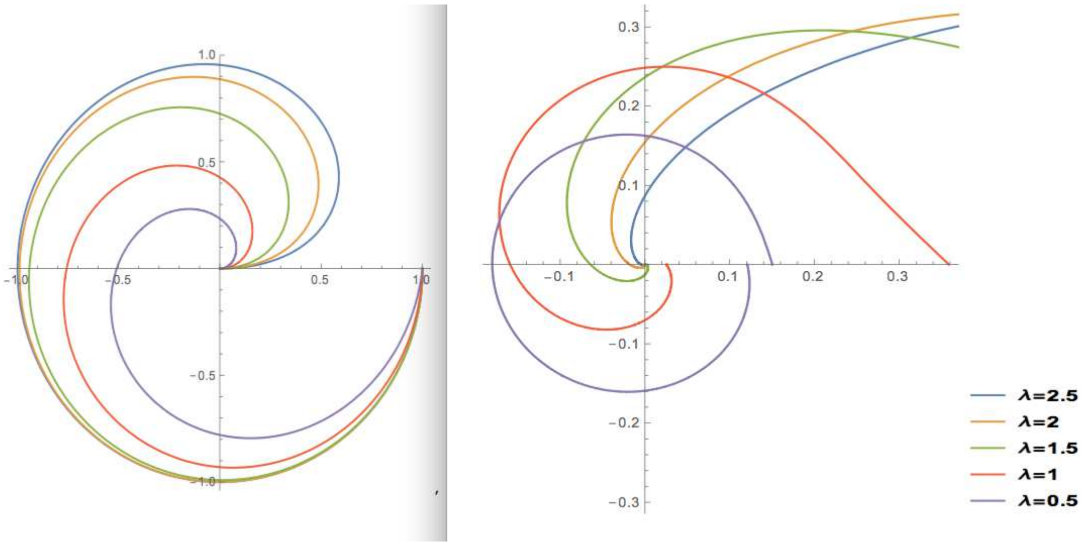

Fig. 1: The CDF and PDF of wrapped Ishita distribution for different values of λ

3 Trigonometric Moments

For a wrapped distribution, the p^{th} trigonometric moment is equal to the value of the characteristic function of the linear distribution, which generates the wrapped distribution, at the integer value p (i.e., $\varphi_\theta(p) = \varphi_X(p)$). Hence, the p^{th} trigonometric moment for wrapped Ishita distribution is

$$\begin{aligned}\varphi_\theta(p) &= \varphi_X(p)\\ &= \frac{\lambda^3}{\lambda^3+2}(\lambda^3 - p^2\lambda + 2 - 2\,ip\lambda^2)(\lambda - ip)^{-3}; i = \sqrt{-1},\ p = \mp 1, \mp 2, \ldots \end{aligned} \tag{6}$$

We can rewrite equation (6) using the fact $(a-ib)^{-r} = (a^2+b^2)^{\frac{-r}{2}} e^{ir\ arctan(b/a)}$ for $a,b,r \in \Re$. Such that:

$$(\lambda^3 - p^2\lambda + 2 - 2\,ip\lambda^2) = ((\lambda^3 - p^2\lambda + 2)^2 + 4p^2\lambda^4)^{\frac{1}{2}} e^{-i\ arctan\left(\frac{2p\lambda^2}{\lambda^3 - p^2\lambda + 2}\right)},$$

and

$$(\lambda - ip)^{-3} = (\lambda^2 + p^2)^{-3/2} e^{3i\ arctan\left(\frac{p}{\lambda}\right)}.$$

Then, we can write the p^{th} trigonometric moment for wrapped Ishita distribution $\varphi_\theta(p)$ as

$$\varphi_\theta(p) = \rho_p e^{i\mu_p}; \quad i = \sqrt{-1},\ p = \mp 1, \mp 2, \ldots, \tag{7}$$

where

$$\rho_p = \frac{\lambda^3}{\lambda^3+2}\sqrt{\frac{(\lambda^3 - p^2\lambda + 2)^2 + 4p^2\lambda^4}{(\lambda^2+p^2)^3}},$$

and

$$\mu_p = 3\ arctan\left(\frac{p}{\lambda}\right) - arctan\left(\frac{2p\lambda^2}{\lambda^3 - p^2\lambda + 2}\right).$$

Also, we can express the p^{th} trigonometric moment of the wrapped Ishita distribution $\varphi_\theta(p)$ in terms of $\alpha_p = E(cos\ p\theta)$ and $\beta_p = E(sin\ p\theta)$ as $\varphi_\theta(p) = \alpha_p + i\beta_p$. For wrapped Ishita distribution the non-central trigonometric moments are given by

$$\begin{aligned}\alpha_p &= \rho_p cos\ \mu_p\\ &= \frac{\lambda^3}{\lambda^3+2}\sqrt{\frac{(\lambda^3 - p^2\lambda + 2)^2 + 4p^2\lambda^4}{(\lambda^2+p^2)^3}} cos\left[3\ arctan\left(\frac{p}{\lambda}\right)\right.\\ &\qquad \left. - arctan\left(\frac{2p\lambda^2}{\lambda^3 - p^2\lambda + 2}\right)\right],\end{aligned} \tag{8}$$

and

$$\begin{aligned}\beta_p &= \rho_p sin\ \mu_p \\ &= \frac{\lambda^3}{\lambda^3+2}\sqrt{\frac{(\lambda^3-p^2\lambda+2)^2+4p^2\lambda^4}{(\lambda^2+p^2)^3}} sin\left[3\ arctan\left(\frac{p}{\lambda}\right)\right. \\ &\left. -arctan\left(\frac{2p\lambda^2}{\lambda^3-p^2\lambda+2}\right)\right]. \end{aligned} \tag{9}$$

While the central trigonometric moments of the wrapped Ishita distribution are given by

$$\begin{aligned}\bar{\alpha}_p &= \rho_p cos(\mu_p - p\mu_1) \\ &= \frac{\lambda^3}{\lambda^3+2}\sqrt{\frac{(\lambda^3-p^2\lambda+2)^2+4p^2\lambda^4}{(\lambda^2+p^2)^3}} cos\left[3\ arctan\left(\frac{p}{\lambda}\right) - arctan\left(\frac{2p\lambda^2}{\lambda^3-p^2\lambda+2}\right)\right. \\ &\left. -3p\ arctan\left(\frac{1}{\lambda}\right) + p\ arctan\left(\frac{2\lambda^2}{\lambda^3-\lambda+2}\right)\right], \end{aligned} \tag{10}$$

and

$$\begin{aligned}\bar{\beta}_p &= \rho_p sin(\mu_p - p\mu_1) \\ &= \frac{\lambda^3}{\lambda^3+2}\sqrt{\frac{(\lambda^3-p^2\lambda+2)^2+4p^2\lambda^4}{(\lambda^2+p^2)^3}} sin\left[3\ arctan\left(\frac{p}{\lambda}\right) - arctan\left(\frac{2p\lambda^2}{\lambda^3-p^2\lambda+2}\right)\right. \\ &\left. -3p\ arctan\left(\frac{1}{\lambda}\right) + p\ arctan\left(\frac{2\lambda^2}{\lambda^3-\lambda+2}\right)\right]. \end{aligned} \tag{11}$$

Table (1) shows the values of non-central and central trigonometric moments for $p = 1,2$ and $\lambda = 0.5, 1, 1.5, 2, 2.5$.

Table 1: Trigonometric Moments for Wrapped Ishita Distribution for $p = 1,2$ and $\lambda = 0.5, 1, 1.5, 2, 2.5$.

Parameter	**Non-central Trigonometric Moments**				**Central Trigonometric Moments**			
λ	α_1	α_2	β_1	β_2	$\bar{\alpha}_1$	$\bar{\alpha}_2$	$\bar{\beta}_1$	$\bar{\beta}_2$
0.5	-0.071	-0.006	0.009	0.004	0.072	-0.006	0	0.003
1	0	-0.008	0.333	-0.123	0.333	0.008	0	0.123
1.5	0.394	-0.151	0.500	-0.330	0.636	-0.285	0	0.224
2	0.666	0.350	0.461	0.450	0.810	0.544	0	-0.169
2.5	0.802	0.517	0.388	0.481	0.891	0.698	0	-0.107

4 Mean Direction and Mean Resultant Length

In circular distribution, the mean direction of the distribution is given by μ_p at $p = 1$. The spread of the circular data around the mean is measured by the mean resultant length which is given by ρ_p at $p = 1$. The closer the mean resultant length is to 1, the more data concentrate towards the mean. The more direction and the mean resultant length for wrapped Ishita distribution are given, respectively, by

$$\mu = \mu_1 = 3\ arctan\left(\frac{1}{\lambda}\right) - arctan\left(\frac{2\lambda^2}{\lambda^3-\lambda+2}\right), \tag{12}$$

and

$$\rho = \rho_1 = \frac{\lambda^3}{\lambda^3+2}\sqrt{\frac{(\lambda^3-\lambda+2)^2+4\lambda^4}{(\lambda^2+1)^3}}. \tag{13}$$

Table (2) shows decreasing in the mean direction μ , and more data concentrate towards the mean as the value of λ increases.

Table 2: Mean Direction and Mean Resultant Length of Wrapped Ishita Distribution for $\lambda = 0.5, 1, 1.5, 2, 2.5$.

Means	**Parameter λ**				
	0.5	1	1.5	2	2.5
Mean Direction μ	3.023	1.571	0.904	0.605	0.451
Mean Resultant Length ρ	0.072	0.333	0.636	0.810	0.891

5 Circular Variance and Circular Standard Deviation

The circular variance for wrapped Ishita distribution is given by

$$\begin{aligned} V &= 1-\rho \\ &= 1-\frac{\lambda^3}{\lambda^3+2}\sqrt{\frac{(\lambda^3-\lambda+2)^2+4\lambda^4}{(\lambda^2+1)^3}}. \end{aligned} \tag{14}$$

However, the circular standard deviation for wrapped Ishita distribution is given by

$$\begin{aligned} \sigma &= \sqrt{-2ln\rho} \\ &= \sqrt{-2ln\left(\frac{\lambda^3}{\lambda^3+2}\sqrt{\frac{(\lambda^3-\lambda+2)^2+4\lambda^4}{(\lambda^2+1)^3}}\right)}. \end{aligned} \tag{15}$$

Table (3) shows decreasing in the variance V and standard deviation σ as the value of λ increases.

Table 3: Circular Variance and Standard Deviation for Wrapped Ishita Distribution for $\lambda = 0.5, 1, 1.5, 2, 2.5$.

Measures of Variation	**Parameter λ**				
	0.5	1	1.5	2	2.5
Circular Variance V	0.928	0.667	0.364	0.190	0.109
Circular Standard Deviation σ	2.297	1.482	0.951	0.650	0.480

6 Skewness and Kurtosis

The skewness coefficient for a circular distribution is calculated by $\zeta_1 = \bar{\beta}_2 V^{-3/2}$, and the kurtosis coefficient is calculated by $\zeta_2 = \left(\bar{\alpha}_2-(1-V)^4\right)V^{-2}$. For wrapped Ishita distribution, the skewness and the kurtosis coefficients are given, respectively, by

$$\begin{aligned} \zeta_1 &= \bar{\beta}_2 V^{-3/2} \\ &= \frac{\lambda^3}{\lambda^3+2}\sqrt{\frac{(\lambda^3-4\lambda+2)^2+16\lambda^4}{(\lambda^2+4)^3}}\left(1-\frac{\lambda^3}{\lambda^3+2}\sqrt{\frac{(\lambda^3-\lambda+2)^2+4\lambda^4}{(\lambda^2+1)^3}}\right)^{\frac{-3}{2}} \\ &\quad \left(sin\left[3\ arctan\left(\frac{2}{\lambda}\right)-arctan\left(\frac{4\lambda^2}{\lambda^3-4\lambda+2}\right)-6\ arctan\left(\frac{1}{\lambda}\right)\right.\right. \\ &\quad \left.\left.+2\ arctan\left(\frac{2\lambda^2}{\lambda^3-\lambda+2}\right)\right]\right), \end{aligned} \tag{16}$$

and

$$\zeta_2 = \left(\bar{\alpha}_2 - (1-V)^4\right) V^{-2}$$
$$= \left(1 - \frac{\lambda^3}{\lambda^3+2}\sqrt{\frac{(\lambda^3-\lambda+2)^2+4\lambda^4}{(\lambda^2+1)^3}}\right)^{-2} \left(\frac{\lambda^3}{\lambda^3+2}\sqrt{\frac{(\lambda^3-4\lambda+2)^2+16\lambda^4}{(\lambda^2+4)^3}}\right.$$
$$cos\left[3\, arctan\left(\frac{2}{\lambda}\right) - arctan\left(\frac{4\lambda^2}{\lambda^3-4\lambda+2}\right) - 6\, arctan\left(\frac{1}{\lambda}\right)\right.$$
$$\left.\left. +2\, arctan\left(\frac{2\lambda^2}{\lambda^3-\lambda+2}\right)\right] - \left(\frac{\lambda^3}{\lambda^3+2}\sqrt{\frac{(\lambda^3-\lambda+2)^2+4\lambda^4}{(\lambda^2+1)^3}}\right)^4\right). \tag{17}$$

Table (4) shows the values of skewness and kurtosis coefficients for $\lambda = 0.5, 1, 1.5, 2, 2.5$.

Table 4: Skewness and Kurtosis Coefficients for Wrapped Ishita Distribution for $\lambda = 0.5, 1, 1.5, 2, 2.5$.

Coefficients	**Parameter λ**				
	0.5	1	1.5	2	2.5
Skewness ζ_1	0.003	0.225	1.022	-2.036	-2.980
Kurtosis ζ_2	-0.007	-0.010	-3.397	3.168	5.684

7 Conclusion

Wrapped Ishita distribution is a new circular distribution generated from wrapping Ishita distribution. We obtained the explicit expressions for the probability density function and the cumulative density function for wrapped Ishita distribution. Also, the trigonometric moments, measures of variation, and some special characteristics for the new distribution were discussed.

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Conflict of Interest

The authors declare that they have no conflict of interest.

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