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On Parabolic Analytic Functions with Respect to Symmetrical Points

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Abstract: Let A be the class of functions f, $f(z)=z+\sum_{m=2}^{\infty}a_mz^m$, analytic in the open unit disc E. Let $S_s^*(h)$ consist of functions $f\in A$ such that $\frac{2zf'(z)}{f(z)-f(-z)}\prec h(z)$, where $\prec$ denotes subordination and $h(z)$ is analytic in E with $h(0)=1$. For $n=0,1,2,3,\ldots$, a certain integral operator $I_n: A\to A$ is defined as $I_nf=f_n^{-1}*f$ such that $(f_n^{-1}*f_n)(z)=\frac{z}{z-1}$, where $f_n(z)=\frac{z}{(1-z)^{n+1}}$, and $*$ denotes convolution. By taking $h(z)=\left[1+\frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2\right]^{\alpha}, 0<\alpha<1$, and using the operator I_n, we define some new classes $UST_s(n,\alpha)$ and $UK_s(n,\alpha)$, and study some interesting properties of these classes. The ideas and techniques of this paper may motivate further research in this field.

Keywords: starlike, convex, integral operator, subordination, arc length, coefficients
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1 Introduction

Let A be the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{j=2}^{\infty}a_jz^j, \tag{1}$$

which are analytic in the open unit disc $E:=\{z:|z|<1\}$. Let S^* and C be the subclasses of A which, respectively, consist of starlike, convex univalent functions.

An analytic function f is subordinate to an analytic function g, written $f(z)\prec g(z)$, if there is an analytic function $w:E\to E$ with $w(0)=0$ satisfying $f(z)=g(w(z))$. Various subclasses of S^* and C can be unified by requiring that either of the quantity $\frac{zf'(z)}{f(z)}$ or $\left\{1+\frac{zf''(z)}{f'(z)}\right\}$ is subordinate to a function $h(z)$ with a positive real part in E, $h(0)=1$, $h'(0)>0$. These unified classes are denoted as $S^*(h)$ and $C(h)$. For recent developments, see [11,12] and the references therein.
We note some of the subclasses as in the following

(i) $S^*(h_{PAR})=UST=\left\{f\in A:\Re\left(\frac{zf'(z)}{f(z)}\right)>\left|\frac{zf'(z)}{f(z)}-1\right|\right\}$, where

$$h_{PAR}(z)=1+\frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2. \tag{2}$$

$UST=S^*(h_{PAR})$ is called the class of the parabolic starlike functions introduced by Ronning [14].

(ii) $S^*(\beta)=S^*\left(\left(\frac{1+z}{1-z}\right)^{\beta}\right)=\left\{f\in A:\left|\arg\frac{zf'(z)}{f(z)}\right|<\frac{\beta\pi}{2}\right\}$. $S^*(\beta)$ is called the class of strongly starlike function of order β, $0<\beta\leq 1$.

(iii) The classes S_γ^*, C_γ of starlike and convex functions of order γ, respectively, are defined as:

$$S_\gamma^*=\left\{f\in A:\Re\frac{zf'(z)}{f(z)}>\gamma\right\},$$

$$C_\gamma=\left\{f\in A:\Re\frac{(zf'(z))'}{f'(z)}>\gamma\right\}.$$

The corresponding classes $C(h_{PAR})$ and $C(\beta)$ of convex functions are defined accordingly.

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Let the parabolic domain Ω_* be defined as follows.

$$\Omega_* = \{u + iv : u > \sqrt{(u-1)^2 + v^2}\}. \tag{3}$$

That is, Ω_* is bounded by parabola $v^2 = 2u - 1$. The function $h_{PAR}(z)$, given by (2), is known to be univalent in E and maps E conformally onto Ω_*.

Let P be the class of Caratheodory functions p, with $p(0) = 1$ and $\Re p(z) > 0, z \in E$.
Then $P_{PAR} \subset P$ is the class of functions $p(z)$ which are subordinate to $h_{PAR}(z)$ in E. Also, we define the class $P_{PAR}(\alpha), 0 < \alpha \leq 1$, which is a subclass of P and consists of analytic functions $p(z)$, $p(0) = 1$ such that $p(z) \prec [h_{PAR}(z)]^\alpha$, where $h_{PAR}(z)$ is given by (2).
We note that $P_{PAR}(1) = P_{PAR}$. We call $UST(\alpha)$ and $UCV(\alpha)$, the classes of strongly uniformly convex functions, respectively. These classes are defined as follows

$$UST(\alpha) = \left\{f \in A : \frac{zf'(z)}{f(z)} \in P_{PAR}(\alpha)\right\},$$

and

$$UCV(\alpha) = \left\{f \in A : \left\{1 + \frac{zf''(z)}{f'(z)}\right\} \in P_{PAR}(\alpha)\right\}.$$

In 1959, Sakaguchi [18] defined the class of starlike functions with respect to symmetrical points. We use this concept and define the following.

Definition 1. *Let $f \in A$. Then $f(z)$ is said to belong to the class $UST_s(\alpha)$ if and only if*

$$\left\{\frac{2zf'(z)}{f(z) - f(-z)}\right\} \in P_{PAR}(\alpha), z \in E.$$

Similarly $f \in UCV_S(\alpha), 0 < \alpha \leq 1$, if and only if, for $z \in E$

$$-\frac{2(zf'(z))'}{(f(z) - f(-z))'}\Big\} \in P_{PAR}(\alpha).$$

The class A is closed under the Hadamard product or convolution $(*)$

$$(f_1 * f_2)(z) = z + \sum_{j=1}^{\infty} a_{j+1,1} a_{j+1,z} z^{j+1},$$

where

$$f_k(z) = z + \sum_{j=1}^{\infty} a_{j+1,m} z^{j+1} \in A, k = 1, 2.$$

Denote by $D^n : A \to A$, the operator defined by

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z), n \in N_0 = \{0, 1, 2 \ldots\}.$$

The symbol D^n is called the Ruscheweyh derivative of nth order.

Let $f_n(z) = \frac{z}{(1-z)^{n+1}}, n \in N_0$,
and let $f_n^{-1}(z)$ be defined such that

$$(f_n * f_n^{(-1)})(z) = \frac{z}{1-z} \tag{4}$$

Analogous to symbol D^n, an integral operator $I_n : A \to A$ is defined as follows; see [7].

$$\begin{aligned} I_n f(z) &= (f_n^{-1} * f)(z) \\ &= \left[\frac{z}{(1-z)^{n+1}}\right]^{-1} * f(z), n \in N_0. \end{aligned} \tag{5}$$

We note that $I_0 f = zf'$ and $I_1 f = f$, see also [8,9].

From (4) and (5), we obtain the following identity for I_n.

$$(n+1)I_n f(z) - nI_{n+1} f(z) = z(I_{n+1} f(z))'. \tag{6}$$

The hypergeometric function ${}_2F_1$ can be used to define $I_n f$ as follows. Since

$$(1-z)^{-a} = {}_2F_1(a, 1; 1; z), a > 1,$$

we have

$$\begin{aligned} \left[\frac{z}{(1-z)^{n+1}}\right]^{-1} &= {}_2F_1(1, 1; a; z) \\ &= (a-1)\int_0^1 (1-t)^{a-2} \frac{dt}{1-tz}. \end{aligned}$$

Therefore

$$I_n f(z) = [z\, {}_2F_1(1, 1; n; z)] * f(z), n \in N_0.$$

We now define the main classes of analytic functions which will be studied in this paper as follows.

Definition 2. *Let $f \in A$. Then $f \in UST_s(n, \alpha)$ if and only if $I_n f \in UST(\alpha)$ for $0 < \alpha \leq 1, n \in N_0$ and $z \in E$.*
We note that $UST_s(1,1) = UST_s$. That is
$f \in UST_s(1,1)$ implies $\frac{2zf'(z)}{f(z) - f(-z)} \prec h_{PAR}(z)$ in E.

Definition 3. *Let $f \in A$. Then $f(z)$ is said to belong to the class $UK_s(n, \alpha)$ if and only if there exists $g \in UST_s(n, \alpha)$ such that $\frac{z(I_n f(z))'}{I_n g(z)} \in P_{PAR}, z \in E$.*

Throughout this paper, we shall assume $n \in N_0$,
$0 < \alpha \leq 1, z \in E$ unless otherwise stated.

2 Preliminaries

Lemma 1([6]). *Let $u_1 + iu_2$ and $v = v_1 + iv_2$ and let Φ be a complex-valued functions satisfying the conditions:*
(i) *$\Phi(u, v)$ is continuous in a domain $D \subset \mathbb{C}^2$,*
(ii) *$(1, 0) \in D$ and $\Phi(1, 0) > 0$,*
(iii) *$\Re\Phi(iu_2, v_1) \leq 0$ whenever $(iu_2, v_1) \in D$ and $v_1 \leq -\frac{1}{2}(1 + u_2^2)$.*
If $h(z) = 1 + \sum_{m=1}^{\infty} c_m z^m$ is a function analytic in E such that $h(z), zh'(z) \in D$ and $\Re(h(z), zh'(z)) > 0$ for $z \in E$, then $\Re h(z) > 0$ in E.

Lemma 2([15]). *Let $p(z)$ be an analytic function in E with $p(0)=1$ and $\Re p(z)>0, z\in E$. Then, for $s>0$ and $\mu\neq -1$ (complex),*

$$\Re\{p(z)+\frac{szp'(z)}{p(z)+\mu}\}>0 \quad \text{for } |z|<r_0,$$

where r_0 is given by

$$r_0=\frac{|\mu+1|}{\sqrt{A+(A^2-|\mu^2-1|^2)^{\frac{1}{2}}}}, \tag{7}$$
$$A=2(s+1)^2+|\mu|^2-1,$$

and this radius is best possible.

The following result is a special case one due to Kanas [4].

Lemma 3. *Let β,δ be any complex numbers with $\beta\neq 0$ and $\Re\left(\frac{\beta}{2}+\delta\right)>0$. If $h(z)$ is analytic in E, $h(0)=1$ and satisfies*

$$\left(h(z)+\frac{zh'(z)}{\beta h(z)+\delta}\right)\prec h_{PAR}(z), \tag{8}$$

where $h_{PAR}(z)$ is given by (2), and $q_*(z)$ is an analytic solution of

$$q_*(z)+\frac{zq_*'(z)}{\beta q_*(z)+\delta}=h_{PAR}(z),$$

then $q_*(z)$ is univalent and $h(z)\prec q_*(z)\prec h_{PAR}(z)$. Here $q_*(z)$ is the best dominant of (2) and is given by

$$q_*(z)=\left[\int_0^1\left(\exp\int_t^{tz}\frac{h_{PAR}(u)-1}{u}\mathrm{d}u\right)\mathrm{d}t\right]^{-1}.$$

Lemma 4([3]). *Let $w(z)$ be analytic in E. If $|w(z)|$ assumes its maximum value on the circle $|z|=r$ at a point z_0, then $z_0w'(z_0)=kw(z_0)$, where $k\geq 1$.*

Lemma 5([17]). *Let $\Phi\in C$ and $g\in S^*$ in E. Then, for F analytic in E with $F(0)=1$, $\frac{\phi*Fg}{\phi*g}$ is contained in the closed convex hull $\overline{C}o$ of $F(E)$.*

Lemma 6. *Let $p\in P$, $z\in E$ and $z=re^{i\theta}$. Then*

(i) $\int_0^{2\pi}|p(re^{i\theta})|^{\lambda}\mathrm{d}\theta<C(\lambda)\frac{1}{(1-r)^{\lambda-1}}$, *where $\lambda>1$ and $C(\lambda)$ is a constant depending only on λ. For this result, we refer to [2].*

(ii) $\int_0^{2\pi}|p(re^{i\theta})|^{2}\mathrm{d}\theta\leq\frac{1+3r^2}{1-r^2}$ *see [13].*

Lemma 7([5]). *Let $q(z)$ be a convex function in E with $q(0)=1$ and let another function $h:E\to\mathbb{C}$ be $\Re h(z)>0$. Let $p(z)$ be analytic in E with $p(0)=1$ such that*

$$(p(z))+h(z)(zp'(z))\prec q(z), \quad z\in E.$$

Then $p(z)\prec q(z)\in E$.

3 The class $UST_s(n,\alpha)$

Theorem 1. *Let $f\in UST_s(n,\alpha)$. Then the odd function*

$$\psi(z)=\frac{1}{2}[f(z)-f(-z)], \tag{9}$$

belongs to $UST(n,\alpha)$.

Proof. From (9), we can write

$$\begin{aligned} I_n\psi(z) &= \tfrac{1}{2}I_n[f(z)-f(-z)]\\ &= \tfrac{1}{2}[I_nf(z)-I_nf(-z)]. \end{aligned}\tag{10}$$

By logarithmic differentiation of (10), we have

$$\frac{z(I_n\psi'(z))}{I_n\psi(z)}=\frac{1}{2}\left[\frac{2z(I_nf(z))'}{(I_nf(z))-(I_nf(-z))}+\frac{2(-z)(I_nf(-z))'}{(I_nf(z))-(I_nf(z))}\right]$$
$$=\tfrac{1}{2}[h_1+h_2(z)]=h(z).$$

Since $f\in UST_s(n,\alpha)$, $h_1,h_2\in P_{PAR}(\alpha)$ in E.
That is, $h_i(z)\prec[h_{PAR}(z)]^{\alpha}$, $i=1,2$, $0<\alpha\leq 1$ and $z\in E$.
This implies that $h(z)\prec h^{\alpha}_{PAR}(z)$, $z\in E$, and therefore $\psi\in UST(n,\alpha)$ in E. The proof is complete. □

Theorem 2. *Let, for $z\in E$, $f\in UST_s(n,\alpha)$ and let $\psi(z)=\frac{1}{2}[f(z)-f(-z)]$. Then $\psi\in UST(n+1,\alpha)$ in E. That is*

$$UST(n,\alpha)\subset UST(n+1,\alpha).$$

*Proof.*Let $f\in UST_s(n,\alpha)$. Then $\psi=\frac{1}{2}[f(z)-f(-z)]$ belongs to the class $UST(n,\alpha)$ by Theorem 1.
Set

$$\frac{z(I_n\psi(z))'}{I_n\psi(z)}=h(z),$$

$h(z)$ is analytic in E with $h(0)=1$.
Using identity (6), we obtain

$$\frac{z(I_n\psi(z))'}{I_n\psi(z)}=\left\{h(z)+\frac{zh'(z)}{h(z)+n}\right\}.$$

Since $\psi\in UST(n,\alpha)$, it follows that

$$\left\{h(z)+\frac{zh'(z)}{h(z)+n}\right\}\prec\phi(z)=(h_{PAR}(z))^{\alpha}$$

in E.
Using Lemma 3, we have

$$h(z)\prec(h_{PAR}(z))^{\alpha}$$

in E, and this proves that $\psi\in UST(n+1,\alpha)$ in E. □

Theorem 3. *Let $f\in UST_s(n+1,\alpha)$ and let, with $\psi=\frac{1}{2}(f(z)-f(-z))$,*

$$g(z)=\frac{n+1}{z^n}\int_0^z t^{n-1}\psi(t)\mathrm{d}t. \tag{11}$$

Then $g\in UST(n,\alpha)$ in E.

Proof. From (11), we have

$$(n+1)\Psi(z) = ng(z) + zg'(z) \tag{12}$$

Using (6) and (12), we can write

$$\begin{aligned}(n+1)I_{n+1}\Psi(z) &= nI_{n+1}g(z) + z(I_{n+1}g(z))' \\ &= (n+1)I_n g(z)\end{aligned}$$

Therefore

$$I_{n+1}\Psi(z) = I_n g(z).$$

Since $f \in UST_s(n+1,\alpha).,\Psi \in UST(n+1,\alpha)$ by Theorem 1 and hence $g \in UST(n,\alpha)$ in E. □

Theorem 4. *Let* $f \in UST_s(n+1,1)$ *and let* $\Psi = \frac{1}{2}(f(z) - f(-z))$. *Then* $I_n\Psi$ *belongs to* $S^*(\frac{1}{2})$ *for* $|z| < R$, *where* R *is given by*

$$R_n = \frac{|\mu+1|}{\sqrt{A + (A^2 - |\mu^2 - 1|^2)^{\frac{1}{2}}}}, \tag{13}$$

$$A = 2(s+1)^2 + |\mu|^2 - 1 \quad \mu = 2n+1, s = 2,$$

and this radius is exact.

Proof. Let

$$\frac{z(I_{n+1}\Psi(z))'}{I_{n+1}\Psi(z)} = \frac{1}{2}(H(z)+1), \quad \Re H(z) > 0 \text{ in } E, \tag{14}$$

since $I_{n+1}\Psi \in UST \subset S_1^*$ see [14].

Using (6) and proceeding as in Theorem 2, we have from (14)

$$\frac{z(I_n\Psi(z))'}{I_n\Psi(z)} = \frac{1}{2}H(z) + \frac{1}{2} + \frac{zH'(z)}{H(z)+2n+1}.$$

That is

$$2\left\{\frac{z(I_n\Psi(z))'}{I_n\Psi(z)} - \frac{1}{2}\right\} = H(z) + \frac{2zH'(z)}{H(z)+2n+1}, \Re H(z) > 0$$

Now, using Lemma 2,

$$\Re\left[2\left\{\frac{z(I_n\Psi(z))'}{I_n\Psi(z)} - \frac{1}{2}\right\}\right] = \Re\left[H(z) + \frac{2zH'(z)}{H(z)+(2n+1)}\right] > 0 \quad \text{for } |z| < R,$$

where

$$R_n = \frac{(2n+2)}{\sqrt{A + (A^2 - |\mu^2 - 1|^2)^{\frac{1}{2}}}},$$

$$\mu = 2n+1, A = 2(s+1)^2 + |\mu|^2 - 1, \quad s = 2. \square$$

We note the following special case.

Let $n = 0$ Then $I_0\Psi = z\Psi'$ and $R_0 = \frac{2}{\sqrt{18+18}} \neq \frac{1}{3}$.

That is , if $I_1\Psi = \Psi \in S^*_{\frac{1}{2}}$ in E,

then $I_0\Psi = z\Psi' \in S^*_{\frac{1}{2}}$ for $|z| < \frac{1}{3}$.

Let $L(r,F)$ denote the length of the image of the circle $|z| = r$ under F.

We prove the following.

Theorem 5. *Let* $f \in UST_s(n,\alpha)$.*Then, for* $0 < r < 1$,

$$L(r,f) = O(1).(\frac{1}{1-r}),$$

where $F = I_n f$ *and* $O(1)$ *is a constant.*

Proof. Since $f \in UST_s(n,\alpha)$, we have with $F = I_n f$,

$$\begin{aligned}\frac{2zF'(z)}{F(z)-F(-z)} &= \frac{2zF'(z)}{\Phi(z)} \\ &= h^\alpha(z), \Re h(z) > 0, \quad \Phi \in UST.\end{aligned}$$

Thus, with $z = re^{i\theta}$, we have

$$\begin{aligned}L(r,F) &= \int_0^{2\pi} |zF'(z)|\, d\theta \\ &= \int_0^{2\pi} |\Phi(z)h^\alpha(z)| d\theta \\ &\le \pi\left[\left(\frac{1}{\pi}\int_0^{2\pi} |\Phi(z)|^{\frac{z}{2-\alpha}} d\theta\right)^{\frac{2-\alpha}{2}} \left(\frac{1}{\pi}\int_0^{2\pi} |h(z)|^2 d\theta\right)^{\frac{\alpha}{2}}\right] \\ &\le \pi\left[\left(\frac{1}{\pi}\int_0^{2\pi} \left|\frac{r}{1-re^{i\theta}}\right|^{\frac{2}{2-\alpha}} d\theta\right)^{\frac{2-\alpha}{2}} \left(\frac{1}{\pi}.\frac{1+3r^2}{1-r^2}\right)^{\frac{\alpha}{2}}\right] \\ &\le C\left[\left(\frac{1}{1-r}\right)^{\frac{2}{2-\alpha}-1}\right]^{\frac{2-\alpha}{2}}.\left(\frac{1}{1-r}\right)^{\frac{\alpha}{2}} \\ &= O(1).\left(\frac{1}{1-r}\right)^\alpha,\end{aligned}$$

where $C, O(1)$ are constants and we have applied Holder's inequality, subordination for the odd functions $\Phi \in UST \subset S^*_{\frac{1}{2}}$ and Lemma 6. □

As an application of Theorem 5, we have following coefficient result.

Corollary 1. *Let* $f \in UST_s(n,\alpha)$ *and let, for* $I_n f = F$, $F(z) = z + \sum_{m=2}^{\infty} A_m z^m$.*Then, by Cauchy Theorem,*

$$\begin{aligned}m|A_m| &= \frac{1}{2\pi r^{m+1}}\left|\int_0^{2\pi} zF'(z)e^{-im\theta} d\theta\right|, z = re^{i\theta} \\ &\le \frac{1}{2\pi r^m} L(r,F)\end{aligned}$$

Now, applying Theorem 5, we obtain

$$A_m = O(1).m^{(\alpha-1)} \quad (m \to \infty)$$

We note that, for $n = 1, \alpha = 1, f \in UST_s$ *and* $f(z)$ *given by (1), we have* $a_m = O(1)$, *where O(1) is a constant.*

We now prove that the class $UST_s(n,\alpha)$ is invariant under convolution with convex univalent functions.

Theorem 6. *Let $f \in UST_s(n,\alpha)$ and let $g \in C$. Then $(f*g) \in UST_s(n,\alpha)$.*

Proof. We note that

$$I_n(f*g) = g*I_nf, g \in C.$$

We consider

$$\begin{aligned}&\frac{2I_n[z\{f*g\}]'}{I_n[(f*g)(z)-(f*g)(-z)]}\\ &= \frac{2z(g*I_nf)'}{g*[I_n\{f(z)-f(-z)\}]}\\ &= \frac{g*\frac{z(I_nf)'}{I_n\Psi}.I_n\Psi}{g*I_n\Psi}, \Psi(z)=\frac{f(z)-f(-z)}{2}.\\ &= \frac{g*H.I_n\Psi}{g*I_n\Psi},\end{aligned}$$

where

$$\frac{z(I_n\Psi)'}{I_n\Psi} \prec h^{\alpha}_{PAR} \prec h_{PAR},$$

which implies $\Psi \in UST \subset S^*$ and $H \in P_{PAR}(\alpha)$.

Now, using Lemma 5, we have

$$\left\{\frac{2z(I_n(f*g))'}{I_n[(f*g)(z)-(f*g)(-z)]}\right\}(E) \subset \overline{Co}H(E).$$

This proves that $(f*g) \in UST_s(n,\alpha)$ in E. □

Applications of Theorem 6.

Let $I_nf_i(z) = F_i(z)$, $1 \le i \le 3$, $I_nf(z) = F(z)$, $f \in UST_s(n,\alpha)$, and let

(i) $F_1(z) = \int_0^z \frac{F(t)}{t}dt$

(ii) $F_2(z) = \int_0^z \frac{F(t)-F(xt)}{t-xt}dt, \quad |x| \le 1, x \ne 1$

(iii) $F_3(z) = \frac{1+c}{z^c}\int_0^z t^{c-1}F(t)dt, \Re(c) > 0$

The proof follows immediately since we can write $F_i = F*g_i$, $1 \le i \le 3$, with

$$g_1(z) = \sum_{j=1}^{\infty}\frac{z^j}{j} = -\log(1-z),$$

$$g_2(z) = \sum_{j=1}^{\infty}\frac{1-x^j}{j(1-x)}z^j = \frac{1}{1-x}\log\frac{1-xz}{1-z}, |x| \le 1, x \ne 1$$

$$g_3(z) = \sum_{j=1}^{\infty}\frac{1+c}{j+c}z^j, \Re(c) > 0,$$

and g_i is convex in E for each i, $1 \le i \le 3$, $g_3(z)$ is convex, see [16].

Theorem 7. *Let $G \in UST_s(n,1)$ and let, for $0 < \lambda \le 1$, $g \in A$ be defined by*

$$g(z) = \frac{1}{\lambda}z^{1-\frac{1}{\lambda}}\int_0^z t^{\frac{1}{\lambda}-2}\psi(t)dt, \tag{15}$$

$$2\psi(z) = G(z) - G(-z).$$

Then

$$\Re\left\{\frac{z(I_ng(z))'}{I_ng(z)}\right\} > \gamma,$$

where (16)

$$\gamma = \frac{1}{(1-\lambda)+\sqrt{\lambda^2+1}}. \tag{17}$$

Proof. Since $G \in UST_s(n,1)$, it follows from Theorem 1 that

$$\psi(z) = G(z) - G(-z) \in UST(n,1),$$

and this implies $I_n\psi \in UST \subset S^*_{\frac{1}{2}}$ in E.

Set

$$\frac{z(I_ng(z))'}{I_ng(z)} = (1-r)h(z)+r,$$

$$h(z) = 1 + c_1z + c_2z^2 + \ldots$$

Then, from (15), we have

$$\begin{aligned}&Re\left[\frac{z(I_ng(z))'}{I_ng(z)}\right]\\ &= Re\left[(1-r)h(z)+r+\frac{\lambda(1-r)zh'(z)}{(1-\lambda)+r\lambda+\lambda(1-r)h(z)}\right] > \frac{1}{2}.\end{aligned} \tag{18}$$

That is

$$Re\left[(1-r)h(z)+r+\frac{\lambda(1-r)zh'(z)}{(1-\lambda)+r\lambda+\lambda(1-r)h(z)}-\frac{1}{2}\right] > 0 \tag{19}$$

We know from the functional $\phi(u,v)$ by taking $u = u_1 + iu_2 = h(z), v = v_1 + iv_2 = zh'(z)$. So, from (18), we have

$$\phi(u,v) = (1-r)u + (r-\frac{1}{2}) + \frac{\lambda(1-r)v}{(1-\lambda)+r\lambda+\lambda(1-r)u}.$$

For

$$D = \mathbb{C}\setminus\left\{-\frac{1-\lambda+r\lambda}{\lambda(1-r)}\right\}\times\mathbb{C},$$

the conditions (i) and (ii) of Lemma 1 are clearly satisfied. We proceed to verify condition (iii).

$$\begin{aligned}&\Re\Phi(iu_2,v_1)\\ &= \frac{2\gamma-1}{2}+\Re\left[\frac{\lambda(1-\gamma)v_1}{1-\lambda+\gamma\lambda+i\lambda(1-\gamma)u_2}\right]\\ &= \frac{2\gamma-1}{2}+\frac{\lambda(1-\gamma)(1-\lambda+\gamma\lambda)v_1}{(1-\lambda+\gamma\lambda)^2+\lambda^2(1-\gamma)^2u_2^2}\\ &\le \frac{2\gamma-1}{2}-\frac{1}{2}\left\{\frac{\lambda(1-\gamma)(1-\lambda+\gamma\lambda)(1+u_2^2)}{(1-\lambda+\gamma\lambda)^2+\lambda^2(1-\gamma)^2u_2^2}\right\}\\ &= \frac{A+Bu_2^2}{2C},\end{aligned}$$

where

$$A = (2\gamma-1)(1-\lambda+\gamma\lambda)^2 - \lambda(1-\gamma)(1-\lambda+\gamma\lambda),$$
$$B = (2\gamma-1)\{\lambda^2(1-\gamma)^2\} - \lambda(1-\gamma)(1-\lambda+\gamma\lambda),$$
$$C = \{(1-\lambda+\gamma\lambda)^2 + \lambda^2(1-\gamma)^2u_2^2\} > 0.$$

Now $\Re\phi(iu_2, v_1) \le 0$ if and only if $A \le 0$ and $B \le 0$. From $A \le 0$, we obtain γ as given by (16) and $B \le 0$ ensures $\gamma \in (0,1)$

Thus all the three conditions of Lemma 1 are satisfied and we apply it to have $Reh(z) > 0$ E. This proves that $I_n g \in S_r^*$ in E and r is given by (16). □

4 The class $UK_s(n,\alpha)$

Theorem 8. *Let $f \in UK_s(n,\alpha)$. Then, with*

$$z = re^i\theta, 0 \le \theta_1 < \theta_2 \le 2\pi,\ F = I_n f,$$

we have

$$\int_{\theta_1}^{\theta_2} \Re\left\{\frac{(zF'(z))'}{F'(z)}\right\} d\theta > -\frac{\pi}{2}.$$

*Proof.*Since $f \in UK_s(n,\alpha)$, there exists $g \in UST_s(n,\alpha)$, such that, with $F = I_n f$, $G = I_n g$, we have

$$zF'(z) = \psi(z)h^{\frac{1}{2}}(z),\ h \in P, \tag{20}$$
$$\psi(z) = \frac{G(z)-G(-z)}{2}.$$

Now by definition $\frac{z\psi'(z)}{\psi(z)} \prec (h_{PAR}(z))^\alpha$. This implies that

$$\left|\arg\frac{z\psi'(z)}{\psi(z)}\right| \le \frac{\alpha\pi}{4}$$

Thus we can write

$$\frac{z\psi'(z)}{\psi(z)} = p^{\frac{\alpha}{2}}(z), \qquad p \in P \tag{21}$$

Logarithmic differentiation of (19) and using (20), we have

$$\frac{(zF'(z))'}{F'(z)} = \frac{1}{2}\frac{zh'(z)}{h(z)} + \frac{\alpha}{2}p(z), \qquad h,p \in P \quad in \quad E \tag{22}$$

Now, for $h \in P$, we have

$$\max_{h\in P}\left|\int_{\theta_1}^{\theta_2}\Re\frac{zh'(z)}{h(-z)}d\theta\right| \le \pi - 2\cos^{-1}\left(\frac{2r}{1-r^2}\right). \tag{23}$$

see [10].

Hence, from (19) and (20), with $0 \le \theta_1 < \theta_2 \le \pi$, we have

$$\int_{\theta_1}^{\theta_2}\Re\frac{(zh'(z))'}{h(-z)}d\theta > -\frac{\pi}{2}$$

This completes the proof. □

Theorem 9. $UK_s(n,1) \subset UK_s(n+1,1)$

Proof. Let $f \in UK_s(n,1)$. Then there exists $g \in UST_s((n,1)$ such that

$$\frac{2zf'(z)}{g(z)-g(-z)} = \frac{zf'(z)}{\psi(z)} \in P_{Par}, \psi \in UST(n,1)$$

By Theorem 2, we note that $g \in UST_s(n,1)$ and consequently $\psi \in UST((n+1,1)$. This implies that $I_{n+1}\psi \in UST \subset S^*_{\frac{1}{2}}$.

Set

$$\frac{(z(I_{n+1}f(z))'}{I_{n+1}\psi(z)} = H(z), \quad \psi(z) = \frac{1}{2}[g(z)-g(-z)],$$

Using identity (6), we have

$$\frac{z(I_n f(z))'}{I_n\psi(z)} = \left\{H(z) + \frac{zH'(z)}{h(z)+n}\right\} \in P_{PAR},$$

where

$$h(z) = \frac{z(I_{n+1}\psi(z))'}{I_{n+1}\psi(z)} \in P \quad in \quad E.$$

Therefore, we have

$$\{H(z) + h_0(z)(zH'(z))\} < h_{PAR}(z) \quad in \quad E,$$

where

$$h_0(z) = \frac{1}{h(z)+n} \in P.$$

Now applying Lemma 7, we have

$$H(z) < h_{PAR}(z), z \in E.$$

This proves that $f \in UK_s(n+1,1)$ in E. □

Remark 1. Let

$$L_n(F) = \frac{n+1}{z^n}\int_0^z t^{n-1}F(t)dt.$$

Then

$$\begin{aligned} L_n(F) &= \left(z\sum_{j=0}^{\infty}\tfrac{n+1}{n+j+1}z^j\right) * F(z) \\ &= \left(z\sum_{j=0}^{\infty}\tfrac{(n+1)_j(1)_j}{(n+2)_j j}z^j\right) * F(z) \\ &= [zF_{21}(1,n+1,n+2;z)] * F(z) \\ &= \tfrac{z}{(1-z)^{n+1}} * \left[\tfrac{z}{(1-z)^{n+2}}\right]^{-1} * F(z) \\ &= f_n(z) * f_{n+1}^{-1}(z) * F(z) \end{aligned}$$

This implies that

$$I_n L_n(F) = I_{n+1}F(z).$$

Thus we can easily drive the following.

Theorem 10. *Let $F \in UK_s(n+1,\alpha)$. Then $L_n(F) \in UK_s(n,\alpha)$.*

We also prove:

Theorem 11. *Let $f \in UK_s(n,1)$ with respect to $g \in UST_s(n,1)$.*
Let

$$\psi(z) = \frac{1}{2}[g(z) - g(-z)].$$

Then

$$\Re\left\{\frac{z(I_{n+1}f(z))'}{I_{n+1}\psi(z)}\right\} > 0, \quad \text{for } z \in E.$$

Proof. Let $f \in UK_s(n,1)$. Then there exists $g \in UST_s(n,1)$, with

$$\psi(z) = \frac{1}{2}[g(z) - g(-z)],$$

such that

$$\Re\{\frac{z(I_n f(z))'}{I_n\psi(z)}\} > 0,$$

where $I_n\psi \in UST \subset S^*(\frac{1}{2})$ in E.
Define $w(z)$ in E such that

$$\frac{z(I_{n+1}f(z))'}{I_{n+1}\psi(z)} = \frac{1-w(z)}{1+w(z)}, \tag{24}$$

where $w(0) = 0$ and $w(z) \neq -1$.
We shall show that $|w(z)| < 1$.
From (23), we have

$$z(I_{n+1}f(z))' = I_{n+1}\psi(z).\frac{1-w(z)}{1+w(z)}. \tag{25}$$

So, from (24) and identity (6), we have

$$\begin{aligned}&(n+1)\frac{z(I_n f(z))'}{I_n\psi(z)}\\ &= \frac{z(I_{n+1}f(z))'}{I_n\psi(z)}\left[\frac{1-w(z)}{1+w(z)}\right]\\ &\quad + \frac{(I_{n+1}f(z))'}{I_n\psi(z)}\left\{\frac{-2zw'(z)}{(1+w(z))^2} + n\left[\frac{1-w(z)}{1+w(z)}\right]\right\}.\end{aligned} \tag{26}$$

We now apply identity (6) for the function ψ and since, by Theorem 1. $UST(n,\alpha) \subset UST(n+1,\alpha)$, there exists an analytic function $w_1(z)$ with $w_1(0) = 0$ and $|w_1(z)| < 1$ such that

$$\frac{I_n\psi(z)}{I_{n+1}\psi(z)} = \frac{1-w_1(z)}{1+w_1(z)}. \tag{27}$$

We note here that, from identity (6), that

$$\Re\left\{\frac{z(I_{n+1}\psi(z))'}{I_{n+1}\psi(z)}\right\} > 0$$

and

$$\Re\left\{\frac{I_n\psi(z)}{I_{n+1}\psi(z)}\right\} > \frac{n}{n+1} > 0$$

are equivalent.
Thus, from (25) and (26), we have

$$\begin{aligned}\frac{z(I_n f(z))'}{I_n\psi(z)} &= \frac{1-w(z)}{1+w(z)}\\ &\quad + \frac{1}{n+1}\left(\frac{1+w_1(z)}{1-w_1(z)}\right)\left(\frac{2zw'(z)}{(1+w(z))^2}\right).\end{aligned} \tag{28}$$

Suppose now that, for $z \in E$,

$$\max_{|z|<|z_0|}|w(z)| = |w(z_0)| = 1, (w(z_0) \neq -1).$$

Then it follows, from Lemma 4, that

$$z_0 w'(z_0) = kw(z_0), \quad \text{where } k \geq 1.$$

Setting $w(z_0) = e^{i\theta}$ and $w_1(z_0) = re^{i\phi}$ in (28), we have

$$\begin{aligned}&\Re\left\{\frac{z_0(I_n f(z_0))'}{I_n\psi(z_0)}\right\}\\ &= \Re\left\{\frac{1}{n+1} - \frac{2k(e^{i\theta}+e^{-i\theta}+2)(1+r^2+2r\cos\phi)}{|1+re^{i\phi}|^2|1+e^{i\theta}|^2}\right\}\\ &= \frac{-4k}{n+1}\left\{\frac{(\cos\theta+1)(1+r^2+2r\cos\phi)}{|1+re^{i\phi}|^2|1+e^{i\theta}|^2}\right\}.\end{aligned}$$

Hence, if $\phi = \frac{\pi}{2}$, we have

$$\Re\left\{\frac{z_0(I_n f(z_0))'}{I_n\psi(z_0)}\right\} < 0,$$

where $I_n\psi \in S^*$ and $k \geq 1$.
This contradicts our hypothesis that $f \in UK_s(n,1)$. Thus $|w(z)| < 1$ and so from (23), we obtain the required result.
□

Theorem 12. *Let $f_i \in UK_s(n,\alpha)$ and let, for $\alpha_1, \alpha_2 \geq 0$, $0 \leq \alpha_1 + \alpha_2 = 1$.*

$$f(z) = \int_0^z (f_1'(t))^{\alpha_1}(f_2'(t))^{\alpha_2}\,dt. \tag{29}$$

Then $f \in UK(n,\alpha)$ in E.

Proof. From (28), we have

$$f'(z) = (f_1'(z))^{\alpha_1}(f_2'(z))^{\alpha_2}.$$

Therefore

$$\begin{aligned}&f_n^{(-1)}(z) * zf'(z)\\ &= f_n^{(-1)}(z) * [(f_1'(z)^{\alpha_1})(f_2'(z)^{\alpha_2})]\\ &= (f_n^{-1} * (f_1'(z))^{\alpha_1}).(f_n^{-1}(z) * f_2'(z)), \quad (\alpha_1+\alpha_2 = 1).\end{aligned}$$

This gives us

$$(I_n f(z))' = [I_n f_1(z)]^{\alpha_1}[I_n f_2(z)]^{\alpha_2}.$$

Let $I_n f = F, I_n f_i = F_i$. Then we have

$$F(z) = \int_0^z (F_1'(t))^{\alpha_1}(F_2'(t))^{\alpha_2}\,dt, \tag{30}$$

where, with

$$\frac{G_i(z) - G_i(-z)}{2} = \psi_i(z), G_i = I_n g \in UST_S(\alpha),$$

$zF_i'(z)=\psi_i(z)H_i(z)$, $\frac{z\psi_i'(z)}{\psi_i(z)}\in P_{PAR}(\alpha)$, $H_i\in P_{PAR}(1)$.
From (30), we have

$$\begin{aligned} zF'(z) &= (\psi_1(z)H_1(z))^{\alpha_1}(\psi_2(z)H_2(z))^{\alpha_2} \\ &= (\psi_1(z))^{\alpha_1}(\psi_2(z))^{\alpha_2}(H_1(z))^{\alpha_1}(H_2(z))^{\alpha_2} \\ &= \psi(z).H(z), \end{aligned}$$

where

$$\psi(z)=(\psi_1(z))^{\alpha_1}(\psi_2(z))^{\alpha_2},$$
$$H(z)=(H_1(z))^{\alpha_1}(H_2(z))^{\alpha_2}.$$

Now it is easy to note that

$$\begin{aligned} &\frac{z\psi'(z)}{\psi(z)} \\ &= \alpha_1\frac{z\psi_1'(z)}{\psi_1(z)}+\alpha_2\frac{z\psi_2'(z)}{\psi_2(z)} \\ &= \alpha_1 p_1(z)+\alpha_2 p_2(z)=p(z), \end{aligned}$$

where $p_i\in P_{PAR}(\alpha)$, $\alpha_1+\alpha_2=1$.
Since $P_{PAR}(\alpha)$, $0<\alpha\le 1$ is a convex set, it follows that $p\in P_{PAR}(\alpha)$ in E.
Therefore $\frac{z\psi'(z)}{\psi(z)}\in P_{PAR}(\alpha)$ in E. Also

$$H(z)=(H_1(z))^{\alpha_1}(H_2(z))^{\alpha_2},$$

where $H_i(z)\prec h_{PAR}(z), i=1,2$.
Since $\alpha_1+\alpha_2=1$, we have $H(z)\prec h_{PAR}(z)$.
Therefore $H\in P_{PAR}$ in E.

Hence, from (31), we have

$$\frac{zF'(z)}{\psi(z)}\in P_{PAR}, \psi_i\in UST(\alpha).$$

This proves that $F=I_nf\in UK$ in E. □

Theorem 13. *Let $f\in UK_s(n+1,1)$. Then I_nf is close-to-convex for $|z|<r_n$, where*

$$r_n=\frac{2(n+1)}{3+\sqrt{9+4n(n+1)}}. \tag{31}$$

Proof. Let $f\in UK_s(n+1,1)$. Then there exists $g\in UST_s(n+1,1)$ such that $\left\{\frac{z(I_{n+1}f(z))'}{I_{n+1}\psi(z)}\prec h_{PAR}(z)\right\}$ in E, where

$$\psi(z)=\frac{g(z)-g(-z)}{2}\in UST(n+1,1).$$

We shall first show that $I_n\psi\in S^*_{\frac{1}{2}}$ in $|z|<r_n$, where r_n is given by (31).
Since $I_{n+1}\psi\in UST\subset S^*_{\frac{1}{2}}$, we can write

$$\frac{z(I_{n+1}\psi(z))'}{I_{n+1}\psi(z)}=h(z), \quad \Re h(z)>\frac{1}{2}.$$

Using identity (6), we have

$$\frac{z(I_n\psi(z))'}{I_{n+1}\psi(z)}=h(z)+\frac{zh'(z)}{h(z)+n},$$

Using well-known [1] distortion results for $h\in P$, we obtain

$$\begin{aligned} \Re\left(\frac{z(I_n\psi(z))'}{I_n\psi(z)}\right) &\ge \Re h(z)\left[1-\frac{2r}{1-r^2}\left\{\frac{1}{\frac{1}{1+r}+n}\right\}\right] \\ &= \Re h(z)\left[1-\frac{2r}{(1-r)+n(1-r^2)}\right] \\ &= \Re h(z)\left[\frac{1-r+n-nr^2-2r}{(n+1)-r-nr^2}\right] \\ &= \Re h(z)\left[\frac{(n+1)-3r-nr^2}{(n+1)-r-nr^2}\right]. \end{aligned} \tag{32}$$

The right hand side of (32) is greater than or equal to zero if $|z|=r<r_n$ where r_n is given by (31). Now, again using identity (6) and $I_n\psi\in S^*_{\frac{1}{2}}\subset S^*$ in $|z|<r_n$, we have

$$\Re\left[\frac{z(I_nf(z))'}{I_n\psi(z)}\right]=\Re\left[H(z)+\frac{zH'(z)}{h_0(z)+n}\right],$$

where

$$\Re H(z)=\Re\left[\frac{z(I_{n+1}f(z))'}{I_{n+1}\psi(z)}\right]>0,$$

$$\Re h_0(z)=\Re\left[\frac{z(I_{n+1}\psi_1(z))'}{I_{n+1}\psi(z)}\right]>\frac{1}{2}.$$

Using distortion results for H and h_0, we get

$$\begin{aligned} &\Re\left[\frac{z(I_nf(z))'}{I_n\psi(z)}\right] \\ &\ge \Re H(z)\left[1-\frac{2r}{1-r^2}.\frac{1}{\frac{1}{1+r}+n}\right] \\ &= \Re H(z)\left[\frac{(1+n)-3r-nr^2}{(1-r)+n(1-r^2)}\right], \end{aligned}$$

and this shows that right hand side is greater than or equal to zero for $|z|=r<r_n$ where r_n is given by (31).
Since $I_n\psi\in S^*$ in $|z|<r_n$, it follows that I_nf is close-to-convex in $|z|<r_n$ and this proves our result. □

Remark 2. Following the similar technique of Theorem 6, we can also prove that the class $UK_s(n+1,\alpha)$ is closed under convolution with convex univalent functions, and consequently it is invariant under the integral operators given in the applications of Theorem 6.

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