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Some Properties of Co-Cohen-Macaulay Modules in the Theory of the Edge Ideal of a Graph Simple

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Abstract: In this article, we study the theory and properties of co-Cohen-Macaulay modules. We show, for example, that the co-localization of co-Cohen-Macaulay modules preserves co-Cohen-Macaulayness under a certain condition. In addition, we give a characterization of co-Cohen-Macaulay modules by vanishing properties of the dual Bass numbers of modules. Moreover, we involve the theory of graphs within such modules achieving some applications for the edge ideal of a graph.

Keywords: co-localization, dual Bass number, co-Cohen Macaulay module, edge ideal of a graph

1 Introduction

Throughout this paper, R is a commutative ring with non-zero identity. Let R be a Noetherian ring, and let M be an R-module; moreover, we consider $\mathfrak{p} \in \operatorname{Spec}(R)$. H. Bass has defined the so-called Bass numbers $\mu_i(\mathfrak{p},M)$ by using the minimal injective resolution of M for all integers $i \geq 0$, and has proved that

$$\mu_i(\mathfrak{p},M) = \dim_{k(\mathfrak{p})} \operatorname{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}),M_\mathfrak{p}).$$

If M is a finitely-generated R-module, the Betti numbers $\beta_i(\mathfrak{p},M)$ is defined by using the minimal free resolution of $M_\mathfrak{p}$ for all $i \geq 0$, and we have

$$\beta_i(\mathfrak{p},M) = \dim_{k(\mathfrak{p})} \operatorname{Tor}_i^{R_\mathfrak{p}}(k(\mathfrak{p}),M_\mathfrak{p}).$$

In addition, E. Enochs and J. Z. Xu defined the dual Bass numbers $\pi_i(\mathfrak{p},M)$ by using the minimal flat resolution of M for all $i \geq 0$ and showed that

$$\pi_i(\mathfrak{p},M) = \dim_{k(\mathfrak{p})} \operatorname{Tor}_i^{R_\mathfrak{p}}(k(\mathfrak{p}),\operatorname{Hom}_R(R_\mathfrak{p},M)),$$

for any cotorsion R-module M. In [1], J. Z. Xu characterized Gorenstein rings and strongly cotorsion modules by vanishing properties of $\pi_i(\mathfrak{p},M)$.

A finitely-generated R-module M over a Noetherian ring R is called Cohen-Macaulay if $\dim_{R_\mathfrak{p}} M_\mathfrak{p} = \operatorname{grade}_{R_\mathfrak{p}} M_\mathfrak{p}$ for any $\mathfrak{p} \in \operatorname{Supp}_R(M)$. Cohen-Macaulay modules over Noetherian rings are important objects in commutative algebra and algebraic geometry, according to [2]. In duality, Z. M. Tang and H. Zakeri introduced the concept of co-Cohen-Macaulay modules and studied the properties of this in [3] and [4]. An elementary and important property of Cohen-Macaulay modules is that the localization preserves the Cohen-Macaulayness. The dual question for Artinian modules is to ask whether the co-localization of co-Cohen-Macaulay modules preserves the co-Cohen-Macaulayness. We show that this statement is true under certain conditions, using the theory of graphs. In addition, we give a characterization of co-Cohen-Macaulay modules by vanishing properties of dual Bass numbers which have relation to the maximal length of co-regular sequence, using the theory of graphs. This is dual to the theory of Cohen-Macaulay modules over Noetherian rings.

In the Section 2, we put some definitions and prerequisites for a better understanding of the theory and results. We introduce preliminaries of the theory of graphs which involve the edge ideal of a graph G; associated to the graph G is a monomial ideal

$$I(G) = (v_iv_j \mid v_iv_j \text{ is an edge of } G),$$

with $v_iv_j = v_jv_i$ and with $i \neq j$, in the polynomial ring $R = K[v_1,v_2,\ldots,v_s]$ over a field K, called the **edge ideal** of G. The preliminaries of the theory of graphs are introduced in Section 2 together with the suitable concepts for the work.

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Here, we use properties of commutative algebra and homological algebra for the development of the results (see [6] and [14]).

Throughout of the paper, we mean by a graph G, a finite simple graph with the vertex set $V(G)$ and without isolated vertices, as it was done in [17].

2 Some prerequisites and preliminaries of the graphs theory

The concept of Krull dimension (Kdim) for Artinian modules was introduced by R.N. Roberts in [10]. Later, D. Kirby [11] changed the terminology of Roberts and referred it to Noetherian dimension (Ndim) to avoid any confusion with well-known Krull dimension defined for finitely-generated modules. Let R be a ring, and let M be an R-module. The Noetherian dimension of M, denoted by $\mathrm{Ndim}(M)$, is defined inductively as it follows: when $M = 0$, we put $\mathrm{Ndim}(M) = -1$. Then, by induction, for an integer $d \geq 0$, we put $\mathrm{Ndim}(M) = d$, if $\mathrm{Ndim}(M) = d$ is false and for every ascending chain $M_0 \subseteq M_1 \subseteq \ldots$ of submodules of M, there exists a positive integer n_0 such that $\mathrm{Ndim}(M_{n+1}/M_n) < d$ for all $n > n_0$. Therefore, $\mathrm{Ndim}(M) = 0$ if and only if M is a non-zero Noetherian module.

L. Melkersson and P. Schenzel introduced the co-localization of modules in [12]. Let R be a ring, $S \subseteq R$ a multiplicative set, and let M be an R-module. The R_S-module $\mathrm{Hom}_R(R_S, M)$ is called the *co-localization* of M with respect to S, and is defined the *co-support* of M by

$$\mathrm{Cos}_R(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathrm{Hom}_R(R_{\mathfrak{p}}, M) \neq 0\}.$$

If R is a Noetherian ring and M an Artinian R-module, then $\mathrm{Att}_R(M)$, $\mathrm{Cos}_R(M)$, $\mathrm{Ann}_R(M)$ have the same minimal elements by [9, Corollary 4.3].

In [13], we define the *co-dimension* of M as

$$\mathrm{Cdim}_R(M) = \sup\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Cos}_R(M)\}.$$

Let R be a ring (not necessarily Noetherian), $S \subseteq R$ a multiplicative set and let M be an Artinian R-module. Then we have

$$\mathrm{Cdim}_{S^{-1}R}\left(\mathrm{Hom}_R(S^{-1}R, M)\right) = \sup\left\{\dim(S^{-1}R/S^{-1}\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Cos}_R(M), \mathfrak{p} \cap S = \emptyset\right\},$$

and

$$\mathrm{Cdim}_{S^{-1}R}\left(\mathrm{Hom}_R(S^{-1}R, M)\right) = \sup\left\{\dim(S^{-1}R/S^{-1}\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Att}_R(M), \mathfrak{p} \cap S = \emptyset\right\}.$$

Let $\mathfrak{p} \in \mathrm{Cos}_R(M)$, we denote

$$\mathrm{ht}_M(\mathfrak{p}) = \sup\{n \mid \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n, \mathfrak{p}_i \in \mathrm{Cos}_R(M) \text{ for } i = 0, 1, \ldots, n\}.$$

It is obvious that $\mathrm{ht}_M(\mathfrak{p}) = \mathrm{Cdim}_{R_{\mathfrak{p}}}(\mathrm{Hom}_R(R_{\mathfrak{p}}, M))$.

We expect that $\mathrm{Ndim}_R(M) = \mathrm{Cdim}_R(M)$, for any Artinian module M. Unfortunately, this equality may not hold in general. In fact, there exists an Artinian module M over a Noetherian local ring $(R, \mathfrak{m})$ such that $\mathrm{Ndim}_R(M) < \mathrm{Cdim}_R(M)$ (see [7, Example 4.1]). However, we have the following proposition.

N.T. Cuong and N.T. Dung showed that the above condition does not hold for any Artinian modules, and they also give some sufficient conditions for that in [8].

Proposition 21([8, Proposition 2.1]) *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let M be an Artinian R-module. If one of the following cases happens:*

(1)*R is complete with respect to $\mathfrak{m}$-adic topology.*
(2)*M contain a submodule which is isomorphic to the injective hull of $R/\mathfrak{m}$.*

Then $\mathrm{Ann}_R((0:_M \mathfrak{p})) = \mathfrak{p}$ *for any* $\mathfrak{p} \in V(\mathrm{Ann}_R(M))$.

Let R be a ring, and let M be an Artinian R-module. R.Y. Sharp in [15] showed that $\mathrm{Supp}_R(M)$ is a finite subset of $\mathrm{Max}(R)$, and if

$$\mathrm{Supp}_R(M) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_s\},$$

then $M = M_1 \oplus M_2 \oplus \ldots \oplus M_s$, where

$$M_i = \left\{x \in M \mid \mathfrak{m}_i = \sqrt{\mathrm{Ann}_R((x))}\right\}.$$

2.1 Edge ideal of a graph

This section is in accordance with [5] and [16].

Let $R = K[v_1, \ldots, v_s]$ be a polynomial ring over a field K, and let $Z = \{z_1, \ldots, z_q\}$ be a finite set of monomials in R. The **monomial subring** spanned by Z is the K-subalgebra,

$$K[Z] = K[z_1, \ldots, z_q] \subset R.$$

In general, it is very difficult to certify whether $K[Z]$ has a given algebraic property - e.g., Cohen-Macaulay, normal - or to obtain a measure of its numerical invariants - e.g., Hilbert function. This arises because the number q of monomials is usually large.

Thus, consider any graph G, simple and finite without isolated vertices, with vertex set $V(G) = \{v_1, \ldots, v_s\}$.

Let Z be the set of all monomials $v_i v_j = v_j v_i$, with $i \neq j$, in $R = K[v_1, \ldots, v_s]$, such that $\{v_i v_j\}$ is an edge of G, i.e., the finite and simple graph G, with no isolated vertices, is such that the square-free monomials of degree two are defining the edges of the graph G.

Definition 22A *walk* of length s in G is an alternating sequence of vertices and edges $w = \{v_1, z_1, v_2, \ldots, v_{s-1}, z_h, v_s\}$, where $z_i = \{v_{i-1} v_i\}$ is the edge joining v_{i-1} and v_i.

Definition 23A walk is **closed** if $v_1 = v_s$. A walk may also be denoted by $\{v_1, \ldots, v_s\}$, the edges are evident by context. A **cycle** of length s is a closed walk, in which the points $v_1, \ldots, v_s$ are distinct.

A **path** is a walk with all the points distinct. A **tree** is a connected graph without cycles and a graph is **bipartite** if all its cycles are even. A vertex of degree one will be called an **end point**.

Definition 24A subgraph $G' \subseteq G$ is called **induced** if $v_iv_j = v_jv_i$, with $i \neq j$, which is an edge of G' whenever v_i and v_j are vertices of G' and v_iv_j is an edge of G.

The **complement** of a graph G, for which we write G^c, is the graph on the same vertex set in which $v_iv_j = v_jv_i$, with $j \neq i$, which is an edge of G^c if and only if it is not an edge of G. Finally, let C_k denote the cycle on k vertices; a **chord** is an edge which is not in the edge set of C_k. A cycle is called **minimal** if it has no a chord.

If G is a graph without isolated vertices, simple and finite, then let R denote the polynomial ring on the vertices of G over some fixed field K.

Definition 25([5]) According to the previous context, the edge ideal of a finite simple graph G, with no isolated vertices, is defined by

$$I(G) = (v_iv_j \mid v_iv_j \text{ is an edge of } G),$$

with $v_iv_j = v_jv_i$, and with $i \neq j$.

3 The edge ideal of a graph together with the co-localization of co-Cohen-Macaulay modules

In this section, we present some results about the co-localization of modules which involve the theory of graphs together with the edge ideal of a graph G, which is simple and finite and with no isolated vertices.

Here, we take K a fixed field and we consider $K[v_1, v_2 \dots, v_s]$ the ring polynomial over the field K. Since K is a field, we have that K is a Noetherian ring and then $K[v_1, \dots, v_s]$ is also a Noetherian ring (Theorem of the Hilbert Basis).

Remark 31 By the previous context, $R = K[v_1, v_2 \dots, v_s]$ is a Noetherian ring. Thus, the edge ideal $I(G)$ is an R-module, and thus we can get characterizations for this module under certain hypothesis.

Cohen-Macaulay modules over Noetherian rings are important objects in commutative algebra. In duality, Z. Tang and H. Zakeri introduced the co-Cohen-Macaulay modules over local rings in [3], [4], and it was generalized to general rings by Z. Tang in [3].

Definition 32([3, Definition 5.3]) Let R be a ring. An Artinian R-module M is called a co-Cohen-Macaulay R-module, if $\operatorname{cograde}_R(J(M), M) = \operatorname{Ndim}_R(M)$, where $\operatorname{cograde}_R(J(M), M)$ is the common length of any maximal M co-regular sequence contained in $J(M) := \bigcap_{\mathfrak{p} \in \operatorname{Supp}_R(M)} \mathfrak{p}$.

Proposition 33*Let $R = K[v_1, \dots, v_s]$ be the ring polynomial, and let $I(G)$ be the edge ideal in R of a finite simple graph G, with no isolated vertices. Suppose that $I(G)$ is an Artinian R-module. If $I(G) = I(G)_1 \oplus \dots \oplus I(G)_s$ and $\operatorname{Supp}_R(I(G)) = \{\mathfrak{m}_1, \dots, \mathfrak{m}_s\}$, where*

$$I(G)_i = \left\{ x \in I(G) \mid \mathfrak{m}_i = \sqrt{\operatorname{Ann}_R((x))} \right\},$$

then $I(G)$ is a co-Cohen-Macaulay R-module if and only if $I(G)_i$ is a co-Cohen-Macaulay R-module and

$$\operatorname{Ndim}_R(I(G)) = \operatorname{Ndim}_R(I(G)_i)$$

for $i = 1, 2, \dots, s$.

*Proof.*Notice that $\operatorname{Ndim}_R(I(G)) = \max\{\operatorname{Ndim}_R(I(G)_i) \mid i = 1, 2, \dots, s\}$ by [10, Proposition 1]. Since $I(G)$ is a co-Cohen-Macaulay R-module, and $J(I(G)_i) = \mathfrak{m}_i$ for any $1 \leq i \leq s$, by [3, Lemma 5.1 and Proposition 5.2], we get

$$\operatorname{Ndim}_R(I(G)) = \operatorname{Ndim}_R(I(G)_i) = \operatorname{cograde}_R(\mathfrak{m}_i, I(G)_i) = \operatorname{cograde}_R(J(I(G)), I(G))$$

for $i = 1, \dots, s$.

Hence, $I(G)_i$ is a co-Cohen-Macaulay R-module and $\operatorname{Ndim}_R(I(G)) = \operatorname{Ndim}_R(I(G)_i)$ for any $i = 1, 2, \dots, s$. It is similar to prove the other side.

It is well known that operation of localization preserves the property of Cohen-Macaulayness of Cohen-Macaulay modules over Noetherian rings. Since co-localization is dual to localization, hence there is a natural question: Whether co-localization preserves the co-Cohen-Macaulayness of co-Cohen-Macaulay modules. The main content of this section is to study the properties of co-localization of co-Cohen-Macaulay modules, and to give a partial arffirmative answer to above question.

Proposition 34*Let $R = K[v_1, \dots, v_s]$ be the ring polynomial, and let $I(G)$ be the edge ideal in R of a finite simple graph G, with no isolated vertices. Suppose that $I(G)$ is an Artinian R-module, such that $\operatorname{Ann}_R(0 :_{I(G)} \mathfrak{p}) = \mathfrak{p}$ for any $\mathfrak{p} \in \operatorname{V}(\operatorname{Ann}_R(I(G)))$. If $\mathfrak{p} \in \operatorname{Cos}_R(I(G))$ and $\frac{x}{s} \in \mathfrak{p}R_\mathfrak{p}$ is a $\operatorname{Hom}_R(R_\mathfrak{p}, I(G))$ co-regular element, then*

$$\operatorname{Cdim}_{R_\mathfrak{p}}\left(0 :_{\operatorname{Hom}_R(R_\mathfrak{p}, I(G))} \frac{x}{s}\right) = \operatorname{Cdim}_{R_\mathfrak{p}}\left(\operatorname{Hom}_R(R_\mathfrak{p}, I(G))\right) - 1.$$

*Proof.*Since $\left(0 :_{\operatorname{Hom}(R_\mathfrak{p}, I(G))} \frac{x}{s}\right) \cong \operatorname{Hom}_R(R_\mathfrak{p}, (0 :_{I(G)} x))$ we have

$$\operatorname{Cdim}_{R_\mathfrak{p}}\left(0 :_{\operatorname{Hom}_R(R_\mathfrak{p}, I(G))} \frac{x}{s}\right) = \operatorname{ht}(\mathfrak{p}/\operatorname{Ann}_R(0 :_{I(G)} x)).$$

Moreover, by [13, Lemma 4.3] we have

$$\operatorname{V}(\operatorname{Ann}_R(0 :_{I(G)} x)) = \operatorname{V}(\operatorname{Ann}_R(I(G)), x).$$

Hence $\sqrt{\mathrm{Ann}(0:_{I(G)} x)} = \sqrt{(\mathrm{Ann}_R(I(G)),x)}$. Since

$$\mathrm{Cdim}_{R_\mathfrak{p}}(\mathrm{Hom}_R(R_\mathfrak{p},I(G))) = \mathrm{ht}(\mathfrak{p}/\mathrm{Ann}_R(I(G))),$$

we only need to show that $\mathrm{ht}(\mathfrak{p}/(\mathrm{Ann}_R(I(G)),x)) = \mathrm{ht}(\mathfrak{p}/\mathrm{Ann}_R(I(G))) - 1$. Since $\frac{x}{s} \in \mathfrak{p}R_\mathfrak{p}$ is a $\mathrm{Hom}_R(R_\mathfrak{p},I(G))$ co-regular element, we get

$$\frac{x}{s} \in \mathfrak{p}R_\mathfrak{p} \setminus \bigcup_{\mathfrak{p}\subseteq\mathfrak{p}} \mathfrak{q}R_\mathfrak{p},$$

for $\mathfrak{q} \in \mathrm{Att}_R(I(G))$. It follows that $x \in \mathfrak{p} \setminus \bigcup_{\mathfrak{q}\subseteq\mathfrak{p}} \mathfrak{q}$, for $\mathfrak{q} \in \mathrm{Att}_R(I(G))$.

Thus, $\mathrm{ht}(\mathfrak{p}/(\mathrm{Ann}_R(I(G)),x)) \leq \mathrm{ht}(\mathfrak{p}/\mathrm{Ann}_R(I(G))) - 1$. On the other hand, we get $\mathrm{ht}(\mathfrak{p}/(\mathrm{Ann}_R(I(G)),x)) \geq \mathrm{ht}(\mathfrak{p}/\mathrm{Ann}_R(I(G))) - 1$, by the theory of system of parameters. Hence, the result follows.

In the next section, we put some applications which show the relationship of co-Cohen-Macaulay modules with the edge ideal $I(G)$.

4 Some applications

The following proposition shows that co-localization preserves co-Cohen-Macaulayness under a certain conditions.

Proposition 41*Let $R = K[v_1,\ldots,v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Suppose that $I(G)$ is a co-Cohen-Macaulay R-module, such that $\mathrm{Ann}_R(0:_{I(G)}\mathfrak{p}) = \mathfrak{p}$ for any $\mathfrak{p} \in \mathrm{V}(\mathrm{Ann}_R(I(G)))$. Then for any $\mathfrak{p} \in \mathrm{Cos}_R(I(G))$, we have that*

$$\mathrm{cograde}_R(\mathfrak{p},I(G)) = \mathrm{cograde}_{R_\mathfrak{p}}(\mathrm{Hom}_R(R_\mathfrak{p},I(G))) = \mathrm{Cdim}_{R_\mathfrak{p}}(\mathrm{Hom}_R(R_\mathfrak{p},I(G))).$$

*Proof.*Let $\mathfrak{p} \in \mathrm{Cos}_R(I(G))$ then we have $\mathrm{cograde}_{R_\mathfrak{p}}(\mathrm{Hom}_R(R_\mathfrak{p},I(G))) < \infty$ and

$$\mathrm{cograde}_R(\mathfrak{p},I(G)) \leq \mathrm{cograde}_{R_\mathfrak{p}}(\mathrm{Hom}_R(R_\mathfrak{p},I(G))) \leq \mathrm{Cdim}_{R_\mathfrak{p}}(\mathrm{Hom}_R(R_\mathfrak{p},I(G))).$$

We only need to show that $\mathrm{cograde}_R(\mathfrak{p},I(G)) = \mathrm{Cdim}_{R_\mathfrak{p}}(\mathrm{Hom}_R(R_\mathfrak{p},I(G)))$. Let $n = \mathrm{cograde}_R(\mathfrak{p},I(G))$. We use induction on n. For the case $n = 0$, there exists $\mathfrak{Q} \in \mathrm{Att}_R(I(G))$ such that $\mathfrak{p} \subseteq \mathfrak{Q}$. Since $\mathfrak{p} \in \mathrm{Cos}_R(I(G))$, then for any $\mathfrak{q} \in \mathrm{Att}_R(I(G))$ with $\mathfrak{q} \subseteq \mathfrak{p}$ we have $\mathfrak{q} = \mathfrak{p} = \mathfrak{Q}$ by [3, Proposition 5.2] and co-Cohen-Macaulayness of $I(G)$. Thus $\mathrm{Att}_{R_\mathfrak{p}}(\mathrm{Hom}_R(R_\mathfrak{p},I(G))) = \{\mathfrak{p}R_\mathfrak{p}\}$. Hence, $\mathrm{Cdim}_{R_\mathfrak{p}}(\mathrm{Hom}_R(R_\mathfrak{p},I(G))) = 0$. Suppose that $n > 0$ and the Proposition holds for $n = 1$. Let $x \in \mathfrak{p}$ be a $I(G)$ co-regular element, then $\mathrm{cograde}_R(\mathfrak{p},(0:_{I(G)} x)) = n-1$. Then we have

$$\mathrm{cograde}_R(\mathfrak{p},(0:_{I(G)} x)) = \mathrm{Cdim}_{R_\mathfrak{p}}\left(\mathrm{Hom}_R(R_\mathfrak{p},(0:_{I(G)} x))\right),$$

by induction hypothesis. On the other hand, by [13, Proposition 3.4], $\frac{x}{1} \in \mathfrak{p}R_\mathfrak{p}$ is a $\mathrm{Hom}_R(R_\mathfrak{p},I(G))$-quasi co-regular element. Since

$$\left(0:_{\mathrm{Hom}_R(R_\mathfrak{p},I(G))} \mathfrak{p}R_\mathfrak{p}\right) \neq 0,$$

we know that $\frac{x}{1}$ is a $\mathrm{Hom}_R(R_\mathfrak{p},I(G))$ co-regular element. Then we have

$$\mathrm{Cdim}_{R_\mathfrak{p}}\left(\mathrm{Hom}_R(R_\mathfrak{p},(0:_{I(G)} x))\right) = \mathrm{Cdim}_{R_\mathfrak{p}}(\mathrm{Hom}_R(R_\mathfrak{p},I(G))) - 1,$$

by Proposition 34. Hence $\mathrm{cograde}_R(\mathfrak{p},I(G)) = \mathrm{Cdim}_{R_\mathfrak{p}}(\mathrm{Hom}_R(R_\mathfrak{p},I(G)))$.

In order to obtain the main result of this section, we first prove the following lemma.

Lemma 42*Let $R = K[v_1,\ldots,v_s]$ be the ring polynomial, and let $I(G)$ be the edge ideal in R of a finite simple graph G, with no isolated vertices. Suppose that $I(G)$ is an Artinian R-module. Then*

$$\mathrm{Cos}_R(I(G)) \cap \mathrm{V}(J(I(G))) = \mathrm{Supp}_R(I(G)).$$

*Proof.*Let $I(G) = N_1 + N_2 + \ldots + N_n$ be a minimal secondary presentation of $I(G)$ such that $\mathrm{Att}_R(I(G)) = \{\mathfrak{p}_1,\ldots,\mathfrak{p}_n\}$, where $\mathfrak{p}_i = \sqrt{\mathrm{Ann}_R(N_i)}$ for $i = 1,2,\ldots,n$. On the other hand, we have decomposition

$$I(G) = I(G)_1 \oplus \ldots \oplus I(G)_s,$$

with $\mathrm{Supp}_R(I(G)) = \{\mathfrak{m}_1,\ldots,\mathfrak{m}_s\}$, where we have that

$$I(G)_i = \left\{x \in I(G) \mid \mathfrak{m}_i = \sqrt{\mathrm{Ann}_R(x)}\right\}.$$

Assume that there exists $\mathfrak{m}_j \in \mathrm{Supp}_R(I(G)) \setminus \mathrm{Cos}_R(I(G))$, then $\mathfrak{p}_i$ is not contained in $\mathfrak{m}_j$ for any $i = 1,\ldots,n$. Hence,

$$\bigcap_{i=1}^{n} \mathfrak{p}_i \text{ is not contained in } \mathfrak{m}_j.$$

Let $x \in \cap_{i=1}^n \mathfrak{p}_i \setminus \mathfrak{m}_j$, then $x^l \cdot I(G) = 0$ for some integer l. Let u be a non-zero element in $I(G)_i$, then there exists $n \geq 0$ such that $\mathfrak{m}_i^n u = 0$. Thus, we have that $(\mathfrak{m}_i^n, x^l)\, u = 0$. Since $\mathfrak{m}_i$ is a maximal ideal, we get $(\mathfrak{m}_i^n, x^l) = R$. Thus, we obtain $u = 0$. This induces a contradiction. Hence, $\mathrm{Supp}_R(I(G)) \subseteq \mathrm{Cos}_R(I(G))$. This implies,

$$\mathrm{Cos}_R(I(G)) \cap \mathrm{V}(J(I(G))) = \mathrm{Supp}_R(I(G)).$$

Now, before of the last theorem, we put a proposition which is pertinent to the study done until the moment.

Proposition 43*Let $R = K[v_1,\ldots,v_s]$ be the ring polynomial, and let $I(G)$ be the edge ideal in R of a finite simple graph G, with no isolated vertices. Suppose that $I(G)$ is an Artinian R-module such that $\mathrm{Ann}_R(0:_{I(G)}\mathfrak{p}) = \mathfrak{p}$ for any $\mathfrak{p} \in \mathrm{V}(\mathrm{Ann}_R(I(G)))$. Then*

$$\max\{\mathrm{cograde}_R(\mathfrak{p},I(G)) \mid \mathfrak{p}\in \mathrm{Cos}_R(I(G))\} =$$

$$\max\{\mathrm{cograde}_R(\mathfrak{m},I(G)) \mid \mathfrak{m}\in \mathrm{Supp}_R(I(G))\}.$$

Proof. Since $\mathrm{Ann}_R\left(0:_{I(G)}\mathfrak{p}\right)=\mathfrak{p}$ for any $\mathfrak{p}\in \mathrm{V}(\mathrm{Ann}_R(I(G)))$ we have

$$\mathrm{cograde}_R(\mathfrak{p},I(G))<\infty,$$

for any $\mathfrak{p}\in \mathrm{Cos}_R(I(G))$. By Lemma 42 we only need to show that for any $\mathfrak{p}\in \mathrm{Cos}_R(I(G))$ there exists $\mathfrak{m}\in \mathrm{Supp}_R(I(G))$ such that

$$\mathrm{cograde}_R(\mathfrak{p},I(G))\le \mathrm{cograde}_R(\mathfrak{m},I(G)).$$

If x is a $I(G)$ co-regular element, then we have

$$x\in \bigcup_{\mathfrak{m}_i\in \mathrm{Supp}_R(I(G))}\mathfrak{m}_i.$$

Let $\mathfrak{p}\in \mathrm{Cos}_R(I(G))$, and x_1, x_2, …, x_t be a maximal $I(G)$ co-regular sequence contained in $\mathfrak{p}$, then there exists $\mathfrak{m}\in \mathrm{Supp}_R(I(G))$ such that $x_i\in\mathfrak{m}$ for $i=1,2,\ldots,t$. Otherwise, for any $k=1,\ldots,s$ there exists $x_{i_k}\notin \mathfrak{m}_k$ for some $i_k=1,\ldots,t$. Then

$$(0:_{I(G)}(x_1,x_2,\ldots,x_t))=(0:_{I(G)_1}(x_1,x_2,\ldots,x_t))\oplus\ldots\oplus(0:_{I(G)_s}(x_1,x_2,\ldots,x_t))=0.$$

This is a contradiction. Hence $\mathrm{cograde}_R(\mathfrak{p},I(G))\le \mathrm{cograde}_R(\mathfrak{m},I(G))$ for some $\mathfrak{m}\in \mathrm{Supp}_R(I(G))$.

The following theorem is the main result of this section. We give a characterization of co-Cohen-Macaulay modules by vanishing properties of dual Bass numbers.

Theorem 44 *Let $R=K[v_1,\ldots,v_s]$ be the ring polynomial, and let $I(G)$ be the edge ideal in R of a finite simple graph G, with no isolated vertices. Suppose that $I(G)$ is an Artinian R-module such that $\mathrm{Ann}_R(0:_{I(G)}\mathfrak{p})=\mathfrak{p}$ for any $\mathfrak{p}\in \mathrm{V}(\mathrm{Ann}_R(I(G)))$. Let $\mathrm{MaxCos}_R(I(G))$ be the maximal elements of $\mathrm{Cos}_R(I(G))$. Then the following conditions are equivalent:*

(1) *$I(G)$ is a co-Cohen-Macaulay R-module.*
(2) $\mathrm{cograde}_{R_\mathfrak{p}}(\mathrm{Hom}_R(R_\mathfrak{p},I(G)))=\mathrm{ht}_{I(G)}\mathfrak{p}$ *for any* $\mathfrak{p}\in \mathrm{Cos}_R(I(G))$, *and* $\mathrm{ht}_{I(G)}\mathfrak{m}=\mathrm{Cdim}_R(I(G))$ *for any* $\mathfrak{m}\in \mathrm{Supp}_R(I(G))$.
(3) $\pi_i(\mathfrak{p},I(G))=0$, *for any* $0\le i<\mathrm{ht}_{I(G)}\mathfrak{p}$ *and for any* $\mathfrak{p}\in \mathrm{Cos}_R(I(G))$ *and* $\mathrm{ht}_{I(G)}\mathfrak{m}=\mathrm{Cdim}_R(I(G))$ *for any* $\mathfrak{m}\in \mathrm{Supp}_R(I(G))$.
(4) $\mathrm{cograde}_R(\mathfrak{m},I(G))=\mathrm{ht}_{I(G)}\mathfrak{m}=\mathrm{Cdim}_R(I(G))$ *for any* $\mathfrak{m}\in \mathrm{Supp}_R(I(G))$.
(5) $\pi_i(\mathfrak{m},I(G))=0$ *for any* $0\le i<\mathrm{ht}_{I(G)}\mathfrak{m}$ *and* $\mathrm{ht}_{I(G)}\mathfrak{m}=\mathrm{Cdim}_R(I(G))$ *for any* $\mathfrak{m}\in \mathrm{Supp}_R(I(G))$.

Proof. Notice that for any $\mathfrak{p}\in \mathrm{Cos}_R(I(G))$, $\mathrm{cograde}_{R_\mathfrak{p}}(\mathrm{Hom}_R(R_\mathfrak{p},I(G)))<\infty$ by [13, Lemma 4.3]. Moreover, $\mathrm{Cdim}_{R_\mathfrak{p}}(\mathrm{Hom}_R(R_\mathfrak{p},I(G)))=\mathrm{ht}_{I(G)}\mathfrak{p}$ and

$$\mathrm{cograde}_R(\mathfrak{m},I(G))=\mathrm{cograde}_{R_\mathfrak{m}}(\mathrm{Hom}_R(R_\mathfrak{m},I(G))),$$

for any $\mathfrak{m}\in \mathrm{Supp}_R(I(G))$. Hence, by [13, Proposition 5.2 and Proposition 5.9] we have $(2)\Leftrightarrow(3)$ and $(4)\Leftrightarrow(5)$. On the other hand, $(2)\Rightarrow(4)\Rightarrow(1)$ is obvious.
$(1)\Rightarrow(2)$. From Proposition 41 we get $\mathrm{cograde}_{R_\mathfrak{p}}(\mathrm{Hom}_R(R_\mathfrak{p},I(G)))=\mathrm{ht}_{I(G)}\mathfrak{p}$ for any $\mathfrak{p}\in \mathrm{Cos}_R(I(G))$. Since

$$\mathrm{cograde}_R(\mathfrak{m},I(G))=\mathrm{cograde}_{R_\mathfrak{m}}(\mathrm{Hom}_R(R_\mathfrak{m},I(G)))\le \mathrm{ht}_{I(G)}\mathfrak{m},$$

for any $\mathfrak{m}\in \mathrm{Supp}_R(I(G))$, we have

$$\mathrm{cograde}_R(J(I(G)),I(G))=\min\{\mathrm{cograde}_R(\mathfrak{m},I(G)) \mid \mathfrak{m}\in \mathrm{Supp}_R(I(G))\},$$

and thus

$$\mathrm{cograde}_R(J(I(G)),I(G))\le \max\{\mathrm{ht}_{I(G)}\mathfrak{m} \mid \mathfrak{m}\in \mathrm{Supp}_R(I(G))\}\le \mathrm{Cdim}_R(I(G)),$$

where the third inequality follows from Lemma 42. Hence, by definition of co-Cohen-Macaulay R-module, we get

$$\mathrm{ht}_{I(G)}\mathfrak{m}=\mathrm{Cdim}_R(I(G)),$$

for any $\mathfrak{m}\in \mathrm{Supp}_R(I(G))$.

5 Conclusion

With the results of this article, we have been able to achieve applications for the theory of the ideal edge of a graph together with concepts involving co-Cohen-Macaulay modules.

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References

[1] E. Enochs, J.Z. Xu, *On invariants dual to the Bass numbers*, Proc. Amer. Math. Soc., 125, 951 - 960, (1997).
[2] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge University Press, London UK: 1 - 465, (1993).
[3] Z. Tang, *Co-Cohen-Macaulay modules and multiplicities for Arinian modules*, J. Suzhou Univ. (natural science), 12, 15 - 26, (1996).
[4] Z. Tang, H. Zakeri, *Co-Cohen-Macaulay modules and modules of generalized fractions*, Comm. Algebra, 22, 2173 - 2204, (1994).
[5] A. Alilooee, A. Banerjee, *Powers of edge ideals of regularity three bipartite graphs*, Journal of Commutative Algebra, 09, 441 - 454, (2017).

[6] M.F. Atiyah and I.G. Macdonald, *Introduction to Commutative Algebra*, University of Oxford, London UK: 1 - 136, (1969).
[7] L.T. Nhan, *On the Noetherian dimension of Artinian modules*, Vietnam J. Math., 30, 121 - 130, (2002).
[8] N.T. Cuong, N.T. Dung, L.T. Nhan, *Top local cohomology and the catenaricity of the unmixed support of a finitely generated module*, Comm. Algebra, 35, 1691 - 1701, (2007).
[9] N.T. Cuong, L.T. Nhan, *On representable linearly compact modules*, Proc. Amer. Math. Soc., 130, 1927 - 1936, (2002).
[10] R.N. Roberts, *Krull dimension for Artinian modules over quasi-local commutative rings*, Quart. J. Math. Oxford, 26, 269 - 273, (1975).
[11] D. Kirby, *Dimension and length for Artinian modules*, Quart. J. Math. Oxford, 41, 419 - 429, (1990).
[12] L. Melkersson, P. Schenzel, *The co-localization of an Artinian module*, Proc. Edinburgh Math. Soc., 38, 121 - 131, (1995).
[13] L. Li, *Vanishing properties of dual Bass numbers*, Algebra Colloq., 21, 167 - 180, (2014).
[14] J.J. Rotman, *An Introduction to Homological Algebra*, University of Illinois, Urbana USA: 1 - 392, (1979).
[15] R.Y. Sharp, *A method for the study of Artinian modules with an application to asymptotic behavior*, Mathematical Sciences Research Institute Publications, 15, 443 - 465, (1989).
[16] A. Simis, W.V. Vasconcelos, R.H. Villarreal, *The integral closure of subrings associated to graphs*, Journal of Algebra, 199, 281 - 289, (1998).
[17] C.H. Tognon, Radwan A. Kharabsheh, *The Theory of the Edge Ideal of a Graph Together with Formal Local Cohomology Modules*, Applied Mathematics & Information Sciences, 13, 411 - 415, (2019).

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