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Some Aspects of Graded Singular Submodules and Goldie Dimension

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Abstract: In this paper we attempt to study singular submodules and Goldie dimension of modules in graded case. We investigate various aspects of graded torsion submodules and graded singular submodules of a graded module. If M is graded module over a ring R then it is seen that under certain conditions $T_g(M)$, the graded torsion submodule of M coincides with $Z_g(M)$, the graded singular submodule of M. We establish some characteristics of graded singular submodules. Defining graded singularity rank function $s-\rho_R(M)$ as the Goldie dimension of M modulo $Z_R(M)$. We prove that $s-\rho_R(M) = s-\rho_R(K) + s-\rho_R(M/K)$, where K is a graded submodule of M and M a graded right R- module with finite Goldie dimension showing thereby that graded singularity rank function is additive.

Keywords: Graded module, graded singular submodule, graded torsion submodule, Goldie dimension.

1 Introduction

In the theory of modules, the concept of torsion submodule, singular submodule and Goldie dimension are well known. The notion of a rank for a suitable kind of module was introduced by Goldie in 1964. This module can be reduced to the usual notion of rank for an abelian group or dimension of a vector space. Moreover, reduction of the study of arbitrary rings to singular and nonsingular cases helps to investigate more about the properties of the rings. Because of its useful consequences the importance of the study of rings and modules with associated graded structures is increasing day by day. As a result the graded analogues of various algebraic concepts like Goldie dimension, primary decomposition, Jacobson radicals, essentiality, singularity and non singularity etc are widely discussed. Motivated by these facts, in this paper our aim is to study some aspects of graded versions of Goldie dimension, torsion submodule and singular submodule. The objective of our work is to investigate various aspects of graded torsion submodules and graded singular submodules. It is seen that under certain condition $T_g(M)$,the graded torsion submodule of a graded module M coincides with the graded singular submodule of M, that is $T_g(M) = Z_g(M)$. We establish some characteristics of graded singular submodules. Defining graded singularity rank function $s-\rho_R(M)$ as the Goldie dimension of M modulo $Z_R(M)$ over R we prove that $s-\rho_R(M) = s-\rho_R(K) + s-\rho_R(M/K)$, where K a graded submodule of M and M a graded right R- module with finite Goldie dimension showing thereby that graded singularity rank function is additive.

2 Definitions and Notations

Throughout our discussion R is G graded ring where G is an ordered abelian group and M a graded R-module.

A graded R module M is said to have finite graded Goldie dimension if it doesnt contain a direct sum of an infinite number of non-zero graded submodules. Here $dimM$ will mean Goldie dimension of M. The graded rank of M, denoted by $\rho_R(M)$, is defined as the Goldie dimension of M over R.

A graded uniform module means a graded module that does not contain a direct sum of two non-zero graded submodules. If M has finite graded Goldie dimension then every graded submodule of M contains a graded uniform sub module. The graded singular submodule $Z_g(M)$ is defined as follows $Z_g(M) = \{x \in M | Ix = 0,$ where I is an essential left graded ideal in $R\}$.

We define the graded singularity rank or s- rank of M, denoted by $s-\rho_R(M)$ as the Goldie dimension of

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$M/Z_R(M)$ over R. An element $c \in h(R)$ is called right regular homogeneous if $r(c) = 0$ and left regular homogeneous if $l(c) = 0$. c is called regular homogeneous if $r(c) = l(c) = 0$. We write $C_G(0)$ for the set of all regular homogeneous elements of R. A homogeneous element x of M is called a torsion element if $xc = 0$ for some regular homogeneous element c of positive degree. We define $T_g(M) = \{xM|cx = 0,$ for some regular homogeneous element $c, degc > 0\}$.
Note: Let M be an R-module and N be an R_e submodule of M. We denote by $N^* = \cap R_g N$ where $g \in G$. Then N^* is the largest R submodule of M contained in N.

3 Preliminaries

In this section we discuss some preliminary results needed for the sequel. Following the same line of proof as in lemma 1.1, [1] we get

Lemma 3.1. Let M be graded right R-module. Let m be a nonzero homogeneous element of M. Let N be a graded essential submodule of M, then there is an essential graded right ideal I of R such that $mI \neq 0$ and $mI \subseteq N$.

*Proof.*We take $I_g = \{r_g \in R_g | mr_g \in N,$ for some $g \in G\}$. We claim that $I = \oplus I_g$ is a graded ideal of R.
Let $r \in I$ such that $r = r_{g1} + ... + r_{gk}$ with $r_{gi} \neq 0$. We want to show that $r_{gi} \in I$, for each i where $i \in \{1, ..., k\}$. Now, $r \in I$,
$\Rightarrow mr \in N$,
$\Rightarrow m(r_{g1} + ... + r_{gk}) \in N$,
$\Rightarrow mr_{g1} + ... + mr_{gk} \in N$,
$\Rightarrow mr_{gi} \in N$, (since N is graded submodule)
$\Rightarrow r_{gi} \in I_g \Rightarrow r_{gi} \in I$.
So, I is a graded ideal of R. Since N is graded essential submodule of M, we must have $mR \cap N \neq 0$. For some $r \in R$, we must have $0 \neq mr \in N$. Thus, we have $r \in I$ so that $mr \neq 0$ and hence, $mI \neq 0$. Let L be a nonzero graded ideal of R. We need to show that $I \cap L \neq 0$. If $mL = 0$, then $mL \subseteq N$. Then, from definition of I, we must have $L \subseteq I$ and hence, we obtain the required result. Let $mL \neq 0$. Since N is graded essential submodule of M, we must have $mL \cap N \neq 0$. For some $0 \neq x \in L$, we have $0 \neq mx \in N$ which implies that $x \in I$. Thus, we have $I \cap L \neq 0$.

Lemma 3.2. Let R be a graded ring with finite Goldie dimension and c be a regular homogeneous element of R. Then cR is graded essential ideal of R.

*Proof.*It is clear that cR is graded ideal of R. To prove the essentiality of cR. First we assume a nonzero graded ideal I of R. Suppose ,if possible $I \cap cR = 0$. Then, the sum $I + cI + c^2I + ...$is direct and since R has finite Goldie dimension, there exists some index n such that $c^nI = 0$ which implies that $I \subseteq ann(c^n)$and hence $I \subseteq ann(c) = 0$ as c is regular homogeneous element. So we have $I = 0$, which is a contradiction. So $I \cap cR \neq 0$ implies that cR is graded essential ideal of R.

Lemma 3.3. The torsion submodule $T_g(M)$ is a graded submodule of M.

*Proof.*Let $x, y \in T_g(M)$. Then $cx = 0, dy = 0$,for some c, d regular homogeneous elements of positive degree. Now, $Rc \leq_e R$, $Rd \leq_e R$. Therefore, $Rc \cap Rd \leq_e R$. This implies $Rc \cap Rd$ contains a regular homogeneous element e, say, with $dege > 0$. Then, $e(x - y) = ex - ey = 0$ which implies $x - y \in T_g(M)$.
Also, let $r \in h(R)$. Then $\exists r' \in h(R), s' \in C_G(0)$ such that $s'r = r'c$. Now, $(s'r)x = (r'c)x = r'(cx) = 0 \Rightarrow s'(rx) = 0$ implies $rx \in T_g(M)$. Thus, $T_g(M)$ is submodule of M.
Next we show $T_g(M)$ is a graded submodule of M. Let $0 \neq x \in T_g(M)$ such that $x = x\sigma_1 + ... + x\sigma_n$, where $x\sigma_i \in M\sigma_i, x\sigma_i \neq 0$ and $\sigma_1 < ...... < \sigma_n$. Then, for some regular homogeneous element c we have $cx = 0$ implies $c \in Ann_R(x)$. Thus, $c \in Ann_R(x\sigma_n)$ [since $Ann_R(x) \subset Ann_R(x\sigma_n)$]. This gives $cx\sigma_n = 0$. Hence, $x\sigma_n \in T_g(M)$.
For the element $x - x\sigma_n$, repeating the same argument we obtain by induction that $x\sigma_1, ..., x\sigma_n \in T_g(M)$.

Note: $T_g(M)$ is called graded torsion submodule of M. As in the ungraded case proposition 1.3 [3], we get the following:

Lemma 3.4. Let Y be a graded submodule of a graded R-module X and Y_1 be graded submodule of X such that Y_1 is maximal with respect to the property $Y \cap Y_1 = 0$. Then $Y \oplus Y_1 \leq_e X$.[Such submodules Y_1 always exist by Zorns Lemma and we call Y_1 graded relative complement for Y]

Lemma 3.5. Let M be a non-zero right graded R-module. If M has finite Goldie dimension then each non-zero graded submodule of M contains a graded uniform submodule and there is a finite number of graded uniform submodules of M whose sum is direct and is a graded essential submodule of M.

*Proof.*As in ungraded case [proposition1.9, [1]].

Lemma 3.6. Let N be a graded R-submodule of M and $N \leq_e M$. Then dimN = dimM.

*Proof.*Let $dimN = k$. Then $\exists$ uniform submodules $u_1, u_2, ..., u_k$ such that $u_1 + u_2 + ... + u_k$ is direct and $u_1 + u_2 + \quad + u_k \leq_e N$. Since $N \leq_e M$, $u_1 + u_2 + ... + uk \leq_e N \leq_e M \Rightarrow u_1 + u_2 + ... + u_k \leq_e M$. Thus, graded uniform submodules $u_1, u_2, ..., u_k$ of M such that $u_1 + u_2 + ... + u_k$ is direct and $u_1 + u_2 + ... + u_k \leq_e M \Rightarrow dimM = K$. So, $dimN = dimM$.

Theorem 3.7. Let G be an abelian group and R be a G-graded prime, graded Goldie ring. Then any essential graded ideal I of R contains a homogeneous regular element.

*Proof.*Theorem 4, [6].

4 Main Results

We now prove our main results.

Theorem 4.1. Let G be an abelian group and R be a G-graded prime, graded Goldie ring. Let M be a graded R -module. Then the graded torsion submodule of M coincides with the graded singular submodule of M, that is $T_g(M) = Z_g(M)$.

*Proof.*Let $x \in Z_g(M)$. Then, $Kx = 0, K \leq_e R$, for some essential graded ideal K of R. Then K contains a regular homogeneous element, c (say). This implies $cx = 0 \Rightarrow x \in T_g(M) \Rightarrow Z_g(M) \subseteq T_g(M)$.
Conversely, let $x \in T_g(M)$. Then, $cx = 0$, where c is a regular homogeneous element. Thus, we have $annc = 0$. So Rc is an essential graded left ideal of R. Thus, $cx = 0 \Rightarrow Rcx = 0$. Also $Rc \leq_e R \Rightarrow x \in Z_g(M)$. Hence, $T_g(M) \subseteq Z_g(M)$. This proves the result.

Theorem 4.2. Let R be a semiprime graded Goldie ring and X is a graded R- module of finite Goldie dimension with a graded submodule Y such that X/Y is torsion-free, then $dimX = dimY + dim(X/Y)$.

*Proof.*Given X is a graded R-module and Y is a graded R-submodule of X. Then by lemma 3.4, $\exists$ a graded submodule Y_1 of X such that $Y \cap Y_1 = 0$ and $Y \oplus Y_1 \leq_e X$. Let $x \in h(X)$ such that $x \notin h(Y)$. Then $\exists$ an essential graded ideal I such that $xI \subseteq Y \oplus Y_1$. [Lemma 3.1] Now I contains a regular homogeneous element c [lemma 3.7]. Then $xc \in Y \oplus Y_1$. If $xc \in Y$, then $xc \in T(X/Y)$, which contradicts X/Y is torsion-free. Let S be a graded submodule of X such that $S \cap Y \oplus Y_1 = Y$ Let $x \in S$ such that $x \notin Y$. Then $xc \in S$ such that $xc \notin Y$. Also, $xc \in Y \oplus Y_1$. Thus, $xc \in S \cap Y \oplus Y_1 = Y$, a contradiction. Hence, $Y = S$ Now $Y \oplus Y_1 \leq_e X \Rightarrow Y \oplus Y_1/Y \leq_e X/Y$ This implies

$$dim(X/Y) = dim(Y \oplus Y_1)/Y = dim(Y_1/Y \cap Y_1) = dim(Y_1)/(0) = dimY_1$$

Since Y and Y_1 are graded R-submodules of finite Goldie dimension, $dim(Y \oplus Y_1) = dimY + dimY1$.
Also $Y \oplus Y_1 \leq_e X \Rightarrow dimX = dim(Y \oplus Y_1) = dimY + dimY_1 = dimY + dim(X/Y)$.

Theorem 4.3. Let R be a semiprime graded Goldie ring. Let M be a graded right R-module with finite Goldie dimension and K, a graded submodule of M. Then $s - \rho_R(M) = s - \rho_R(K) + s - \rho_R(M/K)$.

*Proof.*Let L be a graded submodule of M such that $K \subseteq L$ and $L/K = Z_g(M/K)$. Clearly L/K is a graded R-module. We first show that $K + Z(M) \leq_e L$. Let $0 \neq y \in L$. Now,
$y + K \in Z_g(M/K)$
$\Rightarrow y + K \in T_g(M/K)$
$\Rightarrow d(y+K) = K$ for some regular homogeneous element d.
$\Rightarrow dy + K = K \Rightarrow dy \in K$
Assume, $dy \neq 0, dy \in Ry, dy \in K \Rightarrow dy \in K + Z_g(M)$. Thus, $0 \neq dy \in Ry \cap K + Zg(M)$. This implies $K + Z_g(M) \leq_e L$.

If $dy = 0$, then $y \in T_g(M) \Rightarrow y \in Z_g(M) \Rightarrow y \in K + Z_g(M)$.
Also $y \in Ry$. Thus, $y \in Ry \cap K + Z_g(M)$. This implies $Ry \cap K + Z_g(M) \neq 0$. As a consequence, we get $K + Z_R(M) \leq_e L$. This gives $dimL = dim(K + Z_R(M)), M/L \cong (M/K)/(L/K) = (M/K)/(Z_R(M/K))$, so that M/L is torsion free. Therefore, $dim(M/L) = dimM - dimL$.
As a consequence we get
$s - \rho_R(M/K) = dim(M/K)/(Z_R(M/K)) = dim(M/K)/(L/K) = dim(M/L) = dimM - dimL = dimM - dimZ_R(M) - (dimL - dimZ_R(M))$.

$s - \rho_R(K) = dim(K/Z_R(K)) = dim(K/KZ_R(M)) = dim((K + Z_R(M))/Z_R(M) = dim((K + Z_R(M)) - dimZ_R(M)$ [since $(K + Z_R(M))/Z_R(M)$ is torsion free]
Thus, $s - \rho_R(K) = dimL - dimZ_R(M)$ and hence $s - \rho_R(M/K) = dimM - dimZ_R(M) - s - \rho_R(K) = dimM/Z_R(M) - s - \rho_R(K) = s - \rho_R(M) - s - \rho_R(K)$.
This proves $s - \rho_R(M) = s - \rho_R(K) + s - \rho_R(M/K)$.

As in ungraded case - Theorem 2.5, [5] we get the following result:

Theorem 4.4. Let M be a graded R-module with finite Goldie dimension. N_1 and N_2 are two graded R-submodules of M such that $N = N_1 \cap N_2$ is graded relative complement in M. Then $dimN_1 + dimN_2 = dim(N_1 + N_2) + dimN$.

Corollary 4.5. Let M be a graded R-module with finite Goldie dimension. N_1 and N_2 are two graded R-submodules of M strictly containing $Z_R(M)$ such that $N = N_1 \cap N_2$ is graded relative complement in M. Then

$$s - \rho_R(N_1 + N_2) = s - \rho_R(N_1) + s - \rho_R(N_2) - s - \rho_R(N).$$

*Proof.*From definition of s-rank of a graded R-module, we must have
$s - \rho_R(N_1 + N_2) = dim(N_1 + N_2/Z_R(M))$
$= dim(N_1 + N_2) - dim(Z_R(M))$, [using Theorem 4.2]
$= dimN_1 + dimN_2 - dimN - dim(Z_R(M))$, [using Theorem 4.4]
$= dimN_1 - dim(Z_R(M)) + dimN_2 - dim(Z_R(M)) - dimN + dim(Z_R(M))$
$= dim(N_1/Z_R(M)) + dim(N_2/Z_R(M))dim(N/Z_R(M))$
$= s - \rho_R(N_1) + s - \rho_R(N_2) - s - \rho_R(N)$.

Theorem 4.6. Let G be an abelian group and R be a G-graded prime, graded Goldie ring and M, a graded R -module. Then
(i) If M is torsion, then $s - \rho_R(M) = 0$. (ii) If M is torsion free , then $s - \rho_R(M) = \rho_R(M)$.

Proof.(i) As M is torsion, we have $T_G(M) = M$. Then $s - \rho_R(M) = dim(M/Z_R(M)) = dim(M/T_G(M)) = 0$,[using theorem 4.1].
(ii) As M is torsion-free, we have $T_G(M) = 0$. Then $s - \rho_R(M) = dim(M/Z_R(M)) = dim(M/T_G(M)) = dim(M/(0)) = dim(M) = \rho_R(M)$,[using Theorem 4.1].

Theorem 4.7. Let o(G)=n and R be strongly G-graded ring without n-torsion. Then $s - \rho_{Re}(M) = s - \rho_R(M)$.

*Proof.*As R has no n-torsion, we have $Z_{Re}(M) = Z_R(M)$. Let us suppose that $s - \rho Re(M) = m$, then there exist non-zero Re-submodules $A_i/Z_{Re}(M)$ such that the sum $A_1/Z_{Re}(M) + ... + A_m/Z_{Re}(M)$ is direct modulo $Z_{Re}(M)$. This implies that the sum $A_1/Z_R(M) + ... + A_m/Z_R(M)$ is direct modulo $Z_R(M)$. Thus, we have $dimM/Z_R(M) = m$ which gives that $s - \rho_R(M) = m$. This proves the required result.

In the lemma 4.8 and the Theorem 4.9, we consider R to be almost strongly graded ring graded by a finite group G.

Lemma 4.8. Let M be an R-module. Then M contains an R_e- submodule $Z_{R_e}(N) = N_e$ (say) such that N_e is maximal with respect to $N_e^* = Z(M)$.

*Proof.*We set $Z_{R_e}(N_i) = N_{ei}, i \in I$, an index set and we consider the chain $\{N_{ei} | i \in I\}$ of R_e- submodules of M such that $N_{ei}^* = Z(M)$ for each i. We claim that $(\bigcup_{i\in I} N_{ei})^* = Z(M)$. If $(\bigcup_{i\in I} N_{ei})^* \neq Z(M)$ then there exist an $x \in (\bigcup_{i\in I} N_{ei})^*$ such that $ann(x)$ is not essential in R. We take $x = r_g x_{ei}$ where $r_g \in R_g$ and $x_{ei} \in N_{ei}$. For some nonzero graded ideal I_e of R_e we must have a nonzero graded ideal $\cap R_g I_e$ of R such that $\cap R_g I_e \cap ann(x) \neq 0 \Rightarrow \cap R_g I_e \cap ann(r_g x_{ei}) \neq 0 \Rightarrow I_e \cap ann(r_g x_{ei}) \neq 0 \Rightarrow r_g x_{ei} \notin N_{ei}$ which is a contradiction. So we must have $(\bigcup_{i\in I} N_{ei})^* = Z(M)$. Now by Zorns lemma, we can conclude that there exists an R_e-submodule $Z_{R_e}(N) = N_e$ (say) such that N_e is maximal with respect to $N_e^* = Z(M)$.

Theorem 4.9. Let M be an R-module such that $s - \rho_{R_e}(M) = m$. If $Z_{R_e}(M) = N$ (say) is an R_e-submodule such that N is maximal with respect to $N^* = Z(M)$, then $s - \rho_{R_e}(M) \leq m$.

*Proof.*Let $A_1/Z_{R_e}(M), ..., A_t/Z_{R_e}(M)$ be R_e - submodules of $M/Z_{R_e}(M)$ such that the sum $A_1/Z_{R_e}(M) + ... + A_t/Z_{R_e}(M)$ is direct modulo $Z_{R_e}(M)$. This gives the sum $A_1/Z_R(M) + ... + A_t/Z_R(M)$ is direct modulo $Z_R(M)$. Now, for each $i, 1 \leq i \leq t$, we have $A^*/Z_R(M) \neq Z_R(M)$. If $t > m$, then for some i, we must have

$$\sum_{j\neq i} A_j^*/Z_R(M) \cap A_i^*/Z_R(M) \neq Z_R(M)$$

which gives

$$\sum_{j\neq i} (A_j/Z_R(M) \cap Ai/Z_R(M))^* \neq Z_R(M)$$

as

$$\sum_{j\neq i} (A_j/Z_R(M) \cap Ai/Z_R(M))^* \supseteq \sum_{j\neq i} A_j^*/Z_R(M) \cap A_i^*/Z_R(M).$$

So we have

$$\sum_{j\neq i} A_j/Z_R(M) \cap Ai/Z_R(M) \neq Z_{R_e}(M)$$

and hence we obtain

$$\sum_{j\neq i} A_j/Z_{R_e}(M) \cap Ai/Z_{R_e}(M) \neq Z_{R_e}(M)$$

which contradicts our assumption. Therefore $s - \rho_{R_e}(M) \leq m$.

In [2], for any ring R, the authors have considered $R/Z(R)$ as a right R- module and have defined $G(R)$ to be the right ideal of R containing $Z(R)$ considering $G(R)/Z(R) = ZR(R/Z(R))$. As in ungraded case, we have seen a similar result for a graded ring.

Theorem 4.10. For every graded ring R,G(R) is a graded ideal of R such that $Z_R(R/G(R)) = 0$ and Z(R/G(R))=0.

*Proof.*Let $G(R) \subseteq R$ such that $a \in G(R)$ is homogeneous element of positive degree. Then $a + Z(R) \in G(R)/Z(R) = Z_R(R/Z(R))$. Thus, $annR(a + Z(R)) \leq_e R$.
Also,

$$\begin{aligned} ann_R(a + Z(R)) &= \{b \in h(R) | ab + Z(R) = Z(R)\} \\ &= \{b \in h(R) | ab \in Z(R)\}. \end{aligned}$$

For any $r \in h(R)$, $ann_R(a + Z(R)) \subseteq ann_R(ra + Z(R))$, for if $x \in ann_R(a + Z(R))$, and x is homogeneous, then we have $(a + Z(R))x = Z(R)$ which implies $ax + Z(R) = Z(R)$. This gives $ax \in Z(R)$. Thus, $rax \in rZ(R) = Z(R)$ gives $rax + Z(R) = Z(R)$. This implies that $(ra + Z(R))x = Z(R)$. Therefore, $x \in ann_R(ra + Z(R))$. So $ra \in G(R)$. This proves that $G(R)$ is an ideal of R. Next, we show $G(R)$ is, in fact, a graded ideal. Let $0 \neq x \in G(R)$ such that $x = x\sigma_1 + ... + x\sigma_n$ where $x\sigma_i \in R\sigma_i, x\sigma_i \neq 0$ and $\sigma_1 < .... < \sigma_n$. Then, $ann_R(x + Z(R)) = \{y \in h(R) | yx \in Z(R)\} \leq_e R$.
Also, we have $ann_R(x + Z(R)) \subseteq ann_R(x\sigma_n + Z(R))$ which implies that $ann_R(x\sigma_n + Z(R)) \leq_e R$.
We claim that $x\sigma_n \in G(R)$.
Suppose $x\sigma_n \notin G(R)$. Then $x\sigma_n \notin Z(R)$. So $0 \neq x\sigma_n + Z(R) \in R/Z(R)$ such that $ann_R(x\sigma_n + Z(R)) \leq_e R$. So $0 \neq x\sigma_n + Z(R) \in Z_R(R/Z(R))$ which is a contradiction since $x\sigma_n + Z(R) \notin G(R)/Z(R)$. So $x\sigma_n \in G(R)$.

Repeating the same argument, for the element $x - x\sigma_n$ by induction we obtain that $x\sigma_1, \ldots.., x\sigma_n \in G(R)$.
Let $a + G(R) \in Z_R(R/G(R))$ and $0 \neq H$ be a graded right ideal of R.
If $H \cap Z(R) \neq 0$ and $H \cap Z(R) \subseteq Z(R) \Rightarrow a(H \cap Z(R)) \subseteq aZ(R)Z(R)$. So $0 \neq b \in H$ such that $ab \in Z(R)$. Thus, $b \in \{b \in h(R) | ab \in Z(R)\} = ann_R(a + Z(R))$. This implies $ann_R(a + Z(R)) \cap H \neq 0$.
Suppose, $H \cap Z(R) = 0$ and let $0 \neq b \in H$ such that $ab \in G(R)$,so $b \notin Z(R)$. So there exists a nonzero right graded ideal I of R such that $ann_R(b) \cap I = 0$. Also $ab \in G(R)$ and $G(R)/Z(R) = Z_R(R/Z(R))$. Therefore $ann_R(ab + Z(R)) \cap I \neq 0$. So there exists $0 \neq c \in I$ such that $c \in ann_R(ab + Z(R)) \Rightarrow (ab + Z(R))c = Z(R) \Rightarrow abc \in Z(R)$.
Now $0 \neq b \in H, 0 \neq c \in I \Rightarrow 0 \neq bc \in H$. Then $abc \in Z(R) \Rightarrow abc + Z(R) = Z(R) \Rightarrow (a + Z(R))bc = Z(R) \Rightarrow bc \in ann_R(a + Z(R))$. Thus, $ann_R(a + Z(R)) \cap H \neq 0$. So, $ann_R(a + Z(R)) \leq_e R$. The above implies $a + Z(R) \in Z_R(R/Z(R)) = G(R)/Z(R)$. Thus, $a \in G(R)$. This implies $Z_R(R/G(R)) = 0$.
Let $a + G(R) \in Z(R/G(R))$. Then $ann_{R/G(R)}(a + G(R)) \leq_e R/G(R)$.
Let $ann_{R/G(R)}(a + G(R)) = J/G(R)$. Then, $J/G(R) \leq_e R/G(R) \Rightarrow J \leq_e R \Rightarrow ann_{R/G(R)}(a + G(R)) \leq_e R \Rightarrow (a + G(R)) \in Z_R(R/G(R)) = 0 \Rightarrow (a + G(R)) = 0$.
Hence, $Z(R/G(R)) = 0$.

5 Conclusion

The study of Goldie dimension and singularity can be continued by modifying the notion of singularity and rank function. One can investigate various aspects of graded module with finiteness conditions like ACC on graded essential submodules, ACC on graded supplemented submodules, chain conditions on graded small submodules etc. The study of graded modules with finiteness conditions will lead to the finiteness conditions of graded socle, graded radical of the graded module and various related results can be studied in graded case. This would throw light on the understanding of the theory in a unified and systematic manner and would also provide avenue for looking at their applications.

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