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An axiomatization of the Hirsch-index without adopting monotonicity

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Abstract: The Hirsch-index is an index for measuring and comparing the output of researchers. Under the condition of monotonicity, Woeginger [7] provides a characterization of the Hirsch-index by three axioms in 2008. Replacing monotonicity by expansion consistency, we characterize the Hirsch-index by only two of Woeginger's axioms. Besides, we also introduce an axiom contraction consistency. It is a dual viewpoint of expansion consistency. Based on contraction consistency, an additional characterization of the Hirsch-index is reported.

Keywords: Hirsch-index, monotonicity, consistency.

1. Introduction

In 2005, Jorge Hirsch proposed the Hirsch-index to quantify both the scientific productivity and the scientific impact of a scientist. The Hirsch-index is based on the scientist's most cited papers and on the number of citations that they have received in other people's publications. It reflects both the number of publications and the number of citations per publication.

Under the condition of monotonicity, two characterizations of the Hirsch-index may be found in [5, 7]. [7] provides a characterization of the Hirsch-index if indexes are assumed to be integer-valued. When indexes are allowed to be real-valued, [5] offers a characterization of the Hirsch index.[1]

Monotonicity requires that more citations or papers do not lower the index. Woeginger ([7], p.227) stresses that his axioms should be interpreted within the context of monotonicity. It may be difficult to question monotonicity as an appropriated property of an index, but it is worth providing a characterization of the Hirsch-index without adopting monotonicity. The aim of this note is to do so.

Consistency[2] is a crucial property in axiomatic theory. It says that the alternative chosen for each admissible problem should always be "in agreement" with the alternative chosen for each of the "reduced" problems that result when some agents have received their components of the alternative and left, and the situation is reassessed at that point. This fundamental property has been investigated in various classes of problems such as apportionment problems, bankruptcy problems, bargaining problems, cost allocation problems, fair assignment problems, matching problems, resource allocation problems and taxation problems. The reader is referred to [6] for a survey of this literature.

This note establishes an axiomatization of the Hirsch-index by means of a self-consistency property. We introduce a property—expansion consistency in Section 3. Rough speaking, consistency concerns two problems, an original problem and its "reduced" problem in literature. Expansion consistency is slightly different from consistency. It concerns two vectors, an original vector and its "extended" vector. It requires that if the extended vector results from the original vector by adding some "irrelevant" articles, then their indexes should be consistent. Based on expansion consistency, we present a

[1] Based on the Hirsch-index, [4] presents a characterization of the ranking and [1] introduces two alternate indices that can be used to estimate of the impact of Journals published in Arabic Language as well as scientist' cumulative research contrubutions.

[2] The axiom was originally introduced by [2] under the name of bilateral equilibrium. For discussion of this axiom, see [6].

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In Section 5 we take into account the dual viewpoint of expansion consistency, contraction consistency. It is also a condition of self-consistency. Contraction consistency concerns two vectors, an original vector and its “reduced” vector. It requires that if the reduced vector results from the original vector by deleting some “irrelevant” articles, then their indexes should be consistent. As a by-product, we provide a characterization of the Hirsch-index based on contraction consistency. In Section 6 we introduce an axiom weak contraction consistency. It is logically weaker than contraction consistency. Weak contraction consistency instead of contraction consistency, we obtain the coincident result as that in Section 5. Finally, an axiom quality-quantity rationality is introduced in Section 7. Based on this property, we present three characterizations of the Hirsch-index without adopting monotonicity.

2. Preliminaries

We follow the notation and terminology of [7]. A researcher with $n \geq 0$ publications is formally described by a vector $x = (x_1, x_2, \cdots, x_n)$ with non-negative integer components $x_1 \geq x_2 \cdots \geq x_n$; the kth component x_k of this vector states the total number of citations to this researcher's kth-most important publication. If $n = 0$, the researcher has no publications and the vector is empty. Let X denote the set of all such vectors. We say that a vector $x = (x_1, x_2, \cdots, x_n)$ is *dominated* by a vector $y = (y_1, y_2, \cdots, y_m)$, if $n \leq m$ holds and if $x_k \leq y_k$ for $1 \leq k \leq n$; we will write $x \preceq y$ to denote this situation.

Definition 1.*A scientific impact index (or index, for short) is a function f from the set X into the set $\mathbb{N}$ of non-negative integers that satisfies the following condition:*

–If $x = (0, 0, \cdots, 0)$ or if x is the empty vector, then $f(x) = 0$.

A Hirsch-index of at least k means that there are k distinct publications that all have at least k citations. The following definition provides a formal mathematical description of the Hirsch-index.

Definition 2.*The Hirsch-index (or h-index) is the scientific impact index $h : X \to \mathbb{N}$ that assigns to vector $x = (x_1, x_2, \cdots, x_n)$ the value $h(x) = \max\{k : x_k \geq k\}$.*

Woeginger's theorem is based on the following four properties **Monotonicity**, **A1, B** and **D**. The first property **Monotonicity** requires that more citations or articles do not lower the index. [7] postulates **Monotonicity** in the definition of an index.[3]

[3] See Definition 2.1 in [7].

Monotonicity (MON): If $x \preceq y$, then $f(x) \leq f(y)$.

The property **A1** concerns the addition of a single publication to a publication list. It requires that if the publication is only average with respect to the current index, then it should not raise the index.

A1: If the $(n+1)$-dimensional vector y results from the n-dimensional vector x by adding a new article with $f(x)$ citations, then $f(y) \leq f(x)$.

The property **B** concerns the addition of new citations to old publications. It requires that minor changes in the citation record should not lead to major changes in the index.

B: If the n-dimensional vector y results from the n-dimensional vector x by increasing the number of citations of a single article, then $f(y) \leq f(x) + 1$.

The final property **D** concerns the case where both the number of publication and the number of citations go up. It requires that adding a strong new publication and consistently improving the citations to one's old publications should also raise the index.

D: If the $(n+1)$-dimensional vector y results from the n-dimensional vector x by first adding an article with $f(x)$ citations and afterwards increasing the number of citations of every article by at least one, then $f(y) > f(x)$.

Woeginger [7] offers a characterization of the Hirsch-index as follows.

Theorem 1.(Theorem 4.1, [7]) *A scientific impact index $f : X \to \mathbb{N}$ satisfies the four properties* **MON**, **A1, B** *and* **D**, *if and only if it is the h-index.*

To conclude this section, we illustrate that **MON** is needed in Woeginger's characterization.

Proposition 1*The h-index is not the only one to satisfy* **A1, B**, *and* **D**.

*Proof.*The proof is by way of an example of an index that satisfies the three axioms but differs from the h-index: Let σ^1 be the scientific impact index defined by $\forall\, x = (x_1, x_2, \cdots, x_n)$[4]

$$\sigma^1(x) = \begin{cases} 0 & \text{, if } x = (1, 0, 0, \cdots, 0) \\ h(x) & \text{, otherwise.} \end{cases}$$

Clearly σ^1 differs from the h-index. It is immediate to verify that σ^1 so constructed satisfies **A1, B** and **D**, but it violates **MON**.

[4] $x = (1, 0, 0, \cdots, 0)$ means that $n \geq 2$, $x_1 = 1$ and $x_i = 0\ \forall\, 2 \leq i \leq n$.

3. Main Result

In this section we present a characterization for the Hirsch-index by the following property.

Expansion Consistency (ECON): If the $(n+k)$-dimensional extended vector y results from the n-dimensional vector x by adding k articles with the number of citations of every article being at most $f(x)$, then $f(y) = f(x)$, where $k \geq 1$.

ECON is a condition of self-consistency. It concerns the addition of publications to a publication list. It requires that if the number of citations of every added publication is not above the current index, then the index should not change. That is, if the $(n+k)$-dimensional extended vector y results from the n-dimensional vector x by adding k "irrelevant" articles, then their indexes should be consistent. In fact, if an index satisfies **ECON** for $k = 1$, by a repeated application of **ECON** for $k = 1$ would yield **ECON** for all $k > 1$.

Clearly, if an index satisfies **ECON** then it satisfies **A1**. The converse statement is not true. The counterexample is as follows:

–The minimum-index is the scientific impact index $f_{min} : X \to \mathbb{N}$ that assigns to vector $x = (x_1, x_2, \cdots, x_n)$ the value $f_{min}(x) = x_n$. Then the minimum-index satisfies **A1**, but it violates **ECON**.

Remark 1 *It is easy to see that* **A1** *and* **MON** *together imply* **ECON**. *That is, if an index satisfies* **A1** *and* **MON** *then it also satisfies* **ECON**.

The main result is as follows:

Theorem 2. *A scientific impact index $f : X \to \mathbb{N}$ satisfies* **B**, **D** *and* **ECON** *if and only if it is the h-index.*

Proof. Clearly the h-index satisfies **B**, **D**, and **ECON**. Suppose f is an index satisfying **B**, **D**, and **ECON**. Our argument proceeds in four steps. The Steps (1) and (2) are the same as that in [7]. For completeness, we copy them.
Step (1): We argue that any vector x with at most k non-zero components has $f(x) \leq k$. This follows by induction, starting from Definition 1 and then repeatedly applying property **B**.
Step (2): We consider for every $k \geq 0$ the vector $u^{[k]}$ that consists of exactly k components of value exactly k. We prove by induction on $k \geq 0$ that $f(u^{[k]}) = k$. The statement for $k = 0$ follows from Definition 1. In the inductive step, we derive from the inductive assumption and from property **D** that $f(u^{[k+1]}) > f(u^{[k]}) = k$, whereas the statement in Step (1) yields $f(u^{[k+1]}) \leq k+1$. This yields the desired $f(u^{[k+1]}) = k+1$.
Step (3): Let x be a k-dimensional vector every component of which is at least k. It is easy to see that the k-dimensional vector x results from the $(k-1)$-dimensional vector $u^{[k-1]}$ as in the statement of axiom **D**. Hence by **D**, $f(u^{[k-1]}) < f(x)$. Combining this with Steps (1) and (2), $k-1 = f(u^{[k-1]}) < f(x) \leq k$. Hence $f(x) = k$.
Step (4): We establish $f(x) = h(x)\ \forall\ x$. Let x be an n-dimensional vector in X, and let $k = h(x)$. Let $y = (x_1, x_2, \cdots, x_k)$ denote the vector that consists of the first k components of x. Since these components all are at least k, by Step (3), $f(y) = k$. Since vector x results from vector y by adding components of values at most k, by **ECON**, $f(x) = f(y) = k$. Therefore $f(x) = h(x)$.

Finally, the independence of properties listed in Theorem 2 can be established by adopting the following indexes, the maximum-index, the zero-index and the w-index in [7].

–The maximum-index is the scientific impact index $f_{max} : X \to \mathbb{N}$ that assigns to vector $x = (x_1, x_2, \cdots, x_n)$ the value $f_{max}(x) = x_1$. Then the maximum-index satisfies **D** and **ECON**, but it violates **B**.
–The zero-index assigns to every vector x the value 0. Then the zero-index satisfies **B** and **ECON**, but it violates **D**.
–The w-index is the scientific impact index $w : X \to \mathbb{N}$ that assigns to vector $x = (x_1, x_2, \cdots, x_n)$ the value $w(x) = \max\{k : x_m \geq k - m + 1\ \forall\ m \leq k\}$. Then the w-index satisfies **B** and **D**, but it violates **ECON**.

4. Comparison

In this section we show that our result (Theorem 2) directly implies Woeginger's result (Theorem 1). We firstly investigate the logical relations between **A1** and **ECON**. The logical implications are summarized in the following:

1. **ECON**$\Rightarrow$ **A1**; **ECON**$\not\Leftarrow$ **A1**:
Clearly, **ECON** implies **A1**. The converse statement is not true. The counterexample is as follows:
   –The minimum-index f_{min} satisfies **A1**, but it violates **ECON**.
2. **ECON**$\Leftarrow$ **A1**&**MON**; **ECON**$\not\Rightarrow$ **A1**&**MON**:
In Remark 1 we see that **A1** and **MON** together imply **ECON**. The converse statement is not true. The counterexample is as follows:
   –Let σ^2 be the scientific impact index defined by $\forall\ x$

$$\sigma^2(x) = \begin{cases} 1\ , \text{if } x_1 = 1 \\ 0\ , \text{otherwise.} \end{cases}$$

   Then σ^2 satisfies **ECON**, but it violates **MON**.

By the statement of Point 2, **ECON**$\Leftarrow$ **A1**&**MON**, we conclude that Theorem 2 (our result) directly implies Theorem 1 (Woeginger's result).

5. Contraction Consistency

In this section we take into account the dual viewpoint of **ECON**, **Contraction Consistency**. By investigating the logical relations among **A1**, **ECON** and **Contraction Consistency**, we derive another characterization of the h-index by four properties, **MON**, **Contraction Consistency**, **B** and **D**.

Contraction Consistency (CCON): If the n-dimensional reduced vector x results from the $(n+k)$-dimensional vector y by deleting k articles with the number of citations of every article being at most $(f(y)-1)$, then $f(x) = f(y)$, where $k \geq 1$.

CCON is a condition of self-consistency. It concerns the deletion of publications from a publication list. It requires that if the number of citations of every deleted publication is under the current index, then the index should not change. That is, if the n-dimensional reduced vector x results from the $(n+k)$-dimensional vector y by deleting k "irrelevant" articles, then their indexes should be consistent.[5] In fact, if an index satisfies **CCON** for $k = 1$, by a repeated application of **CCON** for $k = 1$ would yield **CCON** for all $k > 1$.

The following proposition states that **CCON** is a more restrictive variant of **A1**.

Proposition 2*If an index f satisfies* **CCON** *then it satisfies* **A1**.

*Proof.*Let f be an index satisfying **CCON**. Consider two vectors x and y as in the statement of **A1**, and suppose for the sake of contradiction that $f(y) > f(x) \geq 0$. Two cases can be distinguished:
Case (1): $f(x) > x_1$
Let y' be the vector that results from y by removing all articles with value strictly lesser than $f(y)$. Since $f(x) > x_1$ and $f(y) > f(x)$, these imply $y' = \emptyset$. Axiom **CCON** yields $f(y) = f(\emptyset) = 0$. This is a contradiction.
Case (2): $f(x) \leq x_1$
Let y' be the vector that results from y by removing one article of value $f(x)$. This means $y' = x$. Axiom **CCON** yields $f(y) = f(y') = f(x)$. This is a contradiction.

The logical implications are summarized in the following:

1.About **A1** and **CCON**:
(a)**CCON**$\Rightarrow$ **A1**; **CCON**$\not\Leftarrow$ **A1**:
In Proposition 2 we see that **CCON** implies **A1**. The converse statement is not true. The counterexample is as follows:
–Let σ^3 be the scientific impact index defined by $\forall\, x$

$$\sigma^3(x) = \begin{cases} 0 & \text{, if } n = 1 \text{ or } x_i = 0\ \forall\, 2 \leq i \leq n \\ x_1 & \text{, otherwise.} \end{cases}$$

Then σ^3 satisfies **A1**, but it violates **CCON**.
(b)**CCON**$\not\Leftarrow$ **A1**&**MON**; **CCON**$\not\Rightarrow$ **A1**&**MON**:
If an index f satisfies **A1** and **MON** then it may not satisfy **CCON**. Conversely, if an index f satisfies **CCON** then it may not satisfy **A1** and **MON**. The counterexamples are as follows:
–σ^3 satisfies **A1** and **MON**, but it violates **CCON**.
–σ^1 satisfies **CCON**, but it violates **MON**.[6]
2.About **CCON** and **ECON**:
(a)**CCON**$\not\Rightarrow$ **ECON**; **CCON**$\not\Leftarrow$ **ECON**:
If an index f satisfies **CCON** then it may not satisfy **ECON**. Conversely, if an index f satisfies **ECON** then it may not satisfy **CCON**. The counterexamples are as follows:
–f_{min} satisfies **CCON**, but it violates **ECON**.
–σ^3 satisfies **ECON**, but it violates **CCON**.
(b)**CCON**& **MON** $\Rightarrow$ **ECON**; **ECON**& **MON** $\not\Rightarrow$ **CCON**:
If an index f satisfies **CCON** and **MON** then it satisfies **ECON**. This statement is true by combining "**ECON**$\Leftarrow$ **A1**&**MON**" (the Point 2 in Section 4) with "**CCON**$\Rightarrow$ **A1**" (1-(a)). On the other hand, if an index f satisfies **ECON** and **MON** then it may not satisfy **CCON**. The counterexample is as follows:
–σ^3 satisfies **ECON** and **MON**, but it violates **CCON**.

By 2-(b), **CCON**& **MON** $\Rightarrow$ **ECON**, Theorem 2 directly implies the following result:[7]

Theorem 3.*A scientific impact index $f : X \to \mathbb{N}$ satisfies the four properties* **MON**, **CCON**, **B** *and* **D**, *if and only if it is the h-index.*

6. Weak Contraction Consistency

In this section we introduce an axiom **Weak Contraction Consistency**. It is logically weaker than **CCON**. **Weak Contraction Consistency** instead of **CCON**, we obtain the coincident result as Theorem 3.

In order to simplify the axiom of **Weak Contraction Consistency**, an additional piece of notation is needed. Let f be an index and let $y \in X$ be an n-dimensional vector. The set of articles with at least $f(y)$ citations in y is denoted by $P_{y,f} = \{t : y_t \geq f(y)\}$. If $P_{y,f} \neq \emptyset$, we

[5] Clearly the h-index satisfies **CCON**.

[6] We can conclude that the h-index is not the only one to satisfy **CCON**, **B**, and **D**.

[7] Theorem 3 could be also derived by applying Theorem 1 and Proposition 2.

denote $m_{y,f} = \max P_{y,f}$. The **reduced vector** $y|_f$ with respect to f is defined by

$$y|_f = \begin{cases} (y_1, y_2, \cdots, y_{m_{y,f}}) \text{ , if } P_{y,f} \neq \emptyset \\ \text{empty vector} \quad\quad \text{ , otherwise.} \end{cases}$$

Note that $P_{y,f}$ is the set of all "relevant" articles in y with respect to f, and the reduced vector $y|_f$ results from the n-dimensional vector y by deleting all "irrelevant" articles. Hence if all articles are irrelevant in y, then the **reduced vector** $y|_f$ should be the empty vector.

Weak Contraction Consistency (WCCON): For each n-dimensional vector $y \in X$, $f(y) = f(y|_f)$.

WCCON requires that if the reduced vector $y|_f$ results from the n-dimensional vector y by deleting all irrelevant articles, then their indexes should be consistent.

It is easy to see that both **CCON** and **WCCON** concern two vectors, an original vector y and a reduced vector x of the vector y by deleting "irrelevant articles". More precisely, **WCCON** specifies the reduced vector x which is reduced by deleting "all irrelevant articles" in y. Hence, if an index f satisfies **CCON** then it satisfies **WCCON**. The converse statement is not true. The counterexample is as follows:

–Let σ^4 be the scientific impact index defined by $\forall\ x$

$$\sigma^4(x) = \begin{cases} x_2 \text{ , if } n = 2 \\ x_1 \text{ , otherwise.} \end{cases}$$

Then σ^4 satisfies **WCCON**, but it violates **CCON**.

Remark 2*Under the condition of* **MON**, **WCCON** *is an alternative form of* **CCON**. *We have known that if an index satisfies* **CCON** *then it also satisfies* **WCCON**. *Conversely, by* **MON**, *an application of* **WCCON** *would yield* **CCON**.

By Remark 2, Theorem 3 directly implies the following result:

Theorem 4.*A scientific impact index $f : X \to \mathbb{N}$ satisfies the four properties* **MON**, **WCCON, B** *and* **D**, *if and only if it is the h-index.*

7. Quality-Quantity Rationality

The scientific impact index of a researcher involves two factors, the quality and the quantify of publications. It reflects both the number of publications (quantity) and the number of citations (quality). If both the quality and the quantity are greater than and equal to k, then the index should reflect this situation. A such axiom is here formulated as follows:

Quality-Quantity Rationality (QQR): If the n-dimensional vector x with $x_k \geq k$, then $f(x) \geq k$, where $k \leq n$.

Based on **QQR**, we present three characterizations of the Hirsch-index without adopting **MON**.

Theorem 5.

1.*A scientific impact index $f : X \to \mathbb{N}$ satisfies* **B**, **QQR** *and* **ECON** *if and only if it is the h-index.*
2.*A scientific impact index $f : X \to \mathbb{N}$ satisfies* **B**, **QQR** *and* **CCON** *if and only if it is the h-index.*
3.*A scientific impact index $f : X \to \mathbb{N}$ satisfies* **B**, **QQR** *and* **WCCON** *if and only if it is the h-index.*

*Proof.*Clearly the h-index satisfies all axioms. To verify Statement (1), suppose f is an index satisfying **B**, **QQR**, and **ECON**. Our argument proceeds in three steps.

Step (1'): Step (1') is the same as Step (1) in [7]. We argue that any vector x with at most k non-zero components has $f(x) \leq k$. We omit it.
Step (2'): Let x be a k-dimensional vector every component of which is at least k. By applying **QQR**, $f(x) \geq k$. Combining Step (1') with (2'), we derive that $f(x) = k$.
Step (3'): Step (3') is the same as Step (4) in [7]. We omit it. This completes the proof of Statement (1).

Since **WCCON** is logically weaker than **CCON**, it remains to verify Statement (3). Suppose f is an index satisfying **B**, **QQR**, and **WCCON**. By Step (2'), we have known that if x is a k-dimensional vector every component of which is at least k, then $f(x) = k$. Next, we establish $f(x) = h(x)\ \forall\ x$.

Let x be an n-dimensional vector in X, and let $k = h(x)$. By **QQR**, $f(x) \geq k$. If $f(x) = k$, we are done. Suppose $f(x) > k$. Two cases can be distinguished:
Case (1): $f(x) > x_1$
If $f(x) > x_1$, then $x|_f$ is the empty vector. By **WCCON**, $f(x) = f(x|_f) = 0$. The desired contradiction has been obtained.
Case (2): $f(x) \leq x_1$
Let $y^i = (x_1, x_2, \cdots, x_i)$ denote the vector that consists of the first i components of x, where $1 \leq i \leq k$. For each y^i, since these components all are at least i, by Step (2'), $f(y^i) = i$ for $1 \leq i \leq k$. Besides, since $x_1 \geq f(x) > k$, it is easy to see that there exists i such that $y^i = x|_f$. Hence, by **WCCON**, $f(x) = f(y^i) = i$. Thus, $k < f(x) = f(y^i) = i \leq k$. The desired contradiction has been obtained. Therefore $f(x) = h(x)$.

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