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A Jacobi Spectral Collocation Scheme for Solving Abel's Integral Equations

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Abstract: In this article, we construct a numerical technique for solving the first and second kinds of Abel's integral equations. Using the spectral collocation method, the properties of fractional calculus and the Gauss-quadrature formula, we can reduce such problems into those of a system of algebraic equations which greatly simplifies the problem. The proposed numerical technique is based on the shifted Jacobi polynomials and the fractional integral is described in the sense of Riemann-Liouville. In addition, our numerical technique is applied also to solve the system of generalized Abel's integral equations. For testing the accuracy and validity of the proposed numerical techniques, we apply them to solve several numerical examples.

Keywords: Abel's integral equation; Spectral collocation method; Jacobi polynomials; Gauss quadrature; Riemann-Liouville integral

1 Introduction

The Abel's integral equation, a special case of the Volterra integral equations of the first kind, is considered as one of the most important integral equations as it can be derived directly from a mechanical or physical problem, that gives it the ability to describe many engineering and physical phenomena accurately, such as simultaneous dual relations [1], stellar winds [2], water wave [3], spectroscopic data [4] and others [5,6,7], more about the properties of this equation and its different kinds can be found in [8,9].

Finding numerical techniques for solving Abel's integral equations has become an active research undertaking. Liu and Tao [10,11] introduced a mechanical quadrature technique for approximating the solution of the first kind Abel's integral equation. Also, in [12], the authors used the Bernstein polynomials for constructing a numerical solution of the Abel's integral equation, while in [13], the authors introduced the Mikusinskis operator of fractional order for solving it numerically. Recently, Jahanshahi et al. [14] used the properties of the fractional calculus definitions for solving the Abel's integral equation of first kind. Another numerical techniqus have been applied for solving the first kind Abel's integral equation, see [15,16,17].

On the other hand, Yousefi [18] applied the Legendre wavelets method for solving the second kind Abel's integral equation, while Khan and Gondal [19] applied the two-step Laplace decomposition method for approximating its numerical solution. Recently, Kumar et al. [20] applied the homotopy perturbation transform method for finding a numerical solution of the second kind Abel's integral equation. Meanwhile, many researchers have interested in studying the system of generalized Abel's integral equations, see [21]. Kumar et al. [22] applied the homotopy perturbation method for solving the system of Abel's integral equations, while in [23], the authors constructed a numerical method based on the Laguerre polynomials for approximating its solution.

Fractional calculus, the theory of derivatives and integrals with any non-integer arbitrary order, has become the focus of many researchers in recent years due to its high accuracy in modeling many engineering and physical phenomena, such as economics [24], anomalous transport [25], Bioengineering [26] and others [27,28,29]. Therefore, studying the properties of the fractional differential equations and finding effective analytical and numerical techniques for them has become very important topic to be studied, such as the waveform relaxation method [30], the alternating-direction finite difference method [31], the Haar wavelet method [32], the differential transform method [33] and others [34,35].

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One of the best methods for approximating the solution of different kinds of differential equations, is the spectral collocation method [36,37,38]. Besides, the spectral collocation method has high accuracy; it also has exponential rates of convergence. By using the spectral collocation method and in contrast to finite difference and finite element methods, we have numerical solution of better accuracy with far fewer nodes and thus less computational time. The spectral collocation method is used for obtaining the approximate solution of some types of fractional differential equations, such as the fractional Langevin equations [39] and the generalized fractional Pantograph equations [40].

An important role of the Jacobi polynomials has been played in the implementation of spectral methods. Using the Jacobi polynomials in terms of the Jacobi parameters α and β, we have the advantage of obtaining the approximate solution in terms of these parameters. For that reason, instead of finding the approximate solution for each pair of these parameters, we can use the Jacobi polynomials in obtaining our approximate solution. Recently, the Jacobi polynomials have been used as basis functions of the spectral collocation method for approximating the solution of different types of differential equations such as the generalized Fokker-Planck type equations [41], the nonlinear complex generalized Zakharov system [42] and the nonlinear Schrödinger equation [43]. On the other hand, the shifted Jacobi polynomials are used as basis functions for solving types of fractional differential equations such as ordinary fractional differential equations [44], fractional advection-dispersion equations [45] and the multi-term time-space fractional partial differential equations [46].

Our main aim in this article is to construct a numerical method for solving the first kind, the second type and the system of Abel's integral equations. Using the spectral collocation method with some properties of the fractional calculus, we reduce such problems into those consisting of a system of algebraic equations which greatly simplifies the problem.

This article is organized as follows. In the next section, we present some properties of the Jacobi polynomials and shifted Jacobi polynomials. In Sections 3 and 4, the spectral collocation method is used based on the shifted Jacobi polynomials to solve the first and second kinds of Abel's integral equations, respectively, while in Section 5, we apply our technique for approximating the solution of the system of Abel's integral equations. In Section 6, several numerical examples with their approximate solutions obtained using our technique are introduced to show the efficiency of our technique. Concluding remarks are given in Section 7.

2 Properties of shifted Jacobi polynomials

In this section, we reprise some basic properties of orthogonal shifted Jacobi polynomials that are most relevant to spectral approximations. A basic property of the Jacobi polynomials is that they are the eigenfunctions to the singular Sturm-Liouville problem

$$(1-x^2)\psi^{''}(x)+[\vartheta-\theta+(\theta+\vartheta+2)x]\psi^{'}(x)+k(k+\theta+\vartheta+1)\psi(x)=0. \tag{1}$$

The Jacobi polynomials are generated from the three-term recurrence relation:

$$P_{k+1}^{(\theta,\vartheta)}(x)=(a_k^{(\theta,\vartheta)}x-b_k^{(\theta,\vartheta)})P_k^{(\theta,\vartheta)}(x)-c_k^{(\theta,\vartheta)}P_{k-1}^{(\theta,\vartheta)}(x),\quad k\geq 1,$$
$$P_0^{(\theta,\vartheta)}(x)=1,\quad P_1^{(\theta,\vartheta)}(x)=\frac{1}{2}(\theta+\vartheta+2)x+\frac{1}{2}(\theta-\vartheta),$$

where $\theta,\ \vartheta>-1$, $x\in[-1,1]$ and

$$a_k^{(\theta,\vartheta)}=\frac{(2k+\theta+\vartheta+1)(2k+\theta+\vartheta+2)}{2(k+1)(k+\theta+\vartheta+1)},$$
$$b_k^{(\theta,\vartheta)}=\frac{(\vartheta^2-\theta^2)(2k+\theta+\vartheta+1)}{2(k+1)(k+\theta+\vartheta+1)(2k+\theta+\vartheta)},$$
$$c_k^{(\theta,\vartheta)}=\frac{(k+\theta)(k+\vartheta)(2k+\theta+\vartheta+2)}{(k+1)(k+\theta+\vartheta+1)(2k+\theta+\vartheta)}.$$

The Jacobi polynomials satisfy the relations

$$P_k^{(\theta,\vartheta)}(-x)=(-1)^kP_k^{(\theta,\vartheta)}(x),\quad P_k^{(\theta,\vartheta)}(-1)=\frac{(-1)^k\Gamma(k+\vartheta+1)}{k!\Gamma(\vartheta+1)}. \tag{2}$$

Moreover, the qth derivative of $P_j^{(\theta,\vartheta)}(x)$, can be obtained from

$$D^qP_j^{(\theta,\vartheta)}(x)=\frac{\Gamma(j+\theta+\vartheta+q+1)}{2^q\Gamma(j+\theta+\vartheta+1)}P_{j-q}^{(\theta+q,\vartheta+q)}(x). \tag{3}$$

Denoting by $P_{L,k}^{(\theta,\vartheta)}(x) = P_k^{(\theta,\vartheta)}(\frac{2x}{L}-1)$, $L>0$, the shifted Jacobi polynomial of degree k. The explicit analytic form of the shifted Jacobi polynomials $P_{L,k}^{(\alpha,\beta)}(x)$ of degree k is given by

$$\begin{aligned} P_{L,k}^{(\theta,\vartheta)}(x) &= \sum_{j=0}^{k} (-1)^{k-j} \frac{\Gamma(k+\vartheta+1)\Gamma(j+k+\theta+\vartheta+1)}{\Gamma(j+\vartheta+1)\Gamma(k+\theta+\vartheta+1)(k-j)!\,j!L^j} x^j \\ &= \sum_{j=0}^{k} \frac{\Gamma(k+\theta+1)\Gamma(k+j+\theta+\vartheta+1)}{j!(k-j)!\Gamma(j+\theta+1)\Gamma(k+\theta+\vartheta+1)L^j}(x-L)^j, \end{aligned} \tag{4}$$

and this in turn, enables one to get

$$\begin{aligned} P_{L,k}^{(\theta,\vartheta)}(0) &= (-1)^k \frac{\Gamma(k+\vartheta+1)}{\Gamma(\vartheta+1)\,k!}, \\ P_{L,k}^{(\theta,\vartheta)}(L) &= \frac{\Gamma(k+\theta+1)}{\Gamma(\theta+1)\,k!}, \end{aligned}$$

which will be of important use later.

In virtue of (2) and (3), we can deduce that

$$D^q P_{L,k}^{(\theta,\vartheta)}(0) = \frac{(-1)^{k-q}\Gamma(k+\vartheta+1)(k+\theta+\vartheta+1)_q}{L^q\Gamma(k-q+1)\Gamma(q+\vartheta+1)}, \tag{5}$$

$$D^q P_{L,k}^{(\theta,\vartheta)}(L) = \frac{\Gamma(k+\theta+1)(k+\theta+\vartheta+1)_q}{L^q\Gamma(k-q+1)\Gamma(q+\theta+1)}, \tag{6}$$

$$D^m P_{L,k}^{(\theta,\vartheta)}(x) = \frac{\Gamma(m+k+\theta+\vartheta+1)}{L^m\Gamma(k+\theta+\vartheta+1)} P_{L,k-m}^{(\theta+m,\vartheta+m)}(x). \tag{7}$$

Next, let $w_L^{(\theta,\vartheta)}(x) = (L-x)^\theta x^\vartheta$, then we define the weighted space $L^2_{w_L^{(\theta,\vartheta)}}[0,L]$ in the usual way, with the following inner product and norm

$$(u,v)_{w_L^{(\theta,\vartheta)}} = \int_0^L u(x)\,v(x)\,w_L^{(\theta,\vartheta)}(x)\,dx, \quad \|v\|_{w_L^{(\theta,\vartheta)}} = (v,v)^{\frac{1}{2}}_{w_L^{(\theta,\vartheta)}}. \tag{8}$$

The set of shifted Jacobi polynomials forms a complete $L^2_{w_L^{(\theta,\vartheta)}}[0,L]$-orthogonal system. Moreover, and due to (8), we have

$$\|P_{L,k}^{(\theta,\vartheta)}\|^2_{w_L^{(\theta,\vartheta)}} = \left(\frac{L}{2}\right)^{\theta+\vartheta+1} h_k^{(\theta,\vartheta)} = h_{L,k}^{(\theta,\vartheta)}. \tag{9}$$

We denote by $x_{N,j}^{(\theta,\vartheta)}$, $0 \leqslant j \leqslant N$, the nodes of the standard Jacobi-Gauss interpolation on the interval $[-1,1]$, their corresponding Christoffel numbers are $\varpi_{N,j}^{(\theta,\vartheta)}$, $0 \leqslant j \leqslant N$. The nodes of the shifted Jacobi-Gauss interpolation on the interval $[0,L]$ are the zeros of $P_{L,N+1}^{(\theta,\vartheta)}(x)$, which we denote by $x_{L,N,j}^{(\theta,\vartheta)}$, $0 \leqslant j \leqslant N$. Clearly $x_{L,N,j}^{(\theta,\vartheta)} = \frac{L}{2}(x_{N,j}^{(\theta,\vartheta)}+1)$, and their corresponding Christoffel numbers are $\varpi_{L,N,j}^{(\theta,\vartheta)} = (\frac{L}{2})^{\theta+\vartheta+1}\varpi_{N,j}^{(\theta,\vartheta)}$, $0 \leqslant j \leqslant N$. Let $S_N[0,L]$ be the set of polynomials of degree at most N. Thanks to the property of the standard Jacobi-Gauss quadrature, it follows that for any $\phi \in S_{2N+1}[0,L]$, we have

$$\begin{aligned} \int_0^L (L-x)^\theta x^\vartheta \phi(x)dx &= \left(\frac{L}{2}\right)^{\theta+\vartheta+1} \int_{-1}^{1} (1-x)^\theta (1+x)^\vartheta \phi\left(\frac{L}{2}(x+1)\right) dx \\ &= \left(\frac{L}{2}\right)^{\theta+\vartheta+1} \sum_{j=0}^{N} \varpi_{N,j}^{(\theta,\vartheta)} \phi\left(\frac{L}{2}(x_{N,j}^{(\theta,\vartheta)}+1)\right) \\ &= \sum_{j=0}^{N} \varpi_{L,N,j}^{(\theta,\vartheta)} \phi\left(x_{L,N,j}^{(\theta,\vartheta)}\right). \end{aligned} \tag{10}$$

Consequently, the qth-order derivative of shifted Jacobi polynomial can be written in terms of the shifted Jacobi polynomials themselves as

$$D^q P_{L,k}^{(\theta,\vartheta)}(x) = \sum_{i=0}^{k-q} C_q(k,i,\theta,\vartheta) P_{L,i}^{(\theta,\vartheta)}(x), \tag{11}$$

where

$$\begin{aligned} C_q(k,i,\theta,\vartheta) =& \frac{(k+\lambda)_q (k+\lambda+q)_i (i+\theta+q+1)_{k-i-q}\,\Gamma(i+\lambda)}{L^q (k-i-q)!\,\Gamma(2i+\lambda)} \\ & \times {}_3F_2 \left(\begin{matrix} -k+i+q, & k+i+\lambda+q, & i+\theta+1 \\ i+\theta+q+1, & 2i+\lambda+1 & \end{matrix} ;1 \right), \end{aligned} \tag{12}$$

and for the general definition of a generalized hypergeometric series and special ${}_3F_2$, see [47], p. 41 and pp. 103-104, respectively.

3 First kind Abel‘s integral equation

In this section, we use the spectral collocation method to solve the first kind Abel's integral equation in the following form [10,11]

$$\int_0^x \frac{u(t)}{(x-t)^{\mu}} dt = g(x), \quad 0 \le x \le L,\ 0 < \mu < 1, \tag{13}$$

where $g(x)$ is a given function, and $u(t)$ is the unknown function.

First, we state the Riemann-Liouville integral definition as the following form:

Definition 3.1 The integral of order $\mu \geq 0$ (fractional) according to Riemann-Liouville is given by

$$\begin{aligned} J^{\mu} f(x) &= \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} f(t) dt, \qquad \mu > 0,\ x > 0, \\ J^0 f(x) &= f(x), \end{aligned} \tag{14}$$

where

$$\Gamma(\mu) = \int_0^{\infty} x^{\mu-1} e^{-x} dx$$

is the gamma function.

The operator J^{μ} satisfies the following properties

$$J^{\mu} J^{\nu} f(x) = J^{\nu} J^{\mu} f(x) = J^{\mu+\nu} f(x), \qquad J^{\mu} x^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)} x^{\lambda+\mu}. \tag{15}$$

Using definition (14), the Abel's integral equation (13) is transformed to the fractional integral equation in the form:

$$\Gamma(1-\mu) J^{1-\mu} u(x) = g(x), \qquad 0 \le x \le L,\ 0 < \mu < 1. \tag{16}$$

In order to use the spectral collocation method based on the shifted Jacobi polynomials, we approximate $u(x)$ by the shifted Jacobi polynomials as

$$u_N(x) = \sum_{i=0}^{N} a_i P_{L,i}^{(\theta,\vartheta)}(x). \tag{17}$$

In virtue of (4) and (15), we can express the fractional integration of order μ of any shifted Jacobi polynomial $P_{L,i}^{(\theta,\vartheta)}(x)$ as

$$\begin{aligned} J^{\mu}\left(P_{L,i}^{(\theta,\vartheta)}(x)\right) &= P_{L,i}^{(\theta,\vartheta,\mu)}(x) \\ &= \sum_{k=0}^{i} \frac{(-1)^{i-k}\Gamma(i+\vartheta+1)\Gamma(i+k+\theta+\vartheta+1)}{\Gamma(i-k+1)\Gamma(k+\mu+1)\Gamma(k+\vartheta+1)\Gamma(i+\theta+\vartheta+1)L^k} x^{k+\mu}, \end{aligned} \tag{18}$$

therefore, easily we can write

$$J^{1-\mu}u_N(x)=\sum_{i=0}^{N}a_iP_{L,i}^{(\theta,\vartheta,1-\mu)}(x). \tag{19}$$

Therefore, adopting (16) with (19), we can write (13) in the form:

$$\Gamma(1-\mu)\sum_{i=0}^{N}a_iP_{L,i}^{(\theta,\vartheta,1-\mu)}(x)=g(x),\qquad 0\le x\le L,\ 0<\mu<1. \tag{20}$$

Here, we apply the spectral collocation method by setting the residual of the previous equation to be zero at the $N+1$ collocation points as following:

$$\Gamma(1-\mu)\sum_{i=0}^{N}a_iP_{L,i}^{(\theta,\vartheta,1-\mu)}(x_{L,N,k}^{(\theta,\vartheta)})=g(x_{L,N,k}^{(\theta,\vartheta)}),\qquad 0<\mu<1,\ k=0,\cdots,N. \tag{21}$$

The previous equation alternatively may be written in the matrix form:

$$\Gamma(1-\mu)\mathbf{P}^{(1-\mu)}\mathbf{A}=\mathbf{G}, \tag{22}$$

where

$$\mathbf{P}^{(1-\mu)}=\begin{pmatrix} P_{L,0}^{(\theta,\vartheta,1-\mu)}(x_{L,N,0}^{(\theta,\vartheta)}) & P_{L,1}^{(\theta,\vartheta,1-\mu)}(x_{L,N,0}^{(\theta,\vartheta)}) & \dots & P_{L,N}^{(\theta,\vartheta,1-\mu)}(x_{L,N,0}^{(\theta,\vartheta)}) \\ P_{L,0}^{(\theta,\vartheta,1-\mu)}(x_{L,N,1}^{(\theta,\vartheta)}) & P_{L,1}^{(\theta,\vartheta,1-\mu)}(x_{L,N,1}^{(\theta,\vartheta)}) & \dots & P_{L,N}^{(\theta,\vartheta,1-\mu)}(x_{L,N,1}^{(\theta,\vartheta)}) \\ \vdots & \vdots & \ddots & \vdots \\ P_{L,0}^{(\theta,\vartheta,1-\mu)}(x_{L,N,i}^{(\theta,\vartheta)}) & P_{L,1}^{(\theta,\vartheta,1-\mu)}(x_{L,N,i}^{(\theta,\vartheta)}) & \dots & P_{L,N}^{(\theta,\vartheta,1-\mu)}(x_{L,N,i}^{(\theta,\vartheta)}) \\ \vdots & \vdots & \ddots & \vdots \\ P_{L,0}^{(\theta,\vartheta,1-\mu)}(x_{L,N,N-1}^{(\theta,\vartheta)}) & P_{L,1}^{(\theta,\vartheta,1-\mu)}(x_{L,N,N-1}^{(\theta,\vartheta)}) & \dots & P_{L,N}^{(\theta,\vartheta,1-\mu)}(x_{L,N,N-1}^{(\theta,\vartheta)}) \\ P_{L,0}^{(\theta,\vartheta,1-\mu)}(x_{L,N,N}^{(\theta,\vartheta)}) & P_{L,1}^{(\theta,\vartheta,1-\mu)}(x_{L,N,N}^{(\theta,\vartheta)}) & \dots & P_{L,N}^{(\theta,\vartheta,1-\mu)}(x_{L,N,N}^{(\theta,\vartheta)}) \end{pmatrix},$$

$$\mathbf{A}=\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_i \\ \vdots \\ a_{N-1} \\ a_N \end{pmatrix},\qquad \mathbf{G}=\begin{pmatrix} g(x_{L,N,0}^{(\theta,\vartheta)}) \\ g(x_{L,N,1}^{(\theta,\vartheta)}) \\ \vdots \\ g(x_{L,N,i}^{(\theta,\vartheta)}) \\ \vdots \\ g(x_{L,N,N-1}^{(\theta,\vartheta)}) \\ g(x_{L,N,N}^{(\theta,\vartheta)}) \end{pmatrix}.$$

The previous system of algebraic equations can be solved using the Newton's iterative method.

4 Second kind Abel's integral equation

In this section, we apply our technique to solve the second kind Abel's integral equation in the form:

$$u(x)+\int_0^x\frac{u(t)}{(x-t)^{\mu}}dt=g(x),\qquad 0\le x\le L,\ 0<\mu<1, \tag{23}$$

where $g(x)$ is a given function, while $u(t)$ is an unknown function.

As in the previous section, the second kind Abel's integral equation (23) may be transformed into the fractional integral equation:

$$u(x)+\Gamma(1-\mu)J^{1-\mu}u(x)=g(x),\qquad 0\le x\le L,\ 0<\mu<1. \tag{24}$$

After expressing $u(x)$ by shifted Jacobi polynomials as in (17) and approximating the fractional integral of order $1-\mu$ of $u(x)$ as in (19), we can write (24) as in the form:

$$\sum_{i=0}^{N} a_i P_{L,i}^{(\theta,\vartheta)}(x) + \Gamma(1-\mu)\sum_{i=0}^{N} a_i P_{L,i}^{(\theta,\vartheta,1-\mu)}(x) = g(x), \qquad 0 \le x \le L, \quad 0<\mu<1, \tag{25}$$

where $P_{L,i}^{(\theta,\vartheta,1-\mu)}(x)$ is defined as in (18).

Now, we collocate the previous equation at the $N+1$ collocation points as following

$$\sum_{i=0}^{N} a_i \left(P_{L,i}^{(\theta,\vartheta)}(x_{L,N,k}^{(\theta,\vartheta)}) + \Gamma(1-\mu)P_{L,i}^{(\theta,\vartheta,1-\mu)}(x_{L,N,k}^{(\theta,\vartheta)})\right) = g(x_{L,N,k}^{(\theta,\vartheta)}), \quad 0<\mu<1, k=0,\cdots,N. \tag{26}$$

The previous system may be rewritten in a matrix form as

$$\left(\mathbf{P} + \Gamma(1-\mu)\mathbf{P}^{(1-\mu)}\right)\mathbf{A} = \mathbf{G}, \tag{27}$$

where $\mathbf{P}^{(1-\mu)}$, $\mathbf{A}$ and $\mathbf{G}$ are given as in (22) and

$$\mathbf{P} = \begin{pmatrix} P_{L,0}^{(\theta,\vartheta)}(x_{L,N,0}^{(\theta,\vartheta)}) & P_{L,1}^{(\theta,\vartheta)}(x_{L,N,0}^{(\theta,\vartheta)} & \dots & P_{L,N}^{(\theta,\vartheta)}(x_{L,N,0}^{(\theta,\vartheta)}) \\ P_{L,0}^{(\theta,\vartheta)}(x_{L,N,1}^{(\theta,\vartheta)}) & P_{L,1}^{(\theta,\vartheta)}(x_{L,N,1}^{(\theta,\vartheta)} & \dots & P_{L,N}^{(\theta,\vartheta)}(x_{L,N,1}^{(\theta,\vartheta)}) \\ \vdots & \vdots & \ddots & \vdots \\ P_{L,0}^{(\theta,\vartheta)}(x_{L,N,i}^{(\theta,\vartheta)}) & P_{L,1}^{(\theta,\vartheta)}(x_{L,N,i}^{(\theta,\vartheta)} & \dots & P_{L,N}^{(\theta,\vartheta)}(x_{L,N,i}^{(\theta,\vartheta)}) \\ \vdots & \vdots & \ddots & \vdots \\ P_{L,0}^{(\theta,\vartheta)}(x_{L,N,N-1}^{(\theta,\vartheta)}) & P_{L,1}^{(\theta,\vartheta)}(x_{L,N,N-1}^{(\theta,\vartheta)} & \dots & P_{L,N}^{(\theta,\vartheta)}(x_{L,N,N-1}^{(\theta,\vartheta)}) \\ P_{L,0}^{(\theta,\vartheta)}(x_{L,N,N}^{(\theta,\vartheta)}) & P_{L,1}^{(\theta,\vartheta)}(x_{L,N,N}^{(\theta,\vartheta)} & \dots & P_{L,N}^{(\theta,\vartheta)}(x_{L,N,N}^{(\theta,\vartheta)}) \end{pmatrix}.$$

The previous system of algebraic equations can be solved using Newton's iterative method.

5 System of generalized Abel‘s integral equations

In the current section, we apply our technique to solve the second kind Abel's integral equation in the form

$$\begin{aligned} a_{11}(x)\int_0^x \frac{u(t)}{(x-t)^{\mu}}dt + a_{12}(x)\int_x^1 \frac{v(t)}{(t-x)^{\mu}}dt &= g_1(x), \\ a_{21}(x)\int_x^1 \frac{u(t)}{(t-x)^{\mu}}dt + a_{22}(x)\int_0^x \frac{v(t)}{(x-t)^{\mu}}dt &= g_2(x), \end{aligned} \tag{28}$$

where $a_{11}(x)$, $a_{12}(x)$, $a_{21}(x)$, $a_{22}(x)$, $g_1(x)$ and $g_2(x)$ are given functions, while $u(t)$ and $v(t)$ are unknown functions.

Definition 5.1 The left- and right-sided Riemann-Liouville fractional integrals of order μ of any function $f(x)$ for $x \in [0,L]$ are defined, respectively, as

$$\begin{aligned} J_+^{\mu} f(x) &= \frac{1}{\Gamma(\mu)}\int_0^x (x-t)^{\mu-1} f(t)dt, \quad \mu>0, \\ J_-^{\mu} f(x) &= \frac{1}{\Gamma(\mu)}\int_x^L (t-x)^{\mu-1} f(t)dt, \quad \mu>0, \\ J_{\pm}^{0} f(x) &= f(x). \end{aligned} \tag{29}$$

The left- and right-sided Riemann-Liouville fractional integrals operators satisfy the following properties

$$\begin{aligned} J_+^{\mu} x^{\beta} &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\mu)}\, x^{\beta+\mu}, \\ J_-^{\mu} (x-L)^{\beta} &= \frac{(-1)^{\beta}\Gamma(\beta+1)}{\Gamma(\beta+1+\mu)}\,(L-x)^{\beta+\mu}. \end{aligned} \tag{30}$$

The system of generalized Abel's integral equations (28) can be transformed into the following system of fractional integral equations

$$\begin{aligned} a_{11}(x)\Gamma(1-\mu)J_{+}^{1-\mu}u(x)+a_{12}(x)\Gamma(1-\mu)J_{-}^{1-\mu}v(x) &= g_1(x), \\ a_{21}(x)\Gamma(1-\mu)J_{-}^{1-\mu}u(x)+a_{22}(x)\Gamma(1-\mu)J_{+}^{1-\mu}v(x) &= g_2(x). \end{aligned} \tag{31}$$

First, we express $u(x)$ by shifted Jacobi polynomials as in (17) and express $v(x)$ as

$$v_N(x)=\sum_{i=0}^{N} b_i P_{L,i}^{(\theta,\vartheta)}(x). \tag{32}$$

The left- and right-sided Riemann-Liouville fractional integrals of shifted Jacobi polynomials may be given by

$$\begin{aligned} J_{+}^{\mu}\big(P_{L,i}^{(\theta,\vartheta)}(x)\big) &= P_{L,i}^{(\theta,\vartheta,\mu)}(x) \\ &= \sum_{k=0}^{i} \frac{(-1)^{i-k}\Gamma(i+\vartheta+1)\Gamma(i+k+\theta+\vartheta+1)}{\Gamma(i-k+1)\Gamma(k+\mu+1)\Gamma(k+\vartheta+1)\Gamma(i+\theta+\vartheta+1)L^k} x^{k+\mu}, \\ J_{-}^{\mu}\big(P_{L,i}^{(\theta,\vartheta)}(x)\big) &= \widetilde{P}_{L,i}^{(\theta,\vartheta,\mu)}(x) \\ &= \sum_{k=0}^{i} \frac{(-1)^{k}\Gamma(i+\theta+1)\Gamma(i+k+\theta+\vartheta+1)}{\Gamma(i-k+1)\Gamma(k+\mu+1)\Gamma(k+\theta+1)\Gamma(i+\theta+\vartheta+1)L^k} (x-L)^{k+\mu}, \end{aligned} \tag{33}$$

then, easily we can write

$$\begin{aligned} J_{+}^{1-\mu}u_N(x) &= \sum_{i=0}^{N} a_i P_{L,i}^{(\theta,\vartheta,1-\mu)}(x), \\ J_{-}^{1-\mu}u_N(x) &= \sum_{i=0}^{N} a_i \widetilde{P}_{L,i}^{(\theta,\vartheta,1-\mu)}(x), \\ J_{+}^{1-\mu}v_N(x) &= \sum_{i=0}^{N} b_i P_{L,i}^{(\theta,\vartheta,1-\mu)}(x), \\ J_{-}^{1-\mu}v_N(x) &= \sum_{i=0}^{N} b_i \widetilde{P}_{L,i}^{(\theta,\vartheta,1-\mu)}(x). \end{aligned} \tag{34}$$

Therefore, adopting (32)-(34), the system (31) may be written in the form:

$$\begin{aligned} a_{11}(x)\Gamma(1-\mu)\sum_{i=0}^{N} a_i P_{L,i}^{(\theta,\vartheta,1-\mu)}(x)+\Gamma(1-\mu)a_{12}(x)\sum_{i=0}^{N} b_i \widetilde{P}_{L,i}^{(\theta,\vartheta,1-\mu)}(x) &= g_1(x), \\ a_{21}(x)\Gamma(1-\mu)\sum_{i=0}^{N} a_i \widetilde{P}_{L,i}^{(\theta,\vartheta,1-\mu)}(x)+\Gamma(1-\mu)a_{22}(x)\sum_{i=0}^{N} b_i P_{L,i}^{(\theta,\vartheta,1-\mu)}(x) &= g_2(x), \\ 0\le x\le L, \quad 0<\mu<1. \end{aligned} \tag{35}$$

As in the previous section, we collocate the system (35) at $N+1$ collocation points as

$$\begin{aligned} & a_{11}(x_{L,N,k}^{(\theta,\vartheta)})\Gamma(1-\mu)\sum_{i=0}^{N} a_i P_{L,i}^{(\theta,\vartheta,1-\mu)}(x_{L,N,k}^{(\theta,\vartheta)})+\Gamma(1-\mu)a_{12}(x_{L,N,k}^{(\theta,\vartheta)}) \\ & \qquad \times \sum_{i=0}^{N} b_i \widetilde{P}_{L,i}^{(\theta,\vartheta,1-\mu)}(x_{L,N,k}^{(\theta,\vartheta)}) = g_1(x_{L,N,k}^{(\theta,\vartheta)}), \\ & a_{21}(x_{L,N,k}^{(\theta,\vartheta)})\Gamma(1-\mu)\sum_{i=0}^{N} a_i \widetilde{P}_{L,i}^{(\theta,\vartheta,1-\mu)}(x_{L,N,k}^{(\theta,\vartheta)})+\Gamma(1-\mu)a_{22}(x_{L,N,k}^{(\theta,\vartheta)}) \\ & \qquad \times \sum_{i=0}^{N} b_i P_{L,i}^{(\theta,\vartheta,1-\mu)}(x_{L,N,k}^{(\theta,\vartheta)}) = g_2(x_{L,N,k}^{(\theta,\vartheta)}), \\ & 0<\mu<1, \quad k=0,\cdots,N. \end{aligned} \tag{36}$$

The previous system of algebraic equations can be solved using Newton's iterative method.

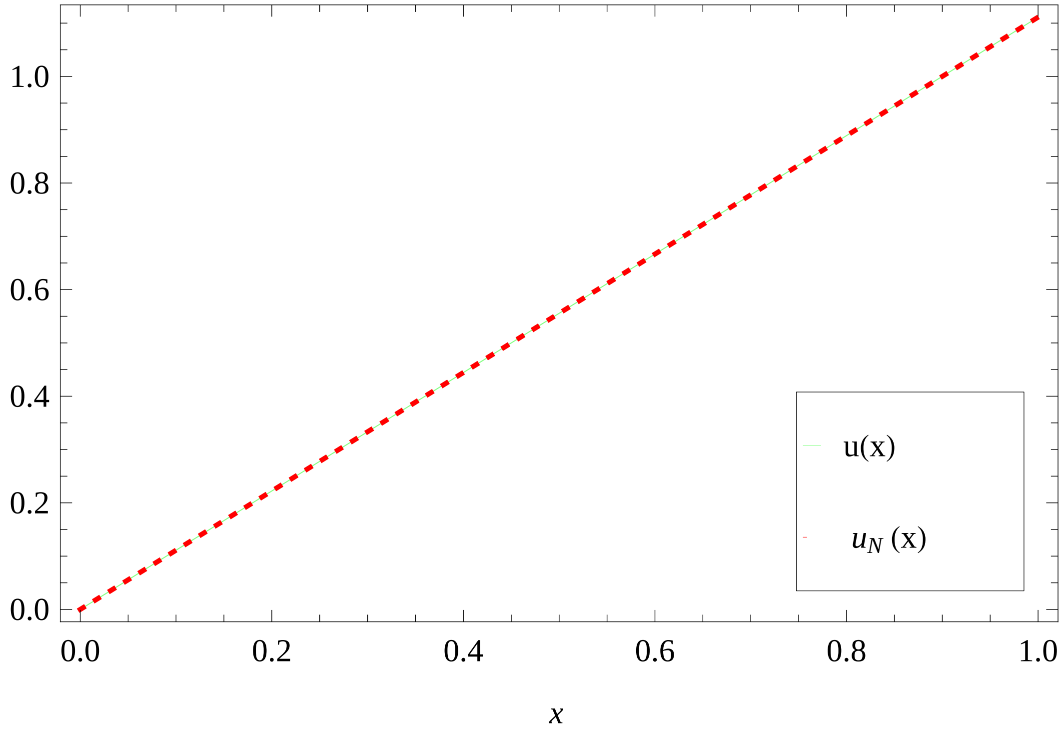

Fig. 1: The exact and numerical solutions curves for Example 6.1 with $\theta = 0$, $\vartheta = \frac{1}{2}$ and $N = 1$.

6 Numerical results

For clarifying the validity and accuracy of the presented algorithm, we have applied it to solve numerical examples of the first kind, the second kind and the system of generalized Abel's integral equations.

6.1 Example 1

Firstly, we introduce the first kind Abel‘s integral equation in the following form [15]

$$\int_0^x \frac{u(t)}{(x-t)^{\frac{1}{3}}} dt = x^{\frac{5}{3}}, \qquad 0 \le x \le 1, \tag{37}$$

and the exact solution is given by $u(x) = \dfrac{10x}{9}$.

Using the technique discussed in Section 3 with different choice of θ, ϑ $\Big($e.g. $\theta = \vartheta = 0$ (shifted Legendre-Gauss collocation method), $\theta = \vartheta = \mp\frac{1}{2}$ (first and second kind shifted Chebyshev-Gauss collocation), $\theta = 0$, $\vartheta = \frac{1}{2}$ and $\theta = 0$, $\vartheta = \frac{1}{2}\Big)$ at $N \ge 1$, we obtained the exact solution $u_N = \dfrac{10x}{9}$. In the case of $\theta = 0$, $\vartheta = \frac{1}{2}$ and $N = 1$, the numerical and exact solutions curves for problem (37) by using our method are shown in Fig. 1.

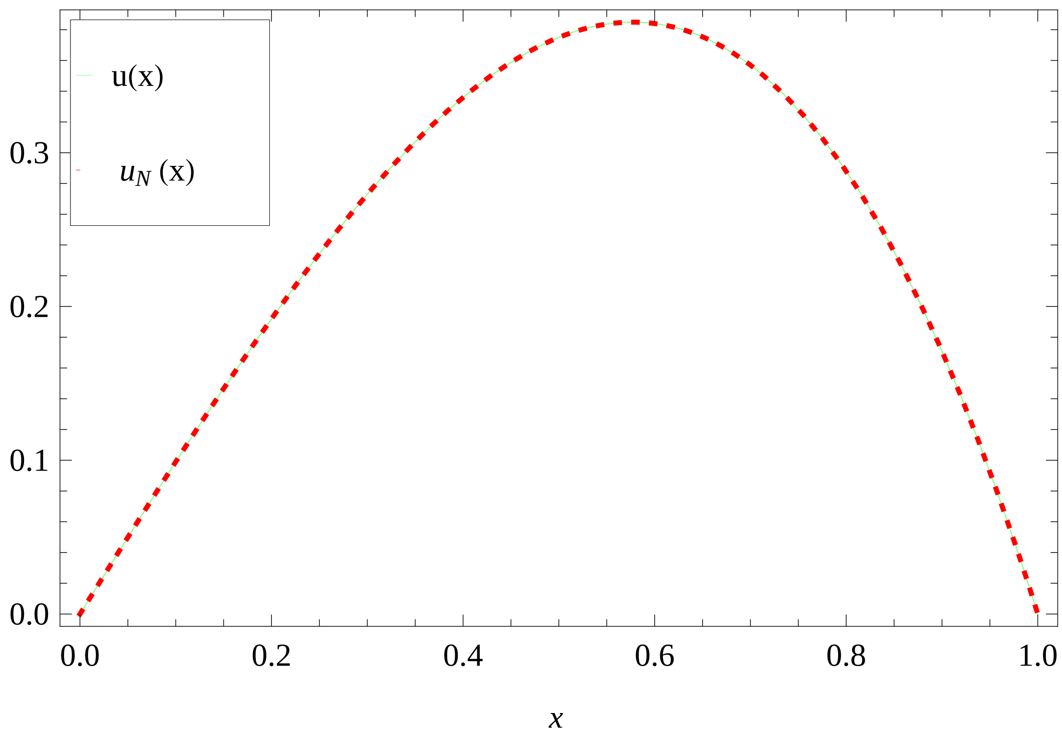

Fig. 2: The exact and numerical solutions curves for Example 6.2 with $\theta = 1$, $\vartheta = -\frac{1}{2}$ and $N = 3$.

6.2 Example 2

Here, we consider the problem

$$\int_0^x \frac{u(t)}{(x-t)^{\frac{1}{2}}} dt = \frac{4}{105} x^{3/2} \left(35 - 24x^2\right), \qquad 0 \le x \le 1, \tag{38}$$

The exact solution of this problem is given by $u(x) = x - x^3$.

Also, the approximate solution for this problem gives the exact solution for different choice of θ, ϑ at $N \ge 3$. The approximate solution obtained using our method at $\theta = 1$, $\vartheta = -\frac{1}{2}$ and $N = 3$ for (38) shown in Figure (2) to make it easier to compare with the exact solution.

6.3 Example 3

Consider the following Abel's integral equation studied in [10,11]

$$\int_0^x \frac{x^2t^2 + t^4 + 1}{(x-t)^{\frac{1}{4}}} u(t) dt = \frac{32768}{100947} x^{\frac{31}{4}} + \frac{262144}{908523} x^{\frac{27}{4}} + \frac{128}{231} x^{\frac{11}{4}} \qquad ,0 \le x \le 1, \tag{39}$$

which having an exact solution given by $u(x) = x^2$.

By applying the technique described in Section 3, we obtain that

$$\int_0^x \frac{(x^2t^2 + t^4 + 1)(\sum_{i=0}^{N} a_i P_{L,i}^{(\theta,\vartheta)}(t))}{(x-t)^{\frac{1}{4}}} dt = \frac{32768}{100947} x^{\frac{31}{4}} + \frac{262144}{908523} x^{\frac{27}{4}} + \frac{128}{231} x^{\frac{11}{4}} \qquad ,0 \le x \le 1, \tag{40}$$

yields the following system of algebraic equations

$$\int_0^{x_{N,k}^L} \frac{((x_{N,k}^L)^2 t^2 + t^4 + 1)(\sum_{i=0}^{N} a_i P_{L,i}^{(\theta,\vartheta)}(t))}{(x_{N,k}^L - t)^{\frac{1}{4}}} dt = \frac{32768}{100947}(x_{N,k}^L)^{\frac{31}{4}} + \frac{262144}{908523}(x_{N,k}^L)^{\frac{27}{4}} + \frac{128}{231}(x_{N,k}^L)^{\frac{11}{4}}, \tag{41}$$

$$0 \le x \le 1, \quad k = 0, \cdots, N.$$

Despite that the best results for the maximum absolute errors achieved by using the numerical methods (mid-point rectangular, trapezoidal quadrature, Asymptotic expansion, combination algorithm, a posteriori estimate and mechanical quadrature methods) given in [10,11] with 40 or 80 steps are bounded between 2.7797×10^{-3} and 1.8779×10^{-8}, we obtained the exact solution using $N \ge 2$ and different choice of θ, ϑ.

6.4 Example 4

Let us consider the following second kind Abel's integral [20]

$$u(x) = x + \frac{4}{2}x^{\frac{3}{2}} - \int_0^x \frac{u(t)}{(x-t)^{\frac{1}{2}}} dt \qquad x \in [0,1], \tag{42}$$

Applying the numerical technique discussed in Section 4 we obtain that $u_N(x) = x$, $N \ge 1$, which is the exact solution of the mentioned problem. Although we got the exact solution using a small number of nodes ($N \ge 1$ and different choice of θ, ϑ) where the best value of the maximum absolute errors obtained in [20] was 5×10^{-6} at $N = 25$.

6.5 Example 5

Here, we tested the following second kind Abel's integral equation [20]

$$u(x) = x^2 + \frac{16}{15}x^{\frac{5}{2}} - \int_0^x \frac{u(t)}{(x-t)^{\frac{1}{2}}} dt \qquad ,0 \le x \le 1,. \tag{43}$$

Kumar et al. [20] introduced this problem and applied the homotopy perturbation transform method, the maximum absolute error achieved in [20] was 5×10^{-7} at $N = 25$.

Taking $N \ge 2$, we used the technique presented in Section 4 with different choices of θ, ϑ $\Big($e.g. $\theta = \vartheta = 0$ (shifted Legendre-Gauss collocation method), $\theta = \vartheta = \mp\frac{1}{2}$ (first and second kind shifted Chebyshev-Gauss collocation), $\theta = 0$, $\vartheta = \frac{1}{2}$ and $\theta = 0$, $\vartheta = \frac{1}{2}\Big)$, the numerical approximation $u_N(x)$ is equal to the exact solution $u(x) = x^2$.

6.6 Example 6

As a system of Abel's integral equations, we consider this problem [23]

$$\int_0^x \frac{u(t)}{(x-t)^{\frac{1}{2}}} dt + \frac{1}{4}\int_x^1 \frac{v(t)}{(t-x)^{\frac{1}{2}}} dt = \frac{16}{15}x^{\frac{5}{2}} + \frac{1}{10}(1-x)^{\frac{5}{2}} + \frac{1}{2}x^2(1-x)^{\frac{3}{2}} + \frac{1}{2}x^2(1-x)^{\frac{1}{2}}$$
$$+ \frac{1}{2}x^3(1-x)^{\frac{1}{2}} + \frac{1}{3}x(1-x)^{\frac{3}{2}} + \frac{1}{14}(1-x)^{\frac{7}{2}} + \frac{3}{10}x(1-x)^{\frac{5}{2}},$$
$$\frac{1}{2}\int_x^1 \frac{u(t)}{(t-x)^{\frac{1}{2}}} dt + 3\int_0^x \frac{v(t)}{(x-t)^{\frac{1}{2}}} dt = \frac{1}{5}(1-x)^{\frac{5}{2}} + x^2(1-x)^{\frac{1}{2}} + \frac{3}{2}x(1-x)^{\frac{3}{2}} + \frac{16}{5}x^{\frac{5}{2}} + \frac{96}{35}x^{\frac{7}{2}},$$

where the exact solution $u(x) = x^2$ and $v(x) = x^2 + x^3$.

Setia and Pandey [23] introduced this problem and used the Laguerre polynomials for approximating its numerical solution. In order to show that our technique discussed in Section 5 is accurate than this introduced in [23], in Fig. 3, we plot the exact and numerical solutions of $u(x)$ and $v(t)$ at $N = 3$ with different choice of θ and ϑ. the numerical approximation $u_N(x)$ is equal to the exact solution $u(x) = x^2$ and $v(x) = x^2 + x^3$.

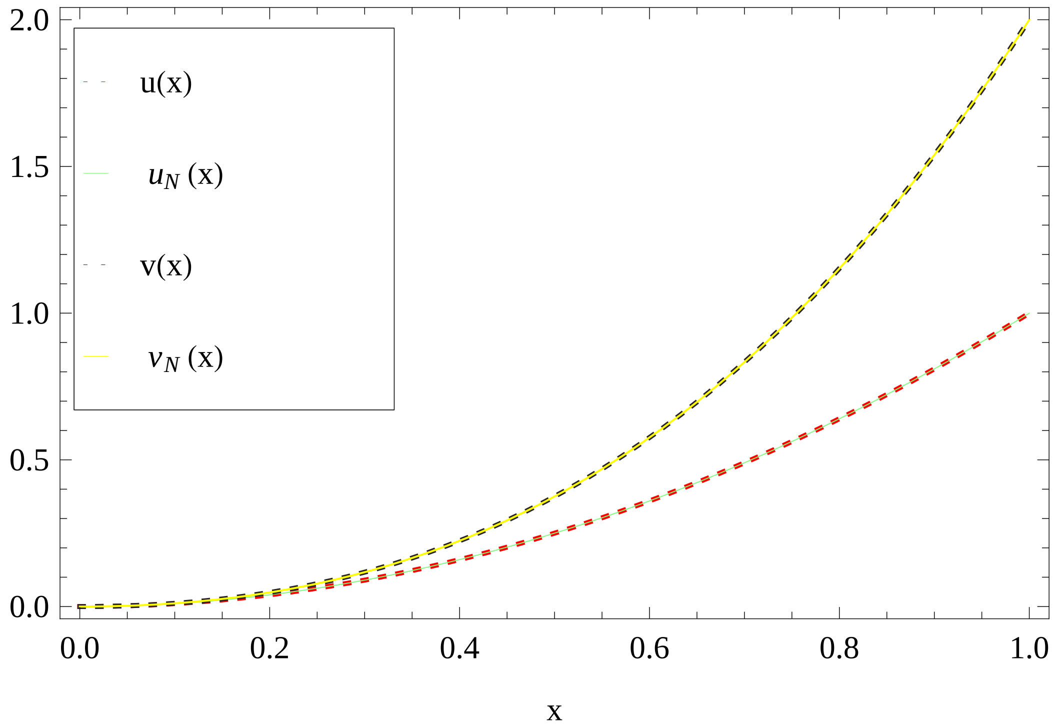

Fig. 3: The exact and numerical solutions curves for Example 6.6 with $\theta = 1,\ \vartheta = -\frac{1}{2}$ and $N = 3$.

6.7 Example 7

Now, we consider the following system of Abel' integral equations [23]

$$
\begin{aligned}
(x^2+1)\int_0^x \frac{u(t)}{(x-t)^{\frac{1}{2}}}dt + \frac{x+1}{4}\int_x^1 \frac{v(t)}{(t-x)^{\frac{1}{2}}}dt &= \frac{1}{14}\sqrt{1-x} + \frac{4}{70}x\sqrt{1-x} + \frac{1}{5}x^2\sqrt{1-x} \\
&\quad + \frac{16}{15}x^{\frac{5}{2}} + \frac{12}{35}x^3\sqrt{1-x} + \frac{8}{35}x^4\sqrt{1-x} + \frac{16}{15}x^{\frac{9}{2}}, \\
\frac{x^2}{2}\int_x^1 \frac{u(t)}{(t-x)^{\frac{1}{2}}}dt + (2-x)\int_0^x \frac{v(t)}{(x-t)^{\frac{1}{2}}}dt &= \frac{1}{5}x^2\sqrt{1-x} + \frac{4}{15}x^3\sqrt{1-x} + \frac{64}{35}x^{\frac{7}{2}} \\
&\quad + \frac{8}{15}x^4\sqrt{1-x} + \frac{32}{35}x^{\frac{9}{2}},
\end{aligned}
$$

with exact solution given by $u(x) = x^2$ and $v(x) = x^3$.

Using the method presented in Section 5, we plot the graph of the numerical and exact solutions of $u(x)$ and $v(x)$ at $N = 3$ with different choice of θ, ϑ in Fig. 4.

7 Conclusion

In this article, a new fast numerical technique is constructed to introduce an approximate solution of the first and second kinds of Abel's integral equations. Our numerical approach is consisting of transforming such problems into a fractional

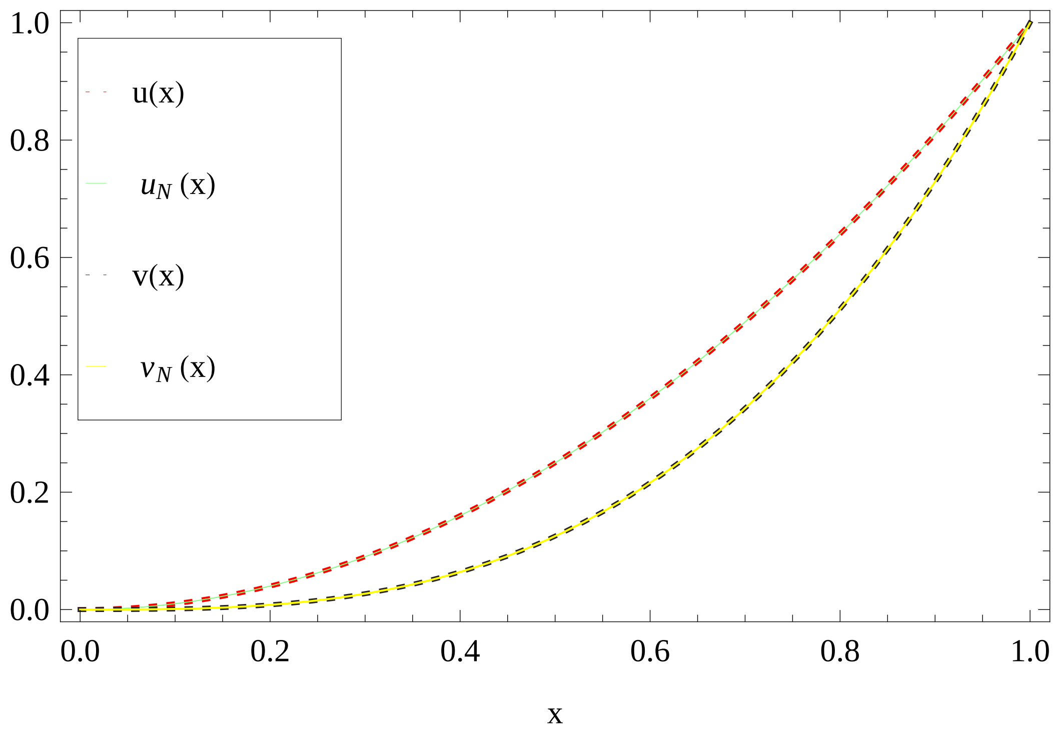

Fig. 4: The exact and numerical solutions curves for Example 6.7 with $\theta = 1$, $\vartheta = -\frac{1}{2}$ and $N = 3$.

integral equation (described in the Riemann-Liouville sense). Using the shifted Jacobi polynomials as basis functions of the spectral collocation method and the Gauss-quadrature formula, the fractional integral equation is reduced into a problem consisting of system of algebraic equations that can be solved using any standard iteration method. The system of generalized Abel's integral equations is investigated also using the proposed technique. The numerical results we have obtained demonstrate the high accuracy of our technique, only a small number of shifted Jacobi polynomials are needed to obtain a satisfactory result.

References

[1] J. Walton, Systems of generalized Abel integral equations with applications to simultaneous dual relations, SIAM J. Math. Anal. **10**, 808-822 (1979).

[2] O. Knill, R. Dgani and M. Vogel, A new approach to Abel's integral operator and its application to stellar winds, Astronom., Astrophys. **274**, 1002-1008 (1993).

[3] S. De, B. N. Mandal and A. Chakrabarti, Use of Abel's integral equations in water wave scattering by two surface-piercing barriers, Wave Motion **47**, 279-288 (2010).

[4] C. J. Cremers and R.C. Birkebak, Application of the Abel integral equation to spectroscopic data, Appl. Opt. **5**, 1057-1064 (1966).

[5] R. Gorenflo and S. Vessella, Abel Integral Equations: Analysis and Applications, Lecture Notes Math., **1461**, Springer, Berlin, 1991.

[6] L. E. Kosarev, Applications of integral equations of the first kind in experiment physics., Comput. Phys. Commun. **20** , 69-75 (1980).

[7] H. Brunner, 1896-1996: One hundred years of Volterra integral equation of the first kind, Appl. Numer. Math. **24**, 83-93 (1997).

[8] A. M. Wazwaz, A First Course in Integral Equations, World Scientific Publishing, Singapore, 1997.

[9] A. D. Polyanin and A.V. Manzhirov, Handbook of Integral Equations, CRC Press, Boca Raton, 1998.

[10] Y-p. Liu and L. Tao, High accuracy combination algorithm and a posteriori error estimation for solving the first kind Abel integral equations, Appl. Math. Comput. **178**, 441-451 (2006).

[11] Y-p. Liu and L. Tao, Mechanical quadrature methods and their extrapolation for solving first kind Abel integral equations, J. Comput. Appl. Math. **201**, 300-313 (2007).
[12] S. Bhattacharya and B. N. Mandal, Use of Bernstein polynomials in numerical solutions of Volterra integral equations, Appl. Math. Sci. **2**, 1773-1787 (2008).
[13] M. Li and W. Zhao, Solving Abel's type integral equation with Mikusinski's operator of fractional order, Adv. Math. Phys. **2013**, DOI:org/10.1155/2013/806984 (2013).
[14] S. Jahanshahi, E. Babolian, D.F.M. Torres and A. Vahidi, Solving Abel integral equations of first kind via fractional calculus, J. King Saud Univ. Sci., DOI:org/10.1016/j.jksus.2014.09.004 (2014).
[15] L. Huang, Y. Huang and X-F. Li, Approximate solution of Abel integral equation, Comput. Math. Appl. **56**, 1748-1757 (2008).
[16] R. K. Pandey, O. P. Singh and V. K. Singh, Efficient algorithms to solve singular integral equations of Abel type, Comput. Math. Appl. **57**, 664-676 (2009).
[17] S. Dixit, O. P. Singh and S. Kumar, A stable numerical inversion of generalized Abels integral equation, Appl. Numer. Math. **62**, 567-579 (2012).
[18] S. A. Yousefi, Numerical solution of Abels integral equation by using Legendre wavelets, Appl. Math. Comput. **175**, 574-580 (2006).
[19] M. Khan and M. A. Gondal, A reliable treatment of Abels second kind singular integral equations, Appl. Math. Lett. **25**, 1666-1670 (2012).
[20] S. Kumar, A. Kumar, D. Kumar, J. Singh and A. Singh, Analytical solution of Abel integral equation arising in astrophysics via Laplace transform, J. Egypt. Math. Soc., DOI:org/10.1016/j.joems.2014.02.004 (2014).
[21] M. Lowengrub and J. Walton, Systems of generalized Abel equations, SIAM J. Math. Anal. **10**, 794-807 (1979).
[22] S. Kumar, O. P. Singh and S. Dixit, Homotopy perturbation method for solving system of generalized Abel's integral equations, Appl. Appl. Math. **6**, 268-283 (2009).
[23] A. Setia and R. K. Pandey, Laguerre Polynomials based numerical method to solve a system of Generalized Abel integral Equations, Procedia Engineering **38**, 1675-1682 (2012).
[24] R. T. Baillie, Long memory processes and fractional integration in econometrics, J. Econometrics **73**, 5-59 (1996).
[25] R. Metzler and J. Klafter, The restaurant at the end of the random walk: Recent developments in the description of anomalous transport by fractional dynamics, J. Phys. A **37**, 161-208 (2004).
[26] R. L. Magin, Fractional Calculus in Bioengineering, Begell House Publishers, 2006.
[27] D. A. Benson, S. W. Wheatcraft and M.M. Meerschaert, Application of a fractional advection-dispersion equation, Water Resour. Res. **36**, 1403-1412 (2000).
[28] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.
[29] A. A. Kilbas, H. M. Srivastava and J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, 2006.
[30] Y.L. Jiang and X. L. Ding, Waveform relaxation methods for fractional differential equations with the Caputo derivatives, J. Comput. Appl. Math. **238**, 51-67 (2013).
[31] H. Wang and N. Du, Fast alternating-direction finite difference methods for three-dimensional space-fractional diffusion equations, J. Comput. . Phys. **258**, 305-318 (2014).
[32] L. Wang, Y. Ma and Z. Meng, Haar wavelet method for solving fractional partial differential equations numerically, Appl. Math. Comput. **227**, 66-76 (2014).
[33] J. Liu and G. Hou, Numerical solutions of the space- and time-fractional coupled Burgers equation by generalized differential transform method, Appl. Math. Comput. **217**, 7001-7008 (2011).
[34] R. K. Pandey, O. P. Singh and V. K. Baranwal, An analytic algorithm for the space-time fractional advection-dispersion equation, Comput. Phys. Commun. **182**, 1134-1144 (2011).
[35] A. H. Bhrawy and M. A. Zaky, Numerical simulation for two-dimensional variable-order fractional nonlinear cable equation, Nonlinear Dyn., DOI:10.1007/s11071-014-1854-7 (2014).
[36] M. A. Abdelkawy, E. A. Ahmed and P. Sanchez, A method based on Legendre pseudo-spectral approximations for solving inverse problems of parabolic types equations, Math. Sci. Lett. **4**, 81-90 (2015).
[37] A. H. Bhrawy, E.H. Doha, D. Baleanu, S. S. Ezz-Eldien, M.A. Abdelkawy, An accurate numerical technique for solving fractional optimal control problems, Proc. Rom. Acad. Ser. A **16**, 47-54 (2015).
[38] E. H. Doha, A. H. Bhrawy, M. A. Abdelkawy, A shifted Jacobi collocation algorithm for wave type equations with non-local conservation conditions, Cent. Eur. J. Phys. **12**, 637-653 (2014).
[39] A. H. Bhrawy, M. A Alghamdi, A shifted Jacobi-Gauss-Lobatto collocation method for solving nonlinear fractional Langevin equation involving two fractional orders in different intervals, Bound. Value Probl. **2012**, 62 (2012).
[40] A. H. Bhrawy, A. A. Al-Zahrani, Y. A. Alhamed and D. Baleanu, A new generalized Laguerre-Gauss collocation scheme for numerical solution of generalized fractional Pantograph equations, Rom. J. Phys. **59**, 646-657 (2014).
[41] A. H. Bhrawy, E. A. Ahmed and D. Baleanu, An efficient collocation technique for solving generalized Fokker-Planck type equations with variable coefficients, Proc. Rom. Acad. Ser. A **15**, 322-330 (2014).
[42] A. H. Bhrawy, An efficient Jacobi pseudospectral approximation for nonlinear complex generalized Zakharov system, Appl. Math. Comput. **247**, 30-46 (2014).
[43] E. H. Doha, A. H. Bhrawy, M. A. Abdelkawy and R. A. V. Gorder, Jacobi-Gauss-Lobatto collocation method for the numerical solution of 1+1 nonlinear Schrödinger equations, J. Comput. Phys. **26**, 244-255 (2014).

[44] E. H. Doha, A. H. Bhrawy and S. S. Ezz-Eldien, A new Jacobi operational matrix: An application for solving fractional differential equations, Appl. Math. Model. **36**, 4931-4943 (2012).
[45] A. H. Bhrawy, M. A. Zaky and J. A. Tenreiro Machado, Efficient Legendre spectral tau algorithm for solving the two-sided spacetime Caputo fractional advection-dispersion equation, J. Vib. Control., doi:10.1177/1077546314566835 (2015).
[46] A. H. Bhrawy and M. A. Zaky, A method based on the Jacobi tau approximation for solving multi-term time-space fractional partial differential equations, J. Comput. Phys. **281**, 876-895 (2015).
[47] Y. Luke, The Special Functions and Their Approximations, vol. 2, Academic Press, New York, 1969.