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Boundary gradient observability for semilinear parabolic systems: Sectorial approach

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Abstract: The aim of this paper is to study the notion of the gradient observability on a subregion Γ of the boundary of the evolution domain Ω for the class of semilinear parabolic systems. We show, under some hypotheses, that the flux reconstruction is guaranteed by means of the sectorial approach combined with fixed point techniques. This leads to several interesting results which are performed through numerical examples and simulations.

Keywords: Distributed systems, parabolic systems, boundary gradient reconstruction, regional boundary observability, fixed point, sectorial operator.

1 Introduction

The regional observability is one of the most important notion of systems theory, it consist to reconstruct the trajectory only in a subregion ω in the whole domain Ω, this concept has been widely developed (see [9]). Afterwards, the concept of regional gradient observability has been developed (see [7]), it concerns the reconstrcution of the state gradient only in a critical subregion interior to the system domain without the knowledge of the state. This concept finds its application in many real world problems. (See [8, 2]).

The goal of this work is to study the regional boundary gradient observability of an important class of semilinear parabolic systems. For the sake of brevity and simplicity we shall focus our attention on the case where the dynamic of the system is a sectorial operator linear and generating an analytical semigroup $(S(t))_{t\geq 0}$ on the Hilbert space X.

Let Ω be an open bounded subset of IR^n, (n=1,2,3), with smooth boundary $\partial\Omega$, we denote $Q=\Omega\times[0,T]$, $\Sigma=\partial\Omega\times\,]0,T[$, and N is a nonlinear operator assumed to be defined so that to ensure the existence and the uniqueness of the solution of the following semilinar parabolic system with Dirichlet conditions:

$$\begin{cases} \dfrac{\partial y(x,t)}{\partial t} = Ay(x,t) + Ny(x,t), & Q, \\ y(\xi,t) = 0 \quad , & \Sigma, \\ y(x,0) = y_0 \quad , & \Omega. \end{cases} \tag{1-1}$$

The regional continuous boundary observability of the gradient concerns the reconstruction of the gradient of the state on a subregion Γ of $\partial\Omega$. Concerning the linear case, research in this problem and other related questions (controllability, stability) has been very active during the last few years and there is an extensive literature on these topics (see [2], [6] and [7]).

Always in the linear framework, we may say that the regional boundary (exact or approximate) gradient observability problem is by now well understood. In contrast, very little is known about the regional semilinear problem, or more generally, in the nonlinear context.

In this paper, some definitions and properties are introduced for the study of this notion. In this way the method of the reconstruction is proven for Lipschitz continuous N satisfying the additional condition:

$$\lim_{|s|\to\infty}\frac{N(s)}{s} \quad \text{exists.}$$

The regional boundary gradient observability is proven in the space $L^2(\Omega)$ (the largest Hilbert space where semilinear problem has some sens), for "large enough" T and $\partial\Omega$. The method of proof is based on sectorial approach (see [3] and [4]) in the internal framework and on a fixed point argument. The plan of the paper is as follows: Section 2 is devoted to the presentation of problem of the boundary regional gradient of the considered system, and then we give definitions and propositions of this new concept. Section 3 concerns the sectorial approach which is illustrated by a numerical example.

2- Problem statement

Let Ω be an open bounded subset of IR^n, n=1,2,3. For T>0. We consider a parabolic system described by

$$\begin{cases} \frac{\partial y(x,t)}{\partial t} = Ay(x,t) \ , & Q\ , \\ y(\xi,t) = 0 \ , & \Sigma, \\ y(x,0) = y_0 \ , & \Omega. \end{cases} \tag{2-1}$$

where A is a second order differential linear operator which generates a strongly continuous semi-group $(S(t))_{t>0}$ in the Hilbert space $L^2(\Omega)$.

The system (2-1) is augmented with the output function given by

$$z(t) = Cy(.,t),\ t \in [0,T], \tag{2-2}$$

where $C: H_0^1(\Omega) \cap H^2(\Omega) \to IR^q$ is a linear operator and depends on the number q and the nature of the considered sensors. The observation space is $O = L^2(0,T;IR^q)$. Let K be the observation operator defined by

$$K: H_0^1(\Omega) \cap H^2(\Omega) \longrightarrow O$$

$$h \longmapsto CS(.)h,$$

We note that K^* is the adjoint of K. The gradient operator is defined by

$$\nabla: H_0^1(\Omega) \cap H^2(\Omega) \longrightarrow (L^2(\Omega))^n$$

$$y \longmapsto \left(\frac{\partial y}{\partial x_1}, \dots, \frac{\partial y}{\partial x_n}\right)$$

The initial state y_0 and its gradient are assumed to be unknown. The trace operator, the trace operator is defined by

$$\gamma : \left(H^1(\Omega)\right)^n \longrightarrow \left(H^{\frac{1}{2}}(\partial\Omega)\right)^n$$
$$z \longmapsto \gamma z = (\gamma_0 z_1, \dots, \gamma_0 z_{1n})$$

with $\gamma_0 : H^1(\Omega) \dashrightarrow H^{\frac{1}{2}}(\partial\Omega)$ is the linear, surjective and zero order trace operator, while γ_0 (resp. γ) is the adjoint of γ_0 (resp. γ).

For $\Gamma \subset \partial\Omega$,
$$\chi_\Gamma : \left(H^{\frac{1}{2}}(\partial\Omega)\right)^n \longrightarrow \left(H^{\frac{1}{2}}(\Gamma)\right)^n$$

$y \longmapsto y_{|\Gamma}$ while χ_Γ^* is the adjoint of χ_Γ, and we consider the operator $H = \chi_\Gamma \gamma \nabla K^*$.

Definition 1 *The system (2-1)-(2-2) is said to be exactly (resp. approximately) gradient observable on Γ if $Im(H) = (H^{\frac{1}{2}}(\Gamma))^n$ (resp. $\overline{Im(H)} = (H^{\frac{1}{2}}(\Gamma))^n$). (For more details see [6] and [7]).*

Remark 2 *If the system is exactly (approximately) gradient observable on Γ then it is exactly (resp. approximately) gradient observable on every subset $\Gamma_1 \subset \Gamma$.*

We consider the following semilinear parabolic system

$$\begin{cases} \frac{\partial y(x,t)}{\partial t} = Ay(x,t) + Ny(x,t), & Q, \\ y(\xi,t) = 0 \quad , & \Sigma, \\ y(x,0) = y_0 \quad , & \Omega. \end{cases} \tag{2-3}$$

augmented with the output function

$$z(t) = Cy(.,t), \ t \in [0,T] \tag{2-4}$$

Where N is a nonlinear operator assumed to be locally Lipschitzian.

Without loss of generality, we denote $y(.,t)$ by $y(t)$ and we give the following definition

Definition 3 *The semilinear system (2-3)-(2-4) is said to be continuously gradient observable on Γ if we can reconstruct the gradient of its state on a subregion Γ of $\partial\Omega$.*

Let the gradient of the initial state be decomposed as follows

$$\nabla y_0 = \begin{cases} y_0^1 & \Gamma, \\ y_0^2 & \partial\Omega\backslash\Gamma. \end{cases}$$

The study of regional boundary gradient observability amounts to solving the following problem:

Problem 4 *Given system (2-3) and output (2-4) on $[0,T]$, is it possible to reconstruct y_0^1 which is the gradient of initial state of (2-3) on Γ?*

3- Sectorial case

In this study, Problem 4 is considered, where A is a linear operator generating an analytical semigroup $(S(t))_{t>0}$ on the Hilbert space $X = L^2(\Omega)$ (see [5]).

Let us consider $A_1 = A + aI$, where a is real and such that $Re(\sigma(A_1)) > \delta > 0$, while $Re(\sigma(A_1))$ denote the real part of the spectrum of A_1.Then for $0 \leq \alpha < 0$, we define the fractional power $(A_1)^\alpha$ as a closed operator with domain $X^\alpha = D(A_1^\alpha)$ (see [3]) which is a dense Banach space on X endowed with the graph norm:

$$\|x\|_{X^\alpha} = \|A_1^\alpha(x)\|_X.$$

Let us consider $W = Im(\chi_\Gamma \gamma \nabla K^*)$ then the objective is to study the Problem 4 in W endowed with the norm

$$\|.\|_W = \|H^*(.)\|_Y.$$

We have

$$\|S(t)\|_{\mathcal{L}(X,X^\alpha)} = ct^\alpha \exp(a - \delta)\, t = g_1(t)$$

We assume that $g_1 \in L^q(0,T)$ for $r, s, q \geq 1$ such that $\frac{1}{q} = 1 + \frac{1}{r} - \frac{1}{s}$ and that the operator $N: L^r(0,T;X^\alpha) \to L^s(0,T;X)$ is well defined and satisfies the following conditions (see [4]):

$$\begin{cases} \|Nx - Ny\|_{L^s(0,T;X)} \leq k(\|x\|, \|y\|)\|x - y\|_{L^r(0,T;X^\alpha)} (\forall\, x, y \in L^r(0,T;X^\alpha)) \\ N(0) = 0 \text{ where } k: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \text{ such that } \lim_{\theta_1,\theta_2 \to 0} k(\theta_1,\theta_2) = 0 \end{cases} \tag{3-1}$$

3-1 Reconstruction method

Let's consider the following map

$$\varphi\left(y_0^1, y(.)\right) = S(t)\nabla^*\gamma^*\chi_\Gamma^* y_0^1 + S(t)\bar{y}_0 + L(t)Ny,$$

where $\bar{y}_0$ is the residual part, and we have the following result.

Proposition 5 *If the hypothesis (3-1) and*

$$\|S(.)\|_{\mathcal{L}(X^\alpha,W)} \leq g_2(.) \in L^r(0,T)$$

Hold, then

- *there exist numbers a and $m > 0$ such that for all y_0^1 in the ball $B(0,m) \subset W$ the function $\varphi(y_0^1,.)$ has a unique fixed point in $B(0,a) \subset L^r(0,T;X^\alpha)$, and.*
- *the map* $$\begin{aligned} f: B(0,m) &\longrightarrow B(0,a) \\ y_0^1 &\longmapsto y(.) \end{aligned}$$

with $y(.)$ is the solution of system (2-3) corresponding to the initial state gradient y_0^1 on Γ and the residual part $\overline{y}_0$ on $\partial\Omega\backslash\Gamma$, satisfies the Lipschitz condition.

Proof.

- Let's consider y_1 and y_2 in $B(0,a) \subset L^r(0,T;X^\alpha)$ and $y_0^1 \in W$ then we have:

$$\left\|\varphi\left(y_0^1, y_2(.)\right) - \varphi\left(y_0^1, y_1(.)\right)\right\|_{L^r(0,T;X^\alpha)} = \|L(.)(Ny_2 - Ny_1)\|_{L^r(0,T;X^\alpha)}$$

$$\leq \|g_1\|_{L^q(0,T)}\, k(\|y_1\|, \|y_2\|)\, \|y_2 - y_1\|_{L^r(0,T;X^\alpha)}$$

$$\leq \|g_1\|_{L^q(0,T)}\, \sup_{\theta_i \leq a} k(\theta_1, \theta_2)\, \|y_2 - y_1\|_{L^r(0,T;X^\alpha)}$$

On the other hand we have

$$\lim_{\theta_1, \theta_2 \to 0} k(\theta_1, \theta_2) = 0.$$

If there exists $a > 0$ such that

$$C_1 = \|g_1\|_{L^q(0,T)}\, \sup_{\theta_i \leq a} k(\theta_1, \theta_2) < 1,$$

then

$$\left\|\varphi\left(y_0^1, y_2(.)\right) - \varphi\left(y_0^1, y_1(.)\right)\right\|_{L^r(0,T;X^\alpha)} \leq C_1 \|y_2 - y_1\|_{L^r(0,T;X^\alpha)}.$$

Then $\varphi\left(y_0^1, y_2(.)\right)$ is Lipschitzian. Further we have

$$\|\varphi(y_0^1, y)\|_{L^r(0,T;X^\alpha)} \leq \left\|S(t)\nabla^*\gamma^*\chi_\Gamma^* y_0^1\right\|_{L^r(0,T;X^\alpha)} + \left\|S(t)\overline{y}_0\right\|_{L^r(0,T;X^\alpha)} + \|L(t)Ny\|_{L^r(0,T;X^\alpha)}$$

$$\leq \|g_2\|_{L^r(0,T)} \left\|y_0^1\right\|_W + \|g_2\|_{L^r(0,T)} \left\|\overline{y}_0\right\|_X$$

$$+ \|g_1\|_{L^q(0,T)} \sup_{\theta \leq a} k(\theta, 0)\, \|y\|_{L^r(0,T;X^\alpha)}$$

Let's consider

$$\left\|y_0^1\right\|_W \leq \frac{a}{\|g_2\|_{L^r(0,T)}} \left(1 - \|g_1\|_{L^q(0,T)} \sup_{\theta \leq a} k(\theta, 0)\right) - \left\|\overline{y}_0\right\|_X$$

Furthermore we have

$$y \in B(0,a) \Longrightarrow \varphi(y_0^1, y) \in B(0,a)$$

- Let's consider y_1 and y_2 be two different solutions of system (2-3) corresponding respectively to the initial gradient y_0^1 and y_0^2 on Γ with the same residual part, then we have

$$\|f(y_0^2) - f(y_0^1)\|_{L^r(0,T;X^\alpha)} \leq \left\|S(t)\nabla^*\gamma^*\chi_\Gamma^*(y_0^2 - y_0^1) + L(.)(Ny_2 - Ny_1)\right\|_{L^r(0,T;X^\alpha)}$$

$$\leq \|g_2\|_{L^r(0,T)} \left\|y_0^2 - y_0^1\right\|_W$$

$$+ \ \|g_1\|_{L^q(0,T)} \sup_{\theta_i \leq a} k(\theta_1,\theta_2) \|y_2 - y_1\|_{L^r(0,T;X^\alpha)}$$

$$\leq \frac{\|g_2\|_{L^r(0,T)}}{1 - C_1} \|y_0^2 - y_0^1\|_W$$

We deduce that f is Lipschitzian.■

If we consider measurements in $B(0,\rho)$, the regional gradient of initial state can be obtained as a solution of a fixed point problem. Let us consider the following map

$$\psi(z, y_0^1) = (H^*)^\dagger \left(z - CS(.)\bar{y}_0 - CS(.)Nf(y_0^1)\right).$$

To find the fixed point of $\psi(z, y_0^1)$, this requires that $CL(.)Nf(y_0^1)$ is in $Im(H^*)$ for all $y_0^1 \in W$.

Then we obtain the following result.

Proposition 6 *We suppose that* $\|CL(.)y\|_Y \leq c_4\|y\|_{L^s(0,T;X)}$ *and* $\|CS(.)\|_Y \leq c_5\|y\|_X$.

If the linear part of the system (2-3) is approximately regionally gradient observable on Γ and (3-1) is satisfied then there exists $\rho(a) > 0$ such that for all $z \epsilon B(0,\rho) \subset Y$, Problem 4 has a unique solution $y_0^1 \in B(0,m)$.

Proof.

Let us consider y_0^1 and y_0^2 in $B(0,m) \subset W$, then we have

$$\left\|\psi(z,y_0^2) - \psi(z,y_0^1)\right\|_W = \left\|(H^*)^\dagger(CL(.)(Nf(y_0^2) - Nf(y_0^1))\right\|_W$$

$$= \left\|(H^*(H^*)^\dagger(CL(.)(Nf(y_0^2) - Nf(y_0^1))\right\|_Y$$

$$= \left\|CL(.)(Nf(y_0^2) - Nf(y_0^1))\right\|_Y,$$

Using (3-1) we obtain

$$\left\|\psi(z,y_0^2) - \psi(z,y_0^1)\right\|_W \leq c_4 k(\left\|f(y_0^2)\right\|, \left\|f(y_0^1)\right\|)\left\|f(y_0^2) - f(y_0^1))\right\|_{L^r(0,T;X^\alpha)}$$

$$\leq c_4 \sup_{\theta_i \leq a} k(\theta_1,\theta_2) \frac{\|g_2\|_{L^r(0,T)}}{1-C_1} \left\|y_0^2 - y_0^1\right\|_W.$$

We know that

$$c_4 \sup_{\theta_i \leq a} k(\theta_1,\theta_2) \frac{\|g_2\|_{L^r(0,T)}}{1 - C_1} < 1,$$

then $\psi(z,.)$ is a contraction on $B(0,m)$.

On the other hand we have

$$\left\|\psi(z,y_0^1)\right\|_W = \left\|z - CS(.)\overline{y}_0 - CL(.)Nf(y_0^1)\right\|_Y$$

$$\leq \|z\|_Y + c_5\left\|\overline{y}_0\right\|_X + ac_4 \ \sup_{\theta\leq a} k(\theta,0),$$

if we consider

$$\|z\|_W \leq m - (ac_4 \ \sup_{\theta\leq a} k(\theta,0) + c_5\left\|\overline{y}_0\right\|_X).$$

In conclusion, $\psi(z,.)$ admits a unique fixed point.■

Proposition 7 The sequence of regional initial gradient

$$\begin{cases} \tilde{y}_{0,0} = 0, \\ \tilde{y}_{0,n+1} = (H^*)^{\dagger}(z - CS(.)\overline{y}_0 - CL(.)Nf(\tilde{y}_{0,n}))' \end{cases}$$

converges to the desired initial state gradient $\tilde{y}_0^1$to be observed on Γ.

Proof. With minor difference techniques, the proof is similar to the state regional case

(see [1], [8]).■

Let's consider $y_n = f(y_0^1)$ and $r_{n+1} = z - CS(.)\overline{y}_0 - CL(.)Ny_n$. Then $z_n = Cy_n = r_n + z + r_{n+1}$ and we have the following algorithm:

Algorithm.

Step 1: The initial state$y_{0,1}$, the subregion Γ, the location of the sensor D the function f, Threshold accuracy $\varepsilon, r_1 = z$.

Step 2: Repeat

- $y_{0,n}^1 = (H^*)^{\dagger} r_n$
- $\dot{y}_n = Ay_n + Ny_n;\ y_n(0) = \nabla^*\gamma^*\chi_\Gamma^* y_{0,n}^1$
- $z_n = Cy_n$
- $r_{n+1} = r_n + z + z_n$

Until ($\|z - z_n\| \leq \varepsilon$)

Step 3: The restriction of $y_{0,n}^1$ to Γ corresponds to $y_0^1$to be reconstructed on Γ.

3-2 Simulation results

Here, we present a numerical example which illustrates the previous algorithm. The obtained results are related to the considered subregion and the location of the sensor.
Let's consider the system described in Ω= [0,1]×[0,1] by the following equation :

$$\begin{cases} \frac{\partial y}{\partial t}(x,t) = 0.01\sum_{i=1}^{2}\frac{\partial^2 y(x,t)}{\partial x_i^2} + \sum_{k,l=1}^{\infty}|\langle y(t),\varphi_{kl}\rangle|\langle y(t),\varphi_{kl}\rangle\varphi_{kl}(x), & Q, \\ y(x,0) = y_0(x) & , \quad \Omega, \\ y(\xi,t) = 0 & , \quad \Sigma. \end{cases} \tag{3-3}$$

Where $x = (x_1, x_2)$ and $(\varphi_{kl})_{kl}$is a complete set of $H^1(\Omega)$. The system (3-3) is augmented with the output function described by a pointwise sensor located in $(b_1, b_2\ )$ where $b_1 = 0.45$, $b_2 = 0.21$ and $T = 2$.

$$z(t) = y(b_1, b_2\ , t), \qquad t \in [0, T]. \tag{3-4}$$

Let's consider

$$\tilde{y}_0(x, y) = (g_1(x, y), g_2(x, y)),$$

with

$$\begin{cases} g_1(x,y) = \sin(y)\cos\left(\frac{y\pi}{2}\right)\left(\cos(x)\, cos\left(\frac{x\pi}{2}\right) - \frac{1}{2}\sin(x)\sin\left(\frac{x\pi}{2}\right)\right) \\ g_2(x,y) = \sin(x)\cos\left(\frac{x\pi}{2}\right)\left(\cos(y)\, cos\left(\frac{y\pi}{2}\right) - \frac{1}{2}\sin(y)\sin\left(\frac{y\pi}{2}\right)\right) \end{cases}$$

be the initial state gradient

to be observed on $\Gamma = \{0\} \times [0,1]$. Using the algorithm in section 3-1, we obtain the following results.

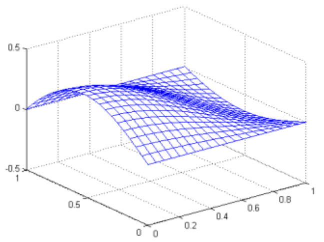

Figure 1: The desired initial gradient in Ω.

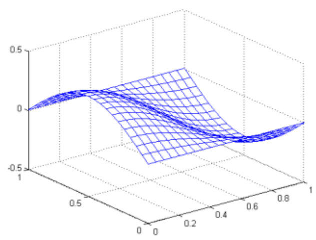

Figure 2: The estimated initial gradient in Ω.

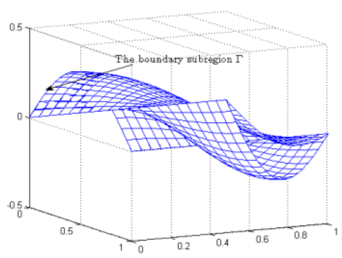

Figure 3: The exact and the estimated initial gradient in Ω.

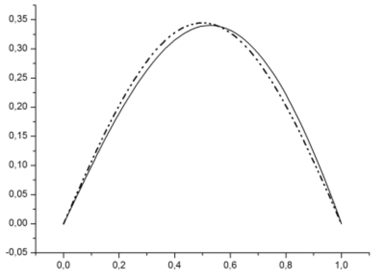

Figure 4: The exact (continuous line) and estimated (dashed line) gradient on Γ.

The initial state gradient is obtained with reconstruction error $\|\tilde{y}_0 - \tilde{y}_{oe}\|^2_{(L^2(\Gamma))^2} = 6.64 \times 10^{-3}$.

The above figure shows that:

1. For a given subregion ω, there is an optimal sensor location (optimal in the sense that it leads to a minimum error corresponding to the best estimation of the solution which is very close to the initial state gradient).
2. The estimated initial state gradient is very close to the exact one, which shows the efficiency of the considered approach.

4- Conclusion

The question of the regional boundary gradient observability for semilinear parabolic systems was discussed and solved using the sectorial approach, which uses sectorial properties of dynamical operators, the fixed point techniques and the properties of the linear part of the considered system. The obtained results lead to an algorithm which is successfully implemented numerically and illustrated with example and simulations, the obtained results are related to the considered subregion and the sensor location. Many questions remain open, such as the case of the regional gradient observability of semilinear systems using Hilbert Uniqueness Method (HUM) and the case of the regional gradient observability of hyperbolic systems. These questions are still under consideration and the results will appear in a separate paper.

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