Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/050404

Analytic solution of space time fractional advection dispersion equation with retardation for contaminant transport in porous media

Ritu Agarwal[1], *Mahaveer Prasad Yadav*[1,*], *Ravi P. Agarwal*[2] *and Dumitru Baleanu*[3,4]

[1] Department of Mathematics, Malaviya National Institute of Technology, Jaipur-302017, India
[2] Department of Mathematics, Texas A&M University - Kingsville, 700 University Blvd. Kingsville, TX 78363-8202,USA
[3] Department of Mathematics, Cankaya University, 06530 Balgat, Ankara, Turkey
[4] Institute of Space Science, R 76900, Magurele-Bucharest, Romania

Received: 2 Apr. 2019, Revised: 2 Jul. 2019, Accepted: 11 Jul. 2019
Published online: 1 Oct. 2019

Abstract: Motivated by recent applications of fractional calculus, in this paper, we derive analytical solutions of fractional advection - dispersion equation with retardation by replacing the integer order partial derivatives with fractional Riesz - Feller derivative for space variable and Caputo fractional derivative for time variable. The Laplace and Fourier transforms are applied to obtain the solution in terms of the Mittag - Leffler function. Some interesting special cases of the time - space fractional advection - dispersion equation with retardation are also considered. The composition formulas for Green function has been evaluated which enables us to express the solution of the space time fractional advection dispersion equation in terms of the solution of space fractional advection dispersion equation and time fractional advection dispersion equation. Furthermore, from this representation we derive explicit formulae, which enable us to plot the probability densities in space for the different values of the relevant parameters.

Keywords: Caputo derivative, Riesz-Feller derivative, Mittag-Leffler functions, Laplace-Fourier transform, H-function.

1 Introduction

Model of advection-dispersion equation used for the prediction of transport of nonreactive dissolved contaminants in the groundwater. Governing equation of advection-dispersion equation in one-dimension is given by

$$\frac{\partial C(x,t)}{\partial t} = -v_x \frac{\partial C(x,t)}{\partial x} + \mathscr{D}_x \frac{\partial^2 C(x,t)}{\partial x^2}. \tag{1}$$

The one-dimensional advection-dispersion transport equation with retardation can be obtained through sorption, radioactive decay, chemical reaction, biological transformations etc. Molecules and ions are exchanged by sorption from solid phase to the liquid phase, including both adsorption and desorption. Adsorption is an excess of contaminant concentration at the surface of a solid by which contaminants move slower than the flowing groundwater, this effect is called retardation process. A retardation factor of 10 implies that contaminant plume moves 10 times slower than the groundwater velocity. Retardation factor is a measure of the retardation effect of the adsorption process on the transport of the chemical in the groundwater. In groundwater, the retardation effect caused by the adsorption process is measured by retardation factor.
The amount of contaminant adsorbed by the solids is a function of the concentration in solution, $S = f(C)$ [1, p. 386]. Governing equation that includes retardation, in a homogeneous saturated media due to adsorption, adds a source-sink term:

$$\frac{\partial C(x,t)}{\partial t} = -v \frac{\partial C(x,t)}{\partial x} + \frac{\rho_b}{n} \frac{\partial S(x,t)}{\partial t} + \mathscr{D}_{\mathscr{L}} \frac{\partial^2 C(x,t)}{\partial x^2}, \tag{2}$$

* Corresponding author e-mail: mahaveeryadav3009@gmail.com

and for linear isotherm $dS(x,t) = K_d\, dC(x,t)$, K_d being distribution coefficient, equation (2) becomes

$$\left(1+\frac{\rho_b}{n}K_d\right)\frac{\partial C(x,t)}{\partial t} = -v\frac{\partial C(x,t)}{\partial x} + \mathscr{D}_{\mathscr{L}}\frac{\partial^2 C(x,t)}{\partial x^2}. \tag{3}$$

$$R\frac{\partial C(x,t)}{\partial t} = -v\frac{\partial C(x,t)}{\partial x} + \mathscr{D}_{\mathscr{L}}\frac{\partial^2 C(x,t)}{\partial x^2}. \tag{4}$$

Here R is called the retardation factor, $\mathscr{D}_{\mathscr{L}}$ is the longitudinal hydrodynamic dispersion coefficient, ρ_b is the bulk mass density of porous media, n is the porosity of the medium, $\frac{\partial S(x,t)}{\partial t}$ is the rate at which constituent is adsorbed, $\frac{\rho_b}{n}\frac{\partial S(x,t)}{\partial t}$ is the change in concentration in groundwater caused by adsorption or desorption, v is the average linear velocity along flow line.

A list of previous work in which analytical solutions have been obtained for advection dispersion equation with retardation factor are as follows:

Genuchten [2] derived an analytic solution for the flow of a chemical in porous medium as affected by adsorption and decay.

Genuchten and Alves [3] derived mathematical models for advection-dispersive equation (4) and solved analytically. Chrysikopoulous et al. [4] derived a closed form analytic solution to the one dimensional advection dispersion equation (4) with spatially variable retardation factor to investigate the transport of sorbing solutes in heterogenous porous media. Van Kooten [5] presented a stochastic nature of first order kinetic process to decouple the advection dispersion equation and the sorption. Liu et al. [6] derived an analytical solution to the one-dimensional solute advection-dispersion equation. Chang et al. [7] applied linear and nonlinear equilibrium - controlled sorption process to study the retardation factor used in the cooper and cadmium transport. Liu and Si [8] developed the orthogonal expansion technique to solve the diffusion in layered porous media.

One of the problems of using advection - dispersion equation to simulate the transport process is the scale dependent dispersion, i.e. the dispersion coefficient increases with contaminant travel distance [9,10]. However, research results of using fractional advection dispersion equation to simulate the data of small scale column test [11] showed that the dispersion coefficient of fractional advection dispersion equation does not change significantly with the length of the column, and the scale effect is reflected by the order of the fractional derivative. Lu et al. [12] found that the dispersion coefficient is still scale dependent when modelling the large scale solute transport with fractional advection dispersion equation.

These days, efforts have been made to generalize the integer order advection dispersion equation to fractional order advection dispersion equation. Yildirim and Kocak [13] solved space time fractional advection dispersion equation by homotopy perturbation method. Doha et al. [14] presented two efficient spectral methods to solve two kinds of space fractional linear advection-dispersion problems.

Huang et al. [15] solved fractional advection dispersion equation by finite element method. Hohener and Atteia [16] presented analytical models which predict concentrations and isotope ratios of organic pollutants in homogenous medium in aquifers. Huang et al. [17] developed a FORTRAN based program named FADEMain for modelling adsorbing contaminant transport in homogenous media.

Recently, Atangana and Kilicman [18] obtained solution of fractional advection dispersion equation

$$\frac{\partial^\alpha C(x,t)}{\partial x^\alpha} - v\frac{\partial^\beta C(x,t)}{\partial x^\beta} - \lambda Rc = R\frac{\partial C(x,t)}{\partial t},\ 0<\beta\le 1<\alpha\le l, \tag{5}$$

using Riemann Liouville and Caputo fractional derivatives, with initial and boundary conditions are as

$$C(x,0)=0,\ C(0,t)=c_0 exp(-\gamma t),\ \frac{\partial C(x,t)}{\partial x}=0 \text{ as } x\to\infty,\ 0<t\le t_0. \tag{6}$$

In our problem, we fractionalize equation (4) with respect to both space and time variables. We use Caputo fractional derivative for time variable and Riesz-Feller derivative of order β and skewness θ for space variable

$$R\,{}_0^C D_t^\alpha C(x,t) = -vD_x\, C(x,t) + \mathscr{D}_{\mathscr{L}}\, {}_x D_\theta^\beta C(x,t),\ x\in\mathbb{R},\ t\in(0,\infty),\ \alpha\in(0,1],\ \beta\in(0,2], \tag{7}$$

with initial and boundary conditions are as

$$C(x,0)=f(x),\ C(x,t)=0 \text{ as } |x|\to\infty, \tag{8}$$

where $\nu > 0$, $\mathscr{D}_{\mathscr{L}} \geq 0$. We have generalized the initial condition for a sufficiently well-behaved function $f(x) \in L^c(\mathbb{R})$, defined and continuous in an open interval except, possibly, at isolated points where these functions can be infinite and $L^c(\mathbb{R})$ is a class of functions for which the Riemann improper integral absolutely converges. Here ${}_a^C D_t^\alpha$ is the Caputo time fractional derivative of order α $(0 < \alpha \leq 1)$, ${}_x D_\theta^\beta$ is the Riesz-Feller space fractional derivative of order β $(0 < \beta \leq 2)$ and skewness θ $(|\theta| \leq \min\{\beta,\, 2-\beta\})$. For $\nu = 0$, our problem reduces to space time diffusion equation.

Since Riesz-Feller derivative is able to model the problems for either sides of domain, no condition is required to be imposed at the source. For $\theta = 0$ case, Riesz-Feller space fractional derivative reduces to symmetric fractional derivative [19, Eq. 2.7]. Many authors have used symmetric and skewed fractional derivative operator in their results (see, e.g.[15], [20], [21], [22]). It has been shown that solutions of fractional Riesz-Feller space derivative [23] and continuous time random walk theory for Levy flights [24,25] gives same results. Because of the added advantage of the Riesz-Feller derivative, it has been used in solving fractional Schrodinger equation also in [26,27,28].

Let $Re(\alpha) > 0$ and $f(t)$ be a piecewise continuous function in interval $J' = (0,\infty)$ also f is integrable on $J = [0,\infty)$. Then

$$ {}_a^{RL}D_t^\alpha f(t) = \left(\frac{d}{dt}\right)^{m+1} \int_a^t (t-\xi)^{m-\alpha} f(\xi)d\xi,\; m \leq \alpha < (m+1),\; t > 0 \tag{9} $$

is the Riemann - Liouville fractional derivative of $f(t)$ of order α [29, Eq. 2.1, p. 82]. Caputo fractional derivative of a function $f(t)$ of order α is given by

$$ {}_a^C D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(\alpha-n)} \int_a^t \frac{f^n(\tau)d\tau}{(t-\tau)^{\alpha+1-n}}, & (n-1) < \alpha < n, \\ \frac{d^n f(t)}{dt^n}, & \alpha = n. \end{cases} $$

This operator was introduced by Caputo [30, Eq. 5, p. 530].

Laplace transform of the Caputo fractional derivative is given by Caputo [31] and Kilbas et al. [32, Eq. 2.4.62, p. 98] as,

$$ L({}_a^C D_t^\alpha f(t)) = s^\alpha f(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0),\; (n-1) < \alpha \leq n, \tag{10} $$

where $Re(s) > 0$ and $Re(\alpha) > 0$.

Riesz - Feller fractional derivative [33] of a sufficiently well-behaved function $f(x)$, is defined via Fourier transform as

$$ F({}_x D_\theta^\beta f(x);k) = -\psi_\beta^\theta(k) f^*(k),\; 0 < \beta \leq 2, \quad |\theta| \leq \min\{\beta, 2-\beta\}, \tag{11} $$

where $\psi_\beta^\theta(k) = |k|^\beta \exp\left[i(\text{sign}\, k)\frac{\theta\pi}{2}\right]$ is the pseudo-differential operator given in [33,34,35,36,37]. When $\theta = 0$, (11) reduces to

$$ F({}_x D_0^\beta f(x);k) = -|k|^\beta, \tag{12} $$

This can be written as $-(k)^\beta = -(k^2)^{\frac{\beta}{2}}$. For $\theta = 0$, we can write

$$ {}_x D_0^\beta \equiv -\left(-\frac{d^2}{dx^2}\right)^{\frac{\beta}{2}}. \tag{13} $$

Saichev and Zaslavsky [19] gave an alternative notation for the symmetric fractional derivative as

$$ {}_x D_0^\beta \equiv \frac{d^\beta}{d|x|^\beta}. \tag{14} $$

Explicit representation of Riesz fractional derivative of a sufficiently well-behaved function $f(x)$, for $0 < \beta < 2$ is given as

$$ {}_x D_0^\beta f(x) = \frac{\Gamma(1+\beta)}{\pi} \sin\left(\frac{\beta\pi}{2}\right) \int_0^\infty \frac{f(x+\xi) - 2f(x) + f(x-\xi)}{\xi^{1+\beta}} d\xi \tag{15} $$

Integral representation for Riesz-Feller derivative of a sufficiently well-behaved function $f(x)$, given as

$$ {}_x D_\theta^\beta f(t) = \frac{\Gamma(1+\beta)}{\pi} \left\{ \sin\left[(\beta+\theta)\frac{\pi}{2}\right] \int_0^\infty \frac{f(t+\xi)-f(t)}{\xi^{1+\beta}} d\xi + \sin\left[(\beta-\theta)\frac{\pi}{2}\right] \int_0^\infty \frac{f(t-\xi)-f(t)}{\xi^{1+\beta}} d\xi \right\}. \tag{16} $$

Using Mellin - Barnes type integral, Fox [38] introduced the H - function defined as follows

$$H_{u,v}^{r,s}(t) = H_{u,v}^{r,s}\left[t\left|\begin{matrix}(a_u,A_u)\\ \\(b_v,B_v)\end{matrix}\right.\right] = H_{u,v}^{r,s}\left[t\left|\begin{matrix}(a_1,A_1),...,(a_u,A_u)\\ \\(b_1,B_1),...,(b_v,B_v)\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_{\omega}\Theta(\xi)t^{-\xi}d\xi, \tag{17}$$

where

$$\Theta(\xi) = \frac{[\Pi_{j=1}^{r}\Gamma(b_j+B_j\xi)][\Pi_{i=1}^{s}\Gamma(1-a_i-A_i\xi)]}{[\Pi_{j=r+1}^{v}\Gamma(1-b_j-B_j\xi)][\Pi_{i=s+1}^{u}\Gamma(a_i+A_i\xi)]}, \tag{18}$$

The empty products are interpreted as unity always; $r,s,u,v \in \mathbb{N}_0$ with $0 \le s \le u,\ 1 \le r \le v,\ A_i,\ B_j \in \mathbb{R}_+,\ a_i,\ b_j \in \mathbb{R}$ or $\mathbb{C}(i = 1,2,...,u;\ j = 1,2,...,v)$ such that

$$A_i(b_j+k) \ne B_j(a_i-l-1),\ (k,l \in N_0;\ i = 1,2,...,s;\ j = 1,2,...,r). \tag{19}$$

The notations here are: $\mathbb{N}_0 = (0,1,2,...)$; $\mathbb{R} = (-\infty,\infty)$, $\mathbb{R}_+ = (0,\infty)$, and $\mathbb{C}$ denotes the complex number field.

The Mittag - Leffler function is a simple generalization of the exponential function and it is an entire function. Its importance is realized due to its direct involvement in the problems of physics, biology, engineering and applied sciences.

Two parameter Mittag - Leffler function is given by Wiman [39] as

$$E_{\alpha,\beta}(t) = \sum_{n=0}^{\infty}\frac{t^n}{\Gamma(n\alpha+\beta)},\ Re(\alpha) > 0,\ Re(\beta) > 0,\ \alpha,\beta \in \mathbb{C},\ t \in \mathbb{C}. \tag{20}$$

On taking $\beta = 1$, in (20), we get corresponding results for one parameter Mittag - Leffler function given by Mittag - Leffler [40].

One can find relation between two parameter Mittag - Leffler function and Fox H-function in terms of Mellin - Barnes integral as

$$E_{\alpha,\beta}(t) = \frac{1}{2\pi i}\int_{\omega}\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta-\alpha s)}t^s ds = H_{1,2}^{1,1}\left[-t\left|\begin{matrix}(0,1)\\ \\(0,1),(1-\beta,\alpha)\end{matrix}\right.\right], \tag{21}$$

where the contour ω goes from $c-i\infty$ to $c+i\infty$, and separates the poles of function $\Gamma(s)$ from those of the function $\Gamma(1-s)$. It is shown by Mathai et al. [41] that the integral converges for all t.

In [42, Eq. 25], inverse Fourier transform of Mittag - Leffler function in two parameter is given as

$$F^{-1}\{E_{\beta,\gamma}(-at^{\beta}|k|^{\alpha});x\} = \frac{1}{\alpha|x|}H_{3,3}^{2,1}\left[\frac{|x|}{a^{\frac{1}{\alpha}}t^{\frac{\beta}{\alpha}}}\left|\begin{matrix}(1,\frac{1}{\alpha}),(\gamma,\frac{\beta}{\alpha})(1,\frac{1}{2})\\ \\(1,\frac{1}{\alpha})(1,1)(1,\frac{1}{2}),\end{matrix}\right.\right] \tag{22}$$

where $\min\{Re(\alpha),Re(\beta),Re(\gamma)\} > 0$ and $\alpha > 0$.

2 Analytic solution of the space time fractional advection dispersion equation with retardation

Theorem 1. *Consider the space time fractional advection dispersion equation*

$$R\,{}_0^C D_t^{\alpha}C(x,t) = -vD_x\,C(x,t) + \mathscr{D_L}\,{}_xD_{\theta}^{\beta}C(x,t),\ x \in \mathbb{R},\ t \in (0,\infty), \tag{23}$$

with initial and boundary conditions

$$C(x,0) = f(x),\ C(x,t) = 0\ \ as\ |x| \to \infty, \tag{24}$$

where ${}_0^C D_t^{\alpha}$ is the Caputo time fractional derivative of order α $(0 < \alpha \le 1)$, ${}_xD_{\theta}^{\beta}$ is the Riesz-Feller space fractional derivative of order β $(0 < \beta \le 2)$ and skewness θ $(|\theta| \le \min\{\beta,\ 2-\beta\})$, $f(x)$ is a sufficiently well-behaved function, $\mathscr{D_L}$ is the longitudinal hydrodynamic dispersion coefficient and v is the average linear velocity along flow line. Then the solution of equation (23) under the above conditions is given by:

$$C(x,t) = \int_{-\infty}^{\infty}G_{\alpha,\beta}^{\theta}(x-y,t)f(y)dy, \tag{25}$$

where

$$G^{\theta}_{\alpha,\beta}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}E_{\alpha}\left[-\Lambda(k)t^{\alpha}\right]dk, \tag{26}$$

is the Green function and $\Lambda(k) = -ik\frac{v}{R} + \frac{\mathscr{D}_{\mathscr{L}}}{R}\psi^{\theta}_{\beta}(k)$.

Proof. Applying Laplace transform for Caputo time fractional derivative with respect to time variable t we obtain,

$$R\left[s^{\alpha}\bar{C}(x,s) - s^{\alpha-1}C(x,0)\right] = -vD_x\bar{C}(x,s) + \mathscr{D}_{\mathscr{L}}\,{}_xD^{\beta}_{\theta}\bar{C}(x,s). \tag{27}$$

Now applying Fourier transform with respect to space variable x

$$R\left[s^{\alpha}\tilde{\bar{C}}(k,s) - s^{\alpha-1}C(k,0)\right] = -ivk\tilde{\bar{C}}(k,s) + \mathscr{D}_{\mathscr{L}}\,\psi^{\theta}_{\beta}(k)\tilde{\bar{C}}(k,s), \tag{28}$$

where

$$\psi^{\theta}_{\beta}(k) = |k|^{\beta}\exp i(\mathrm{sign}k)\frac{\theta\pi}{2} \quad \text{and} \quad i = \sqrt{-1}. \tag{29}$$

Applying initial and boundary conditions we get,

$$R\left[s^{\alpha}\tilde{\bar{C}}(k,s) - s^{\alpha-1}\tilde{f}(k)\right] = -ivk\tilde{\bar{C}}(k,s) + \mathscr{D}_{\mathscr{L}}\,\psi^{\theta}_{\beta}(k)\tilde{\bar{C}}(k,s), \tag{30}$$

$$\tilde{\bar{C}}(k,s) = \frac{s^{\alpha-1}\tilde{f}(k)}{s^{\alpha}+\Lambda(k)}, \tag{31}$$

where $\Lambda(k) = -ik\frac{v}{R} + \frac{\mathscr{D}_{\mathscr{L}}}{R}\psi^{\theta}_{\beta}(k)$. Inversion of Laplace transform of equation (31), using the result in [42, Eq. 18]

$$L^{-1}\left(\frac{s^{\beta-1}}{s^{\alpha}+a}\right) = t^{\alpha-\beta}E_{\alpha,\,\alpha-\beta+1}(-at^{\alpha}),\ Re(s)>0,\ \ Re(\alpha)>0,\ \ Re(\alpha-\beta)>-1, \tag{32}$$

gives

$$\tilde{C}(k,t) = E_{\alpha}\left[-\Lambda(k)t^{\alpha}\right]\tilde{f}(k). \tag{33}$$

Further, taking inverse Fourier transform of (33), we get

$$C(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}E_{\alpha}\left[-\Lambda(k)t^{\alpha}\right]\tilde{f}(k)dk. \tag{34}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}E_{\alpha}\left[-\Lambda(k)t^{\alpha}\right]\int_{-\infty}^{\infty} e^{iky}f(y)dydk. \tag{35}$$

$$= \int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ik(x-y)}E_{\alpha}\left[-\Lambda(k)t^{\alpha}\right]dk\right)f(y)dy. \tag{36}$$

$$= \int_{-\infty}^{\infty} G^{\theta}_{\alpha,\beta}(x-y,t)f(y)dy, \tag{37}$$

where

$$G^{\theta}_{\alpha,\beta}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}E_{\alpha}\left[-\Lambda(k)t^{\alpha}\right]dk, \tag{38}$$

is the Green function and $\Lambda(k) = -ik\frac{v}{R} + \frac{\mathscr{D}_{\mathscr{L}}}{R}\psi^{\theta}_{\beta}(k)$.

3 Composition rule for the Green function

Here we present composition of two Green functions in the integral form corresponding to space-fractional and time-fractional diffusion equations. Equation (38) can be written as (see, [19])

$$\tilde{\bar{G}}^{\theta}_{\alpha,\beta}(k,s) = \frac{s^{\alpha-1}}{s^{\alpha}+\frac{\mathscr{D}_{\mathscr{L}}}{R}\psi^{\theta}_{\beta}(k)-ik\frac{v}{R}} \tag{39}$$

$$= s^{\alpha-1}\int_0^{\infty} e^{-u\left(s^{\alpha}+\frac{\mathscr{D}_{\mathscr{L}}}{R}\psi^{\theta}_{\beta}(k)-ik\frac{v}{R}\right)}du \tag{40}$$

$$= 2\int_0^{\infty} \tilde{G}^{\theta}_{1,\beta}\left(k,\frac{u\mathscr{D}_{\mathscr{L}}}{R}\right)\mathfrak{F}\left(\delta\left(k+\frac{vu}{2\pi R}\right)\right)\bar{G}^{0}_{2\alpha,2}(u,s)\,du, \tag{41}$$

where $\tilde{G}^{\theta}_{1,\beta}(k,\frac{u\mathscr{D}_{\mathscr{L}}}{R}) = e^{-\frac{u\mathscr{D}_{\mathscr{L}}}{R}\psi^{\theta}_{\beta}(k)}$ is the Fourier transform w.r.t. space variable x, $\mathfrak{F}\left(\delta\left(k+\frac{vu}{2\pi R}\right)\right)$ is the Fourier transform of the Dirac-delta function and $\bar{G}^{0}_{2\alpha,2}(u,s) = s^{\alpha-1}e^{-us^{\alpha}}$ is the Laplace transform w.r.t. time variable t given in [43].
By inverting the Laplace and Fourier transforms and using the convolution theorem for Fourier transform, we obtain

$$G^{\theta}_{\alpha,\beta}(x,t) = \frac{1}{2\pi}\int_0^{\infty}\left\{\int_{-\infty}^{\infty} G^{\theta}_{1,\beta}\left(p,\frac{u\mathscr{D}_{\mathscr{L}}}{R}\right)\delta\left(x-p+\frac{vu}{2\pi R}\right)dp\right\}G^{0}_{2\alpha,2}(u,t)du, \tag{42}$$

where $G^{\theta}_{1,\beta}$ is the Green function for the space fractional advection dispersion equation of order β and $G^{0}_{2\alpha,2}$ is the Green function for the time fractional advection dispersion equation of order 2α. Thus, the general Green function of the fractional space time advection dispersion equation with retardation can be expressed as an integral involving the Green functions: one corresponding to space fractional advection dispersion equation with retardation and other corresponding to time fractional advection dispersion equation with retardation.
The M function denoted by $M_{\nu}(z)$, $\nu\in(0,1)$ and $z\in\mathbb{C}$ defined as

$$M_{\nu}(z) = \sum_{n=0}^{\infty}\frac{(-z)^n}{n!\Gamma[-\nu n+(1-\nu)]} \tag{43}$$

$$= \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-z)^{n-1}}{(n-1)!}\Gamma(\nu n)\sin(\pi\nu n), \tag{44}$$

which is in the Wright function form. The Green function can be interpreted in terms of M function according to (Eq. 4.4, 4.23 [43]) as

$$G^{0}_{2,2\alpha}(u,t) = \frac{1}{2}\left(\frac{u\mathscr{D}_{\mathscr{L}}}{R}\right)^{-\alpha}M_{\alpha}\left(\frac{u}{t^{\alpha}}.\right) \tag{45}$$

The L function defined as according to equation (11)

$$\tilde{L}^{\theta}_{\beta}(k) = e^{-\psi^{\theta}_{\beta}(k)}, \tag{46}$$

and the Green function given in [43, eq. 4.4] can be written in L function form as

$$G^{\theta}_{1,\beta}(x,t) = t^{-\frac{1}{\beta}}L^{\theta}_{\beta}\left(\frac{x}{t^{\frac{1}{\beta}}}\right),\ x\in\mathbb{R},\ t\geq 0. \tag{47}$$

Relationship between M and L functions is given by [43, Eq. 4.32]

$$\frac{1}{c^{\frac{1}{\nu}}}L^{-\nu}_{\nu}\left(\frac{r}{c^{\frac{1}{\nu}}}\right) = \frac{c\nu}{r^{\nu+1}}M_{\nu}\left(\frac{c}{r^{\nu}}\right),\ 0<\nu<1,\ c>0,\ r>0. \tag{48}$$

Now composition of Green function (42), that involve L and M functions, for $x>0$ is obtained as

$$G^{\theta}_{\alpha,\beta}(x,t) = \frac{1}{2\pi}\int_0^{\infty}t^{-\alpha}\left\{\int_{-\infty}^{\infty}\left(\frac{u\mathscr{D}_{\mathscr{L}}}{R}\right)^{-\frac{1}{\beta}}L^{\theta}_{\beta}\left(p\left(\frac{u\mathscr{D}_{\mathscr{L}}}{R}\right)^{-\frac{1}{\beta}}\right)\delta\left(x-p+\frac{vu}{2\pi R}\right)dp\right\}M_{\alpha}\left(\frac{u}{t^{\alpha}}\right)du. \tag{49}$$

$$G^{\theta}_{\alpha,\beta}(x,t) = \frac{1}{2\pi}\int_0^{\infty}\left\{\int_{-\infty}^{\infty}\left(\frac{u\mathscr{D}_{\mathscr{L}}}{R}\right)^{-\frac{1}{\beta}} L^{\theta}_{\beta}\left(p\left(\frac{u\mathscr{D}_{\mathscr{L}}}{R}\right)^{-\frac{1}{\beta}}\right)\delta\left(x-p+\frac{vu}{2\pi R}\right)dp\right\}\left[\frac{t}{u\alpha}u^{-\frac{1}{\alpha}}L^{-\alpha}_{\alpha}\left(\frac{t}{u^{\frac{1}{\alpha}}}\right)\right]du. \quad (50)$$

By putting $y = u^{-\frac{1}{\alpha}}$ and $t = 1$ (50) can be written in reduced Green function $K^{\theta}_{\alpha,\beta}(x)$ as

$$G^{\theta}_{\alpha,\beta}(x,1) = K^{\theta}_{\alpha,\beta}(x) = \frac{1}{2\pi}\int_0^{\infty}\left\{\int_{-\infty}^{\infty} y^{\frac{\alpha}{\beta}}\left(\frac{\mathscr{D}_{\mathscr{L}}}{R}\right)^{-\frac{1}{\beta}} L^{\theta}_{\beta}\left(py^{\frac{\alpha}{\beta}}\left(\frac{\mathscr{D}_{\mathscr{L}}}{R}\right)^{-\frac{1}{\beta}}\right)\delta\left(x-p+\frac{vy^{-\alpha}}{2\pi\mathscr{D}_{\mathscr{L}}}\right)dp\right\}L^{-\alpha}_{\alpha}(y)dy. \quad (51)$$

For $v = 0$, one obtain

$$G^{\theta}_{\alpha,\beta}(x,1) = K^{\theta}_{\alpha,\beta}(x) = \frac{1}{2\pi}\int_0^{\infty} y^{\frac{\alpha}{\beta}}\left(\frac{\mathscr{D}_{\mathscr{L}}}{R}\right)^{-\frac{1}{\beta}} L^{\theta}_{\beta}\left(xy^{\frac{\alpha}{\beta}}\left(\frac{\mathscr{D}_{\mathscr{L}}}{R}\right)^{-\frac{1}{\beta}}\right)L^{-\alpha}_{\alpha}(y)dy, \quad (52)$$

which is similar to the form obtained by Mainardi et al. [43]

4 Special cases

1.On taking $\theta = 0$, as an particular case of Theorem 1, the solution of symmetric fractional advection dispersion with retardation

$$R\,{}^{C}_{a}D^{\alpha}_{t}C(x,t) = -vD_xC(x,t) + \mathscr{D}_{\mathscr{L}}\,{}_xD^{\beta}_{0}C(x,t), \quad (53)$$

subject to initial and boundary conditions

$$C(x,0) = f(x),\ C(x,t) = 0 \ \text{ as } |x| \to \infty, \quad (54)$$

is obtained as

$$C(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}E_{\alpha}\left[\left(ik\frac{v}{R} - \frac{\mathscr{D}_{\mathscr{L}}}{R}|k|^{\beta}\right)t^{\alpha}\right]\tilde{f}(k)dk. \quad (55)$$

2.In equation (53), if we take $v = 0$, $\frac{\mathscr{D}_{\mathscr{L}}}{R} = \eta$ and $f(x) = \delta(x)$, we get the solution of the fractional diffusion equation discussed by Saxena et al. [44] as

$$C(x,t) = \frac{1}{\beta|x|}H^{2,1}_{3,3}\left[\frac{|x|}{\eta^{\frac{1}{\beta}}t^{\frac{\alpha}{\beta}}}\,\middle|\,\begin{matrix}(1,\frac{1}{\beta}),(1,\frac{\alpha}{\beta})(1,\frac{1}{2})\\ (1,1)(1,\frac{1}{\beta})(1,\frac{1}{2})\end{matrix}\right]. \quad (56)$$

Taking $\beta = 2$ in equation (56), Debnath [45, p. 152] obtained explicit form of the solution as $C(x,t) = \frac{1}{2|x|}H^{1,0}_{1,1}\left[\frac{|x|}{\eta^{\frac{1}{2}}t^{\frac{\alpha}{2}}}\,\middle|\,\begin{matrix}(1,\frac{\alpha}{2})\\ (1,1)\end{matrix}\right]$

3.When $\alpha = 1$, $f(x) = \delta(x)$ and $R = 1$ in (53), then by properties of the H-function given in [46], equation (56) changes to the following form given in [47,48] as

$$C(x,t) = \frac{1}{\beta|x|}H^{1,1}_{2,2}\left[\frac{|x|}{\eta^{\frac{1}{\beta}}t^{\frac{1}{\beta}}}\,\middle|\,\begin{matrix}(1,\frac{1}{\beta}),(1,\frac{1}{2})\\ (1,1),(1,\frac{1}{2})\end{matrix}\right],\ \beta > 0. \quad (57)$$

5 Applications

Corollary 1. *Consider the space time fractional advection dispersion equation*

$$R\,{}^{C}_{a}D^{\alpha}_{t}C(x,t) = -vD_x\,C(x,t) + \mathscr{D}_{\mathscr{L}}\,{}_xD^{\beta}_{\theta}C(x,t), \quad x\in\mathbb{R}, \quad t\in\mathbb{R}^{+}, \quad (58)$$

with initial and boundary conditions

$$C(x,0) = \delta(x),\ C(x,t) = 0 \ \ as\ |x| \to \infty, \tag{59}$$

where $\delta(x)$ *is Dirac-delta function and*

$$0 < \alpha \le 1,\ \ 0 < \beta \le 2,\ \ \ |\theta| \le \min\{\beta,\ 2-\beta\}.$$

Then from Theorem 1, solute concentration $C(x,t)$ *is given by*

$$C(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} E_\alpha\left[-\Lambda(k)t^\alpha\right] dk, \tag{60}$$

where $\Lambda(k) = -ik\frac{v}{R} + \frac{\mathscr{D}_\mathscr{L}}{R}\psi_\beta^\theta(k)$.

If we take $R = 1$, above example, reduces to the result derived by Huang and Liu [49]. When we take $\theta = 0$ and suitable substitution for constants above example reduces to the result given by Camargo et al. [50, p. 043514-7].

Corollary 2. *Consider the space time fractional advection dispersion equation*

$$R\,{}_a^CD_t^\alpha C(x,t) = -vD_x\,C(x,t) + \mathscr{D}_\mathscr{L}\,{}_xD_\theta^\beta C(x,t),\ \ \ x \in \mathbb{R},\ \ \ t \in \mathbb{R}^+, \tag{61}$$

with initial and boundary conditions

$$C(x,0) = e^{-x},\ C(x,t) = 0 \ \ as\ |x| \to \infty, \tag{62}$$

and

$$0 < \alpha \le 1,\ \ 0 < \beta \le 2,\ \ \ |\theta| \le \min\{\beta,\ 2-\beta\}.$$

Then from Theorem 1, solute concentration $C(x,t)$ *is given by*

$$C(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} E_\alpha\left[-\Lambda(k)t^\alpha\right]\int_{-\infty}^{\infty} e^{iky}e^{-y}dydk. \tag{63}$$

$$C(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} E_\alpha\left[-\Lambda(k)t^\alpha\right] g(k)dk. \tag{64}$$

where $g(k) = \frac{1}{2\pi}\left[\frac{e^{-(1+ik)}-1}{1+ik}\right]$. *and* $\Lambda(k) = -ik\frac{v}{R} + \frac{\mathscr{D}_\mathscr{L}}{R}\psi_\beta^\theta(k)$.

For $\alpha = 1,\ \ R = 1$ and $\beta = 2$, (61) reduces to the advection - dispersion transport equation considered by Pandey et al. [51].

6 Illustration and discussions

To obtain the effect of the fractional order derivatives to the advection dispersion transport equation with retardation, we compare the analytic solution obtained with the experimental data. The figures are plotted via MATLAB. Figures are the simulation of the concentration of the NH_4^+–N as a function of time and distance from a point or line source in an aquifer with known properties. The aquifer properties are: $D_\mathscr{L} = 6.50\ \mathrm{cm}^2/\mathrm{h},\ v = 1.689$ cm/h, $R = 7.220$, for reference purpose have been taken from [17, p. 299]. Typical values for R for organics, often encountered in field sites, range from 2 to 10 [52]. Graphs are plotted for fractional and integer order values of α and β in the advection - dispersion equation (23) for $x = 32.5,\ \ f(x) = \delta(x)$.

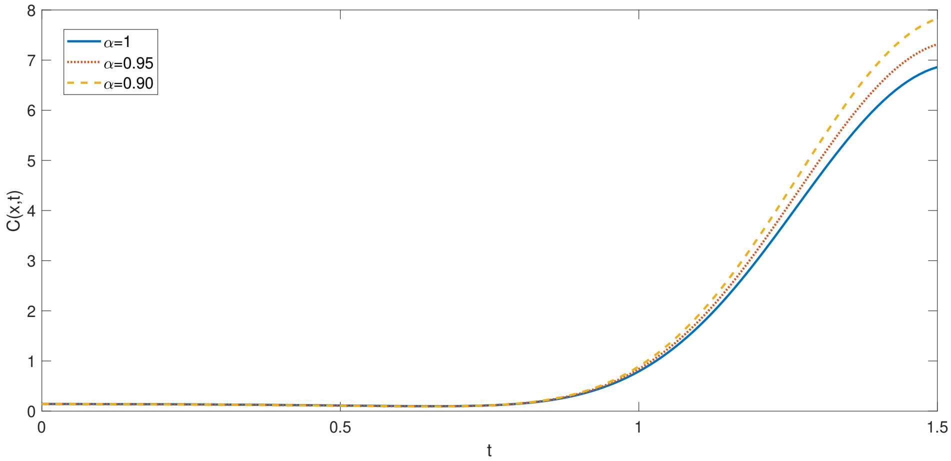

Fig. 1: Concentration profile of NH_4^+–N varying with time for $\beta = 2$.

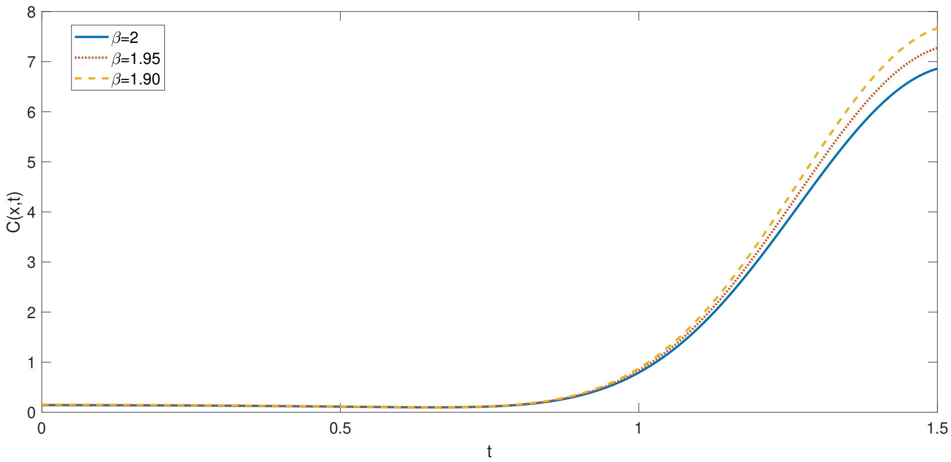

Fig. 2: Concentration profile of NH_4^+–N varying with time for $\alpha = 1$.

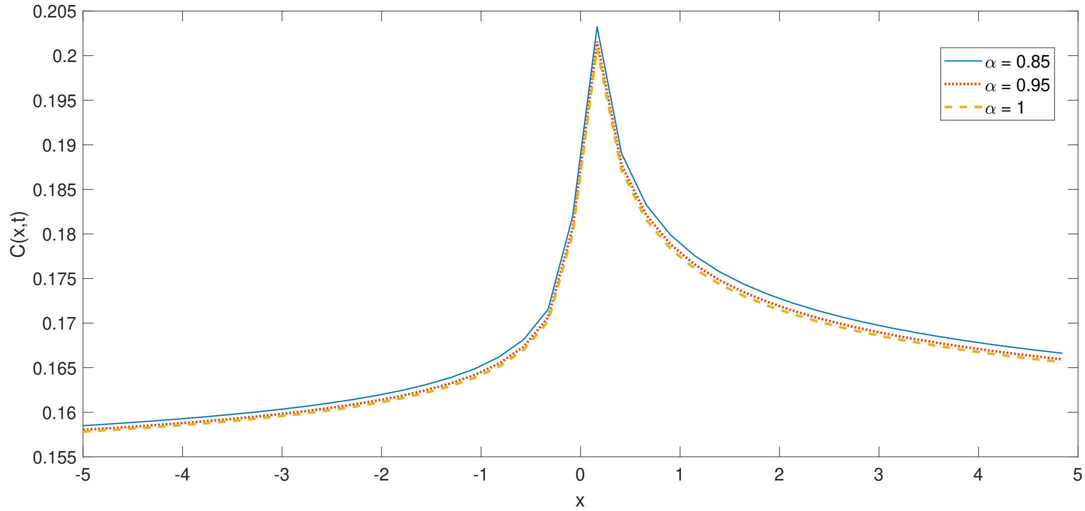

Fig. 3: Graphical interpretation of the solution of space time fractional advection dispersion equation with retardation for $\alpha = 0.85,\ 0.95,\ 1$ with $\beta = 2$ and $\theta = 0$ respectively.

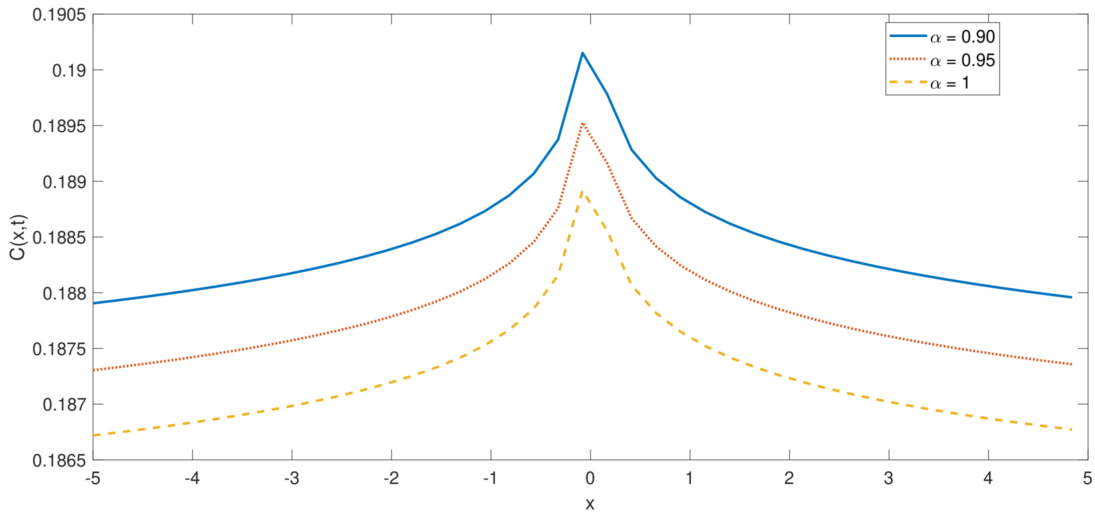

Fig. 4: Graphical interpretation of the solution of space time fractional advection dispersion equation with retardation for $\alpha = 0.90,\ 0.95,\ 1$ with $\beta = 2$ and $\theta = 0.25$ respectively.

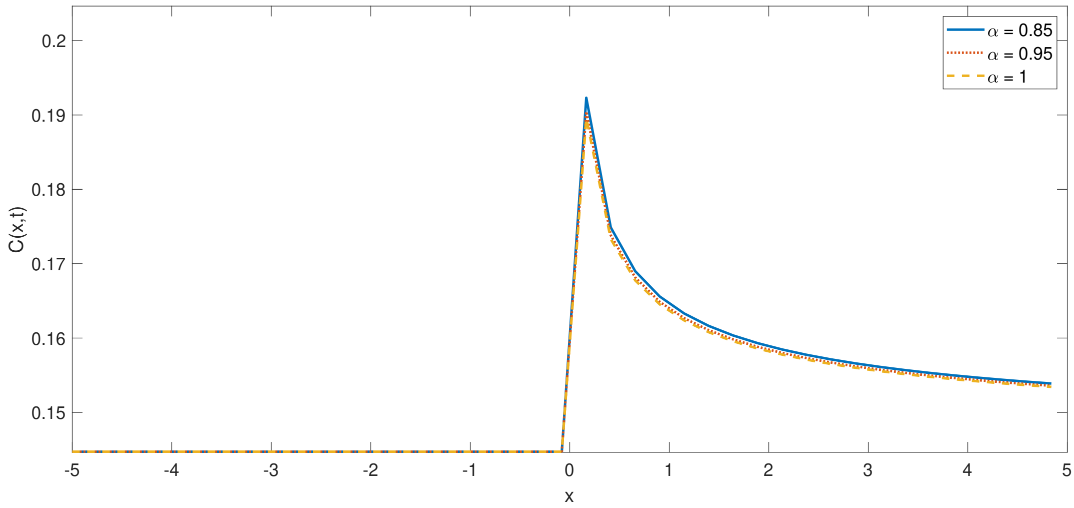

Fig. 5: Graphical interpretation of the solution of space time fractional advection dispersion equation with retardation for $\alpha = 0.85,\ 0.95,\ 1$ with $\beta = 2$ and $\theta = 0.50$ respectively.

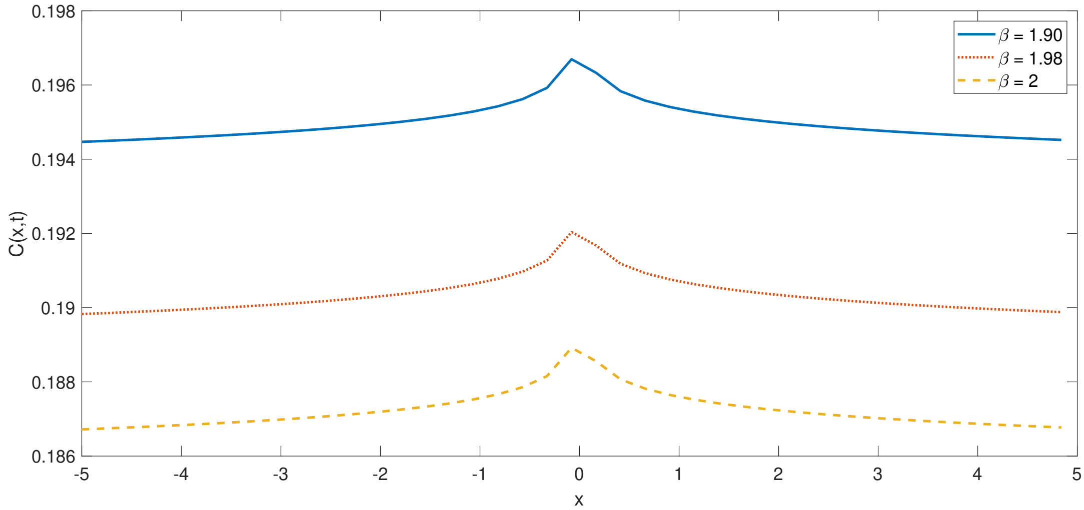

Fig. 6: Graphical interpretation of the solution of space time fractional advection dispersion equation with retardation for $\beta = 1.90,\ 1.98,\ 2$ with $\alpha = 1$ and $\theta = 0$ respectively.

7 Conclusion

In this paper, we obtain solution of space time fractional advection-dispersion equation with retardation by applying Caputo and Riesz-Feller fractional derivative. Compared to advection - dispersion equation model, fractional advection - dispersion equation model provide better simulation for the concentration in breakthrough curves for adsorbing contaminant NH_4^+–N. To show the effect of the Riesz - Feller space fractional derivative, we have obtained graphs for different values of θ which are symmetric for $\theta = 0$ and skewed for $\theta \neq 0$. The advantage of applying Riesz-Feller derivative to obtain the solution of the fractional advection dispersion equation is that, it includes the fundamental solution for space-time fractional diffusion equation. The Riesz fractional derivative includes both left and right Riemann-Liouville derivatives that allow the modeling of flow systems from either side of the domain. Since in our model, the space variable is varying over $\mathbb{R}$, use of Riesz-Feller derivative is justified to cover both sides of the domain.

References

[1] D. K. Todd and L. W. Mays, *Groundwater hydrology,* John Wiley and Sons, Inc (2005).
[2] M. T. Genuchten, Analytical solutions for chemical transport with simultaneous adsorption, zero - order production and first - order decay, *J. Hydr.* **49**, 213 - 233 (1981).
[3] M. T. Genuchten and W. J. Alves, Analytical solutions of the one-dimensional convective-dispersive solute transport equation,*U.S. Department of Agriculture, Bulletin No. 1661*, 151 (1982).
[4] C. V. Chrysikopoulous, P. K. Kitanidis and P. V. Roberts, Analysis of one dimensional solute transport through porous media with spatially variable retardation factor, *Water Resou Res.* **26**, 437-446 (1990).
[5] J. J. A. Kooten, A method to solve the advection-dispersion equation with a kinetic adsorption isotherm, *Adv. Wat. Res.* **19**, 193 - 206 (1996).
[6] C. Liu, W. P. Ball and J. H. Ellis, An analytic solution to the one dimensional solute advection dispersion equation in multi layer porous media, *Trans. Por. Med.* **30**, 25-43 (1998).
[7] C. M. Chang, M. K. Wang, T. W. Chang, C. Lin and Y. R. Chen, Transport modeling of copper and cadmium with linear and nonlinear retardation factors, *Chemosphere* **43**, 1133 - 1139 (2001).
[8] G. Liu and B. C. Si, Analytical modeling of one-dimensional diffusion in layered systems with position-dependent diffusion coefficients, *Adv. Wat. Res.* **31**, 251-268 (2008).
[9] L. Pang and B. Hunt, Solutions and verification of a scale-dependent dispersion model, *J. Contamin. Hydr.* **53**, 21 - 39 (2001).
[10] R. Zhang, K. Huang and J. Xiang, Contaminant movement through homogeneous and heterogeneous soil columns, *Adv. Wat. Res.* **17**, 317 - 324 (1994).
[11] Y. Pachepsky, B. David and R. Walter, Simulating scale-dependent contaminant transport in soils with the fractional advective-dispersive equation, *Soil Sci. Soc. Am. J.* **64**, 1234 - 1243 (2000).
[12] S. Lu, F. J. Molz and G. J. Fix, Possible problems of scale dependency in applications of the three-dimensional fractional advection-dispersion equation to natural porous media, *Water Resour. Res.* **38**,(2002).
[13] A. Yildirim and H. Kocak, (2009), Homotopy perturbation method for solving the space time fractional advection dispersion equation, *Adv.Wat. Res.* **32**, 1711 - 1716 (2009).
[14] E. H. Doha, W. M. Abd-Elhameed, N. A. Elkot and Y. H. Youssri, Integral spectral Tchebyshev approach for solving space Riemann-Liouville and Riesz fractional advection-dispersion problems, *Adv. Differ. Equ.* **284**, 1-23 (2017).
[15] Q. Huang, G. Huang and H. Zhan, A finite element solution for the fractional advection - dispersion equation, *Adv. Wat. Res.* **31**, 1578 - 1589 (2008).
[16] P. Hohener and O. Atteia, Multidimensional analytical models for isotope ratios in groundwater pollutant plumes of organic contaminants undergoing different biodegradation kinetics, *Adv. Wat. Res.* **33**, 740 - 751 (2010).
[17] H. Guanhua, H. Quanzhong, Z. Hongbin, J. Chen, Y. Xiong and F. Shaoyuan, Modeling contaminant transport in homogeneous porous media with fractional advection-dispersion equation,*Sci. China Ser. D Earth Sci.* **48**, 295 - 302 (2005).
[18] A. Atangana and A. Kilicman, Analytical solutions of the space-time fractional derivative of advection dispersion equation, *Math. Probl. Eng.* **2013**, Article Id 853127 (2013).
[19] A. Saichev and G. Zaslavsky, Fractional kinetic equations : solutions and applications, *Chaos* **7**, 753 - 764 (1997).
[20] K. Wang and H. Wang, A fast characteristic finite difference method for fractional advection–diffusion equations, *Adv. Wat. Res.* **34**, 810 - 816 (2011).
[21] C. Shen and M. S. Phanikumar, An efficient space-fractional dispersion approximation for stream solute transport modeling, *Adv. Wat. Res.* **32**, 1482 - 1494 (2009).
[22] X. Zhang, M. Lv, J. W. Crawford and I. M. Young (2007), The impact of boundary on the fractional advection dispersion equation for solute transport in soil: Defining the fractional dispersive flux with the Caputo derivatives, *Adv. Wat. Res.* **30**, 1205 - 1217 (2007).
[23] A. Compte, Stochastic foundations of fractional dynamics, *Phys. Rev. E*, **53**, 4191 - 4193 (1996).
[24] R. Metzler and J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, *Phys. Rep.* **339**, 1 - 77 (2000).
[25] R. Metzler and J. Klafter, The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics, *J. Phy. A Math. Gen.* **37**, R161 - R208 (2004).
[26] N. H. Sweilam and M. M. A. Hasan, Numerical solutions for 2-D fractional Schrodinger equation with the Riesz-Feller derivative, *Math. Comput. Simul.* 140, 53-68 (2017).
[27] N. H. Sweilam and M. M. A. Hasan, Numerical studies for the fractional Schrodinger equation with the quantum Riesz-Feller derivative, *Progr. Fract. Differ. Appl.* **4**, 231-245 (2016).
[28] N. H. Sweilam, T. A. Assiri and M. M. A. Hasan, Numerical solutions of nonlinear fractional Schrodinger equations using nonstandard discretizations, *Numer. Meth. Part. Differ. Equ.* **33**(5), 1399-1419 (2017).

[29] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, John Willy and Sons, Inc, New York 1993.
[30] M. Caputo, Linear models of dissipation whose Q is almost frequency independent – II, *Geophys. J. R. Astr. Soc.,* **13**, 529 - 539 (1967).
[31] M. Caputo, *Elasticita e dissipazione*, Zanichelli, Bologna 1969.
[32] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations,* Elsevier, Amsterdam, 2006.
[33] W. Feller, On a generalization of Marcel Riesz' potentials and the semi-groups generated by them, *Meddeladen Lund Universitets Matematiska Seminarium (Comm. Sem. Mathem. Universite de Lund), Tome suppl. dediea M. Riesz, Lund,* 73 - 81 (1952).
[34] W. Feller, *An introduction to probability theory and its applications,* New York, Wiley, (1971).
[35] R. Gorenflo and F. Mainardi, Random walk models for space-fractional diffusion processes, *Fract. Calc. Appl. Anal.* **1**, 167 - 191 (1998).
[36] R. Gorenflo and F. Mainardi, Approximation of Levy-Feller diffusion by random walk, *Zaa* **18**, 231 - 246 (1991).
[37] R. Gorenflo and F. Mainardi, Random walk models approximating symmetric space-fractional diffusion processes, *Probl. Mathemat. Phys.* (Siegfried Prossdorf Memorial Volume), 120 - 145 (2001).
[38] C. Fox, The G and H functions as symmetrical Fourier kernels, *Trans. Amer. Math. Soc.* **98**, 395-429 (1961).
[39] A. Wiman, Uber den fundamental satz in der theorie der funcktionen, $E_\alpha(x)$, *Acta Math.* **29**, 191-201 (1905).
[40] G. M. Mittag - Leffler, Sur la nouvelle fonction $E_\alpha(x)$, *C. R. Acad. Sci. Paris (Ser. II)* **137**, 554 - 558 (1903).
[41] R. K. Saxena, A. M. Mathai and H. J. Haubold, *The H - function - theory and applications*, Springer, New York, 2010
[42] H. J. Haubold, A. M. Mathai and R. K. Saxena, Solution of reaction-diffusion equations in terms of the H-function, *Bull. Astro. Soc. India.* **35**, (2007).
[43] F. Mainardi, Y. Luchko and G. Pagnini, The fundamental solution of the space-time fractional diffusion equation, *Fract. Calc. Appl. Anal.* **4**, 153 - 192 (2001).
[44] R. K. Saxena, A. M. Mathai and H. J. Haubold,Fractional reaction-diffusion equations, *Astrophys. Space Sci.* **305**, 289 - 296 (2006).
[45] L. Debnath, *Nonlinear partial differential equations for scientists and engineers*, Second Edition, Birkhauser, Basel,2005.
[46] A. M. Mathai and R. K. Saxena, *The H-function with applications in statistics and other disciplines*, John Wiley and Sons, New York, London, Sydney, 1978.
[47] D. Del-Castillo-Negrete, B. A. Carreras and V. E. Lynch, Front dynamics in reaction-diffusion systems with Levy flights: A fractional diffusion approach, *Phys. Rev. Lett.* **91**, 2003.
[48] S. Jespersen, R. Metzler and H. C. Fogedby, Levy flights in external force fields: Langevin and fractional Fokker-Planck equations and their solutions, *Phys. Rev. E* **59**, 2736 - 2745 (1999).
[49] F. Huang and F. Liu, The fundamental solution of the space time fractional advection - dispersion equation, *J. Appl. Math. Comput.* **18**, 339–350 (2005).
[50] F. R. Camargo, R. Charnet and C. de Oliveira, On some fractional Green's functions, *J. Math. Phys.* **50**, 043514 (2009).
[51] R. K. Pandey, O. P. Singh, V. K. Baranwal and M. P. Tripathi, An analytic solution for the space–time fractional advection–dispersion equation using the optimal homotopy asymptotic method, *Comput. Phys. Commun.* **183**, 2098 - 2106 (2012).
[52] P. V. Roberts, M. N. Goltz and D. M. Mackay, A natural gradient experiment on solute transport in a sand aquifer 3. retardation estimates and mass balances for organic solutes, *Water Resou. Res.* **22**, 2047 - 2058 (1986).