Mathematical Sciences Letters
An International Journal

http://dx.doi.org/10.18576/msl/050312

Exact Solutions of the Cubic Nonlinear Schrödinger Equation with a Trapping Potential by Reduced Differential Transform Method

N. Taghizadeh[*] *and S. R. Moosavi Noori*

Department of Mathematics, Faculty of Mathematical Sciences, University of Guilan, P.O.Box 1914, Rasht, Iran.

Received: 11 Dec. 2015, Revised: 21 Mar. 2016, Accepted: 23 Mar. 2016
Published online: 1 Sep. 2016

Abstract: In this paper, the nonlinear Schrödinger (NLS) equation with cubic nonlinearity will be studied. The reduced differential transform method (RDTM) will be used to obtain approximate analytical solutions for this equation. The proposed technique, which does not require linearization, discretization or perturbation, gives the solution in the form of convergent power series with elegantly computed components. Therefore, the solution procedure of the RDTM is simpler than other traditional methods. This method can successfully be applied to a large class of problems.

Keywords: Nonlinear Schrödinger equation with cubic nonlinearity, Reduced differential transform method, Exact solutions, Approximate solutions..

1 Introduction

A considerable amount of research work has been devoted for the study of nonlinear Schrödinger (NLS) equations with a variety of nonlinearities [1-5]. The NLS equation describes numerous nonlinear physical phenomena in the field of nonlinear science such as optical solitons in optical fibres, solitons in the mean-field theory of Bose-Einstein condensates, rogue waves in the nonlinear oceanography, etc. One of the NLS equation with trapping potential, Gross-Pitaevskii equation attracts extreme interests recently. This equation describing the dynamics of Bose-Einstein Condensate at extremely low temperature. More details are presented [6-8].

Many powerful methods, numerical and analytical, have been presented to solve NLS equations like the inverse scattering method [9, 10], Lax pairs [11], Backlund transformation, Hirota bilinear forms [12], Adomian decomposition method (ADM) [13, 14], homotopy perturbation method (HPM) [15-17], the variational iteration method (VIM) [18, 19], differential transform method (DTM) [20-22], and homotopy analysis method (HAM) [23], first integral method [24, 25] and some others.

The reduced differential transform method (RDTM) was first proposed by Keskin [26] to look for exact solutions of PDEs. In recent years, Keskin and Oturanc [27-29] developed the reduced differential transform method (RDTM) for the fractional differential equations and showed that RDTM is the easily useable semi analytical method and gives the exact solution for both the linear and nonlinear differential equations. The solution obtained by the reduced differential transform method is an infinite power series for initial value problems, which can be, in turn, expressed in closed form, the exact solution. Recently, this useful method is widely used in many papers such as in [30-36] and the reference therein.

In this paper, we consider the nonlinear Schrödinger equation with a cubic nonlinearity, with the following initial condition

$$i\frac{\partial u(X,t)}{\partial t} = -\frac{1}{2}\nabla^2 u + V_d(X)u + \beta_d|u|^2 u,$$
$$X \in R^d, t \geq 0, i^2 = -1, u(X,0) = f(X) X \in R^d.$$

Where $u(X,t)$ is a complex function, $V_d(X)$ is the trapping potential and β_d is a constants [37].

The aim of this paper is to obtain the exact solution of this equation by using the RDTM and it is shown that the computational size of this method is small compared with those of ADM, HPM, VIM and DTM.

[*] Corresponding author e-mail: taghizdeh@guilan.ac.ir

The paper is arranged as follows. In Section 2, we describe briefly the reduced differential transform method (RDTM). In Section 3, we apply this method to the cubic nonlinear Schrödinger (CNLS) equations. In Section 4, some conclusions are given.

2 Analysis of the Method

Consider a function of two variables $u(x,t)$ and suppose that it can be represented as a product of two single-variable functions, i.e., $u(x,t) = f(x)g(t)$. Based on the properties of one dimensional differential transform, the function $u(x,t)$ can be represented as follows:

$$u(x,t) = \left(\sum_{i=0}^{\infty} F(i)x^i\right)\left(\sum_{j=0}^{\infty} G(j)t^j\right) = \sum_{k=0}^{\infty} U_k(x)t^k, \tag{1}$$

where $U_k(x)$ is called t-dimensional spectrum function of $u(x,t)$. The basic definitions of RDTM are introduced as follows [26-29]:

Definition 2.1. If function $u(x,t)$ is analytic and differentiated continuously with respect to time t and space x in the domain of interest, then let

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}, \tag{2}$$

where the t-dimensional spectrum function $U_k(x)$ is the transformed function. In this paper, the lowercase $u(x,t)$ represents the original function, while the uppercase $U_k(x)$ stands for the transformed function.

Definition 2.2. The differential inverse transform of $U_k(x)$ is defined as follows:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k. \tag{3}$$

Then, combining Eqs. (2) and (3) we write

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} t^k, \tag{4}$$

from the above definitions, it can be found that the concept of the RDTM is derived from the power series expansion. To illustrate the basic concepts of the RDTM, consider the following nonlinear partial differential equation written in an operator form

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = g(x,t), \tag{5}$$

with initial condition

$$u(x,0) = f(x), \tag{6}$$

where $L = \frac{\partial}{\partial t}$, R is a linear operator which has partial derivatives, $Nu(x,t)$ is a nonlinear operator and $g(x,t)$ is an inhomogeneous term.
According to the RDTM, we can construct the following iteration formula:

$$(k+1)U_{k+1}(x) = G_k(x) - RU_k(x) - NU_k(x), \tag{7}$$

where $U_k(x)$, $RU_k(x)$, $NU_k(x)$ and $G_k(x)$ are the transformations of the functions $Lu(x,t)$, $Ru(x,t)$, $Nu(x,t)$ and $g(x,t)$ respectively.
From initial condition (6), we write

$$U_0(x) = f(x). \tag{8}$$

Substituting (8) into (7) and by straightforward iterative calculation, we get the following $U_k(x)$ values. Then, the inverse transformation of the set of values $\{U_k(x)\}_{k=0}^{n}$ gives the n-terms approximation solution as follows:

$$\tilde{u}_n(x,t) = \sum_{k=0}^{n} U_k(x)t^k. \tag{9}$$

Therefore, the exact solution of the problem is given by

$$u(x,t) = \lim_{n\to\infty} \tilde{u}_n(x,t). \tag{10}$$

The fundamental mathematical operations performed by RDTM can be readily obtained and are listed in Table 1.

3 Applications

In this section, we illustrate the RDTM for solving the cubic nonlinear Schrödinger equations.
Example 1: We first consider the following one-dimensional Schrödinger equation

$$i\frac{\partial u(x,t)}{\partial t} = -\frac{1}{2}\frac{\partial^2 u}{\partial x^2} + (cos^2 x)u + |u|^2 u, \quad t \geq 0, \tag{11}$$

subject to the initial condition

$$u(x,0) = sinx. \tag{12}$$

According to the RDTM and Table 1, the differential transform of Eq. (11) reads

$$(k+1)U_{k+1}(x) = \frac{1}{2}i\frac{\partial^2}{\partial x^2}U_k(x) - i(cos^2 x)U_k(x) + N(U_k(x)), \tag{13}$$

where the t-dimensional spectrum functions $U_k(x)$ is the transformed function. $N(U_k(x))$ is the transformed form of the nonlinear terms.
From initial condition (12), we write

$$U_0(x) = sinx. \tag{14}$$

Table 1. Reduced differential transformation

Functional Form	Transformed Form
$u(x,t)$	$U_k(x) = \frac{1}{k!}[\frac{\partial^k}{\partial t^k}u(x,t)]_{t=0}$
$w(x,t) = u(x,t) \pm v(x,t)$	$W_k(x) = U_k(x) \pm V_k(x)$
$w(x,t) = \alpha u(x,t)$	$W_k(x) = \alpha U_k(x)$ (α is a constant)
$w(x,t) = x^m t^n$	$W_k(x) = x^m \delta(k-n), \qquad \delta(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$
$w(x,t) = x^m t^n u(x,t)$	$W_k(x) = x^m U_{k-n}(x)$
$w(x,t) = u(x,t)v(x,t)$	$W_k(x) = \sum_{r=0}^{k} V_r(x)U_{k-r}(x) = \sum_{r=0}^{k} U_r(x)V_{k-r}(x)$
$w(x,t) = \frac{\partial^r}{\partial t^r}u(x,t)$	$W_k(x) = (k+1)\ldots(k+r)U_{k+r}(x) = \frac{(k+r)!}{k!}U_{k+r}(x)$
$w(x,t) = \frac{\partial}{\partial x}u(x,t)$	$W_k(x) = \frac{\partial}{\partial x}U_k(x)$

Substituting Eq. (14) into Eq. (13) and by straightforward iterative steps, we can obtain

$$U_1(x) = -\frac{3i}{2}sinx,$$
$$U_2(x) = \frac{(-\frac{3i}{2})^2}{2!}sinx,$$
$$U_3(x) = \frac{(-\frac{3i}{2})^3}{3!}sinx, \quad \cdots$$

and so on, in the same manner, the rest of components can be obtained by using MAPLE software. Taking the inverse transformation of the set of values $\{U_k(x)\}_{k=0}^{n}$ gives n-terms approximation solutions as follows:

$$\begin{aligned}\tilde{u}_n(x,t) &= \sum_{k=0}^{\infty} U_k(x)t^k \\ &= U_0(x) + U_1(x)t + U_2(x)t^2 + U_3(x)t^3 + ... \\ &= (1 - \frac{3it}{2} + \frac{(-\frac{3it}{2})^2}{2!} + \frac{(-\frac{3it}{2})^3}{3!} + ...)sinx.\end{aligned}$$

Therefore, the exact solution of the problem is readily obtained as follows:

$$u(x,t) = e^{-\frac{3it}{2}}sinx. \tag{15}$$

Example 2: We next consider the following two-dimensional Schrödinger equation

$$i\frac{\partial u(X,t)}{\partial t} = -\frac{1}{2}(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + V(X)u + |u|^2 u, \tag{16}$$

where $X = (x,y) \in [0,2\pi] \times [0,2\pi]$ and $V(X) = 1 - sin^2 x sin^2 y$,
with initial conditions

$$u(X,0) = sinxsiny. \tag{17}$$

According to the RDTM and Table 1, the differential transform of Eq. (16) reads

$$(k+1)U_{k+1}(X) = \frac{1}{2}i(\frac{\partial^2}{\partial x^2}$$
$$+\frac{\partial^2}{\partial y^2})U_k(X) - iV(X)U_k(X) + N(U_k(X)), \tag{18}$$

where the t-dimensional spectrum functions $U_k(X)$ is the transformed function. $N(U_k(X))$ is the transformed form of the nonlinear terms.
From initial conditions (17), we write

$$U_0(X) = sinxsiny. \tag{19}$$

Substituting Eq. (19) into Eq. (18) and by straightforward iterative steps, we can obtain

$$U_1(X) = -2isinxsiny,$$
$$U_2(X) = \frac{(-2i)^2}{2!}sinxsiny,$$
$$U_3(X) = \frac{(-2i)^3}{3!}sinxsiny, \quad \cdots$$

and so on, in the same manner, the rest of components can be obtained by using MAPLE software. Taking the inverse transformation of the set of values $\{U_k(X)\}_{k=0}^n$ gives n-terms approximation solutions as follows:

$$\begin{aligned}\tilde{u}_n(X,t) &= \sum_{k=0}^{\infty} U_k(X)t^k \\ &= U_0(X) + U_1(X)t + U_2(X)t^2 + U_3(X)t^3 + ... \\ &= (1 - 2it + \frac{(-2it)^2}{2!} + \frac{(-2it)^3}{3!} + ...)sinxsiny.\end{aligned}$$

Therefore, the exact solution of the problem is readily obtained as follows:

$$u(x,y,t) = e^{-2it}sinxsiny. \tag{20}$$

Example 3: We finally consider the following three-dimensional Schrödinger equation

$$i\frac{\partial u(X,t)}{\partial t} = -\frac{1}{2}(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}) + V(X)u + |u|^2 u \tag{21}$$

where $X = (x,y,z) \in [0,2\pi] \times [0,2\pi] \times [0,2\pi]$ and $V(X) = 1 - sin^2 x sin^2 y sin^2 z$,
with initial conditions

$$u(X,0) = sinxsinysinz. \tag{22}$$

According to the RDTM and Table 1, the differential transform of Eq. (21) reads

$$\begin{aligned}&(k+1)U_{k+1}(X) \\ &= \frac{1}{2}i(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})U_k(X) - iV(X)U_k(X) + N(U_k(X)),\end{aligned} \tag{23}$$

where the t-dimensional spectrum functions $U_k(X)$ is the transformed function. $N(U_k(X))$ is the transformed form of the nonlinear terms.
From initial conditions (22), we write

$$U_0(X) = sinxsinysinz. \tag{24}$$

Substituting Eq. (24) into Eq. (23) and by straightforward iterative steps, we can obtain

$$\begin{aligned}U_1(X) &= -\frac{5i}{2}sinxsinysinz, \\ U_2(X) &= \frac{(\frac{-5i}{2})^2}{2!}sinxsinysinz, \\ U_3(X) &= \frac{(\frac{-5i}{2})^3}{3!}sinxsinysinz, \quad \cdots\end{aligned}$$

and so on, in the same manner, the rest of components can be obtained by using MAPLE software. Taking the inverse transformation of the set of values $\{U_k(X)\}_{k=0}^n$ gives n-terms approximation solutions as follows:

$$\begin{aligned}\tilde{u}_n(X,t) &= \sum_{k=0}^{\infty} U_k(X)t^k \\ &= U_0(X) + U_1(X)t + U_2(X)t^2 + U_3(X)t^3 + ... \\ &= (1 - \frac{5it}{2}t + \frac{(\frac{-5it}{2})^2}{2!} \\ &+ \frac{(\frac{-5it}{2})^3}{3!} + ...)sinxsinysinz.\end{aligned}$$

Therefore, the exact solution of the problem is readily obtained as follows:

$$u(x,y,z,t) = e^{\frac{-5it}{2}}sinxsinysinz. \tag{25}$$

Comparing our results with the solutions obtained by HPM and DTM [15, 20], we can see that the results are the same.

4 Conclusion

In this work, we obtained exact solutions of nonlinear Schrödinger equations with cubic nonlinearity by using the reduced differential transform method. The results indicate the efficiency and reliability of the method and furthermore the comparison of the methods with other analytical methods available in the literature shows that although the results of these methods are the same, RDTM is much easier, more convenient and efficient than them and is a powerful mathematical tool for handling linear and nonlinear PDEs.

Acknowledgements

The authors are very grateful to the referees for their valuable suggestions and opinions.

References

[1] A.H. Bhrawy, M.A. Abdelkawy and A. Biswas, Optical solitons in (1+1) and (2+1) dimensions, *Optik.* 125(4) (2014) 1537-1549.

[2] A.H. Bhrawy, A.A. Alshaery, E.M. Hilal, K.R. Khan, M.F. Mahmood and A. Biswas, Optical soliton perturbation with spatio-temporal dispersion inparabolic and dual-power law media by semi-inverse variational principle, *Optik.* 125(17) (2014) 4945-4950.

[3] A.H. Bhrawy, A.A. Alshaery, E.M. Hilal, W. Manrakhan, M. Savescu and A. Biswas, Dispersive optical solitons with SchrödingerHirota equation, *J. Nonlinear Opt. Phys. Mater.* 23(1) (2014) 1450014.

[4] A.H. Bhrawy, A.A. Alshaary, E.M. Hilal, D. Milovic, L. Moraru, M. Savescu and A. Biswas, Optical solitons with polynomial and triple power law nonlinearities and spatio-temporal dispersion, *P ROMANIAN ACAD A* 15(3) (2014) 235-240.

[5] A.H. Bhrawy, A.A. Alshaery, E.M. Hilal, Z. Jovanoski and A. Biswas, Bright and dark solitons in a cascaded system, *Optik.* 125(20) (2014) 6162-6165.

[6] W. Bao, D. Jaksch, P.A. Markowich, Numerical solution of the Gross-Pitaevskii equation for Bose-Einstein condensation, *J. Comput. Phys.* 187(1) (2003) 318-342.

[7] W. Bao, H. Wang and P.A. Markowich, Ground, symmetric and central vortex states in rotating Bose-Einstein condensates. *Commun. Math. Sci*, 3(1) (2005) 57-88.

[8] J.C. Bronski, L.D. Carr, B. Deconinck and J.N. Kutz, Bose-Einstein condensates in standing waves: The cubic nonlinear Schrödinger equation with a periodic potential. *Phys. Rev. Lett* 86(8) (2001) 1402.

[9] M. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981).

[10] P.G. Drazin, R.S. Johnson, *Solitons: An Introduction*, (Cambridge, New York, 1993).

[11] P.D. Lax, Periodic solutions of the Korteweg-de Vries equation, *Comm. Pure. Appl. Math.* 28 (1975) 141-188.

[12] R. Hirota, *Direct Methods in Soliton Theory*, (Springer, Berlin, 1980).

[13] S. Maitama, A new approach to linear and nonlinear Schrödinger equations using the natural decomposition method. *Int. Math. Forum* 9(17) (2014) 835-847.

[14] S. A. Khuri, A new approach to the cubic Schrödinger equation: an application of the decomposition technique. *Appl. Math. Comput,* 97(2) (1998) 251-254.

[15] J. Biazar and H. Ghazvini, Exact solutions for non-linear Schrödinger equations by Hes homotopy perturbation method,*Phys. Lett. A.* 366 (2007) 79-84.

[16] D. Baleanu, A.K. Golmankhaneh and A. K. Golmankhaneh, Solving of the fractional non-linear and linear Schrödinger equations by homotopy perturbation method. *Rom. J. Phys*, 54(10) (2009) 823-832.

[17] A. Sadighi and D.D. Ganji, Analytic treatment of linear and nonlinear Schrödinger equations: a study with homotopy-perturbation and Adomian decomposition methods. *Phys. Lett. A*, 372(4) (2008). 465-469.

[18] B. Jazbi and M. Moini, The variational iteration method for solving linear and nonlinear Schrödinger equations. *IJNAO*, 2(1) (2009) 43-51.

[19] N. H. Sweilam, Variational iteration method for solving cubic nonlinear Schrödinger equation. *J. Comput. Appl. Math*, 207(1) (2007) 155-163.

[20] Y. Do and B. Jang, Nonlinear Klein-Gordon and Schrödinger equations by the projected differential transform method. *Abstract. Appl. Analys*, 2012, (2012) doi:10.1155/2012/150527.

[21] A.R. Kanth and K. Aruna, Two-dimensional differential transform method for solving linear and non-linear Schrödinger equations. *Chaos, Solitons Fractals*, 41(5) (2009) 2277-2281.

[22] J. Biazar, M. Eslami, Differential transform method for nonlinear fractional gas dynamics equation, *Int. J. Phys. Sci* 6(5) (2011) 1203-1206.

[23] N.A. Khan, M. Jamil and A. Ara, Approximate solutions to time-fractional Schrödinger equation via homotopy analysis method. *ISRN Mathematical Physics*, 2012 (2012).

[24] I. Aslan, Travelling wave solutions to nonlinear physical models by means of the first integral method. *Pramana, J. phys.* 76(4) (2011) 533-542.

[25] M. Mirzazadeh, M. Eslami and A. Biswas, Solitons and periodic solutions to a couple of fractional nonlinear evolution equations, *Pramana J. Phys.* 82(3) (2014) 465-476.

[26] Y. Keskin, *Ph.D Thesis*, (Selcuk University, in Turkish, 2010).

[27] Y. Keskin, G. Oturance, Reduced differential transform method for partial differential equations, *Int. J. Nonlin. Sci. Num.* 10(6) (2009) 741-749.

[28] Y. Keskin, G. Oturance, Reduced differential research transform method for fractional partial differential equations, *Nonlinear. Sci. Letters A.* 1(1) (2010) 61-72.

[29] Y. Keskin, G. Oturance, Application of reduced differential transformation method for solving Gas Dynamics equation, *Int. J. Contemp. Math. Sci.* 5(22) (2010) 1091-1096.

[30] M.A. Abdou, Approximate Solutions of System of PDEEs Arising in Physics, *Int. J. Nonlinear Sci.* 12(3) (2011) 305-312.

[31] M.O. Al-Amr, New applications of reduced differential transform method, *Alexandria Engineering J.* 53 (2014) 243-247.

[32] R.A. Oderinu, The reduced differential transform method for the exact solutions of Advection, Burgers and coupled Burgers equations, Theory. *Appl. Math. Comput. Sci.* 2(1) (2012) 10-14.

[33] M. Rawashdeh, N.A. Obeidat, On finding exact and approximate solutions to some PDEs using the reduced differential transform method, *Appl. Math. Info. Sci.* 8(5) (2014) 2171-2176.

[34] R. Abazari, Numerical simulation of coupled nonlinear Schrödinger equation by RDTM and comparison with DTM, *j. appl. sci*, 11(20) (2011) 3454-3463.

[35] M. Rawashdeh, Using the reduced differential transform method to solve nonlinear PDEs arises in biology and physics, *World Appl. Sci. J,* 23(8) (2013) 1037-1043.

[36] M. Rawashdeh, Approximate solutions for coupled systems of nonlinear PDEs using the reduced differential transform method, *Mathematical and Computational Applications*, 19(2) (2014) 161-171.

[37] H.Q. Wang, Numerical studies on the split-step finite difference method for nonlinear Schrodinger equations, *Appl. Math. Comput.* 170 (2005) 17-35.

Nasir Taghizadeh is Professor of pure mathematics and faculty of mathematical sciences, Guilan university, Guilan, Iran. His research interests are theory of partial differential equations. His research involves using analytical methods to obtain approximate and exact solutions to partial differential equations arising from nonlinear PDEs problems in engineering and physics. He has published research articles in a well-recognized international journals of mathematical and engineering sciences.

Seyyedeh Roodabeh Moosavi Noori is currently a Ph.D. student at Guilan university, Guilan, Iran. Her research involves using analytical methods to obtain approximate and exact solutions of linear and nonlinear PDEs in the area of physics and engineering.