Applied Mathematics & Information Sciences **2**(3) (2008), 309-316
– An International Journal

Optical Production of the Husimi Function of Two Gaussian Functions

J. R. Moya-Cessa[1], L. R. Berriel-Valdos[2] and H. Moya-Cessa[2]

[1]CIO, Apdo. Postal I-948, León, Gto. 37000, Mexico

Email Address: moyac@cio.mx

[2]INAOE, Coordinación de Optica, Apdo. Postal 51 y 216, 72000 Puebla, Pue., Mexico

Received January 2, 2008; Accepted March 14, 2008

The intensity distribution of the Husimi function (HF) and the squared modulus of the Wigner function (WF) are detected in the phase space of an astigmatic optical processor. These results, obtained in the laboratory, are compared against numerical results generated by using analytical calculation for the HF and WF. The signal function is the superposition of two Gaussian functions with a separation between them, having the same amplitude but a different variance.

Keywords: Quasiprobability distribution functions, signal analysis.

1 Introduction

In 1932 Wigner introduced a distribution function in the context of quantum mechanical correction to thermodynamic equilibrium [1]. It is a bilinear function and its importance is due to the fact that the WF describes a signal in the time and frequency domains, called phase space, simultaneously [2]. The WF can describe signals having two or more variables, or their Fourier transforms. The WF of a signal ψ(q), $W_\psi(q,p)$, has the important property that its marginal probability in each coordinate is given by the integration of $W_\psi(q,p)$ in its conjugate coordinate, i.e., $P(q) = \int W_\psi(q,p)dp$, and $P(p) = \int W_\psi(q,p)dq$. At present, the WF has applications in many fields of physics and engineering: in quantum mechanics it is useful for identifying non-classic states, such as Fock states; in optics, to describe a signal in the space and in the spacial frequencies simultaneously so as in optical information processing [3]. This is because the WF of a deterministic signal (totally coherent light) relates Fourier optics and geometrical optics, and the Wigner function of a stochastic signal (partially coherent light) relates radiometry and partial coherence light theory.

While the WF is one of many distribution functions [3], there are still some functions not introduced in the classical (optics) world, namely the Glauber Sudarshan P-function

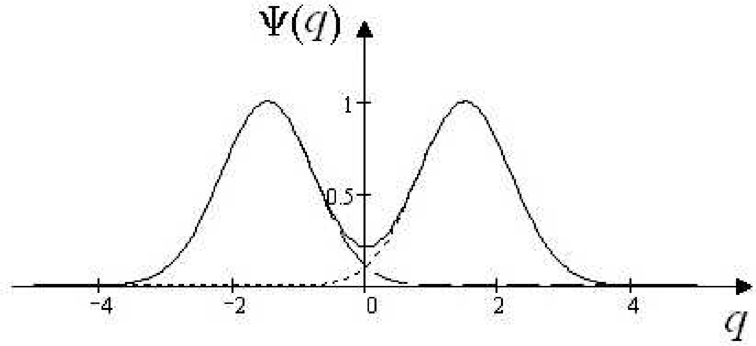

Figure 1.1: The sum of two Gaussian functions having the same amplitude. They having a difference in variance and a separation $q_0 = 1.5$.

[4], [5] and the Husimi Q-function [6, 7]. It is therefore the purpose of this manuscript to present how the Husimi function (HF) may be obtained in optics. The HF has a one-to-one correspondence with the WF and to the optical signal (density matrix in quantum mechanics) [8]. One of the important properties of the HF is that its distribution in the space is a radiometric observable that can be measured directly. In quantum mechanics, the HF is refereed as a classical quasi-probability distribution since it is a real and non-negative quantity. In classical optics, it is directly proportional to the intensity distribution detected in a one-dimensional Fourier optical transformer, i.e., the HF of a signal is the squared modulus of the Fourier transform of the signal function times a weighted Gaussian function.

Here we will obtain the Husimi and (squared) Wigner functions for two Gaussian functions [9] having different variances and positions and will compare them against experimental results. The optical detection of these bilinear functions in the laboratory was made using an astigmatic processor.

2 The Wigner Distribution Function

From its definition, we have that the Wigner function of a signal is a 1D Fourier transform. Then, the optical description is in the focal plane, called Fourier plane, of a cylindrical lens when used in a 1D astigmatic transform system illuminated with a collimated beam. Because it is detected the intensity of light, in the Fourier plane we have the squared absolute values of the Fourier transform.

The Wigner distribution function in the space and spatial-frequency domain can be written as (we use Dirac notation, see [10])

$$W_{\hat{\rho}}(q,p) = \int_{-\infty}^{\infty} \exp(ipq')\langle q - \frac{q'}{2}|\hat{\rho}|q + \frac{q'}{2}\rangle dq', \tag{2.1}$$

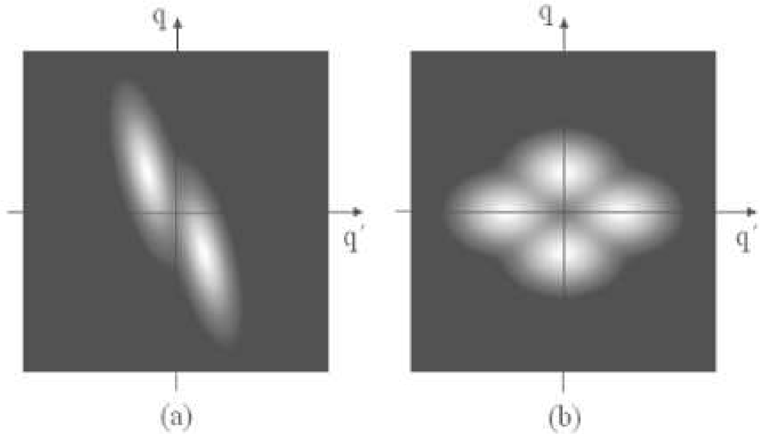

Figure 2.1: Bilinear functions $\Psi(q,q')$ and $\Psi_W(q,q')$. They were calculated numerically.

where q is the spatial variable and p is the spatial-frequency, and $\hat{\rho}$ is the matrix of density. For a spatial signal ψ(q) we have

$$\int_{-\infty}^{\infty} \exp(ipq')\langle q - \frac{q'}{2}|\hat{\rho}|q + \frac{q'}{2}\rangle dq' = \int_{-\infty}^{\infty} \exp(ipq')\psi(q - \frac{q'}{2})\psi^*(q + \frac{q'}{2})dq', \quad (2.2)$$

where the symbol * means conjugation: $\langle q|\psi\rangle = \psi(q)$ and $\langle\psi|q\rangle = \psi^*(q)$ with $\psi(q)$ the signal. The density operator, $\hat{\rho}$, may be given in general by

$$\hat{\rho} = \sum_n \rho_n |\psi_n\rangle\langle\psi_n|, \quad (2.3)$$

i.e., a sum of different states. From Eq. (2.2) we can see that the Wigner function is the Fourier transform of the function $\psi(q - q'/2)\psi^*(q + q'/2)$ having the variables q and p as the canonic variables. In physics they represent position and momentum but in optics they are associated with position and spatial frequency.

The functions associated with the variables q and p are related by

$$\Psi(p) = F[\psi(q)] \qquad \text{and} \qquad \psi(q) = F^{-1}[\Psi(p)], \quad (2.4)$$

where $F[\cdot]$ means the Fourier transform and $F^{-1}[\cdot]$ means the inverse Fourier transform. Given a signal, $\psi(q)$, and its Fourier transform, $\Psi(p)$, we have

$$W_\psi(q,p) = W_\Psi(q,p). \quad (2.5)$$

From Equations (2.2) and (2.4) we have that $W_\psi(q,p)$ is the Fourier transform, in the variable q', of

$$r(q;q') = \psi(q + \frac{q'}{2})\psi^*(q - \frac{q'}{2}) \quad (2.6)$$

or, equivalently, $W_\Psi(q,p)$ is the Fourier transform, in the variable p', of

$$R(p;p') = \Psi(p+\frac{p'}{2})\Psi^*(p-\frac{p'}{2}), \tag{2.7}$$

where $r(q;q')$ and $R(p;p')$ are related through a 2D Fourier transform. So, the WF can only be reconstructed by a holographic process and reconstructed holographically. Through an astigmatic optical processor that can be detected is the squared modulus of $W_\psi(q,p)$ or $W_\Psi(q,p)$.

3 The Husimi Distribution Function

The HF of a given signal is defined as

$$Q_\psi(q,p) = \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle, \tag{3.1}$$

where $|\alpha\rangle$ is a so-called coherent state [7]. The above equation written in normal notation is

$$Q_\psi(q,p) = \frac{1}{\pi}|\int_{-\infty}^{\infty}\psi_\alpha(q,p;q')\psi(q')dq'|^2, \tag{3.2}$$

where $\psi_\alpha(q,p;q')$ is given by [7]

$$\psi_\alpha(q,p;q') = \pi^{-\frac{1}{4}}\exp[-\frac{(q'-q)^2}{2} + ip(q'-\frac{q}{2})], \tag{3.3}$$

which can directly be detected through an astigmatic optical processor since it is the squared modulus of a Fourier transform. The term $\exp(-ipq')$ in Eq.(3.3) being the kernel of the Fourier transform and the function to be transformed is

$$\psi(q,q') = \sqrt[4]{\pi}\psi(q')\exp[-\frac{(q'-q)^2}{2}], \tag{3.4}$$

that is the equivalent expression of $r_\psi(q,q')$ given in Eq. (2.6) for the Wigner function.

3.1 The Wigner function

The signal considered in this work is two Gaussian functions having unitary amplitudes and centered a quantity q_0 and $-q_0$ from the origin

$$\psi(q) = \exp[-b(q-q_0)^2] + \exp[-(q+q_0)^2] \tag{3.5}$$

where b is the inverse of the relative standard variation, i.e., the standard variation of the first Gaussian function divided by that of the second Gaussian function. Fig. 1.1 shows the function $\psi(q)$ for $q_0 = 1.5$ and $b = 1$ that it has a minimum at origin. By substituting Eq. (3.5) in (3.2) we have

$$W_\psi(q,p) = \sqrt{\frac{2\pi}{b}}\exp[-\frac{4b^2(q-q_0)^2+p^2}{2b}] + \sqrt{2\pi}\exp[-\frac{4(q+q_0)^2+p^2}{2}$$

$$+ 4\sqrt{\frac{\pi}{b+1}}\exp[-\frac{4b^2q^2+p^2}{b+1}]\cos[2p(\frac{b-1}{b+1}q+q_0)]. \tag{3.6}$$

The first and the second terms of this equation are the Wigner functions of the each terms of Eq. (12), respectively. The third term is the interference term, which gives the negative values in the Wigner function. The intensity distribution in the phase space is then

$$I_W(q,p) = W_\psi^2(q,p). \tag{3.7}$$

3.2 The Husimi function

For the Husimi function, by substituting Eqs. (3.3) and (3.5) in Eq. (3.2), we have

$$Q_\psi(q,p) = \frac{2}{\sqrt{\pi}}\Big[\frac{1}{2b+1}\exp[-\frac{2b(q-q_0)^2+p^2}{2b+1}] + \frac{1}{3}\exp[-\frac{2(q+q_0)^2+p^2}{3}]$$
$$+ \frac{2}{\sqrt{3(2b+1)}}\exp[-\frac{1}{3}\frac{(b-1)(q-q_0)^2+2(2b+1)(q^2+q_0^2)+(b+2)p^2}{2b+1}]$$
$$\cdot\cos[\frac{2p}{3}\frac{(b-1)q-(5b+1)q_0}{2b+1}]\Big]. \tag{3.8}$$

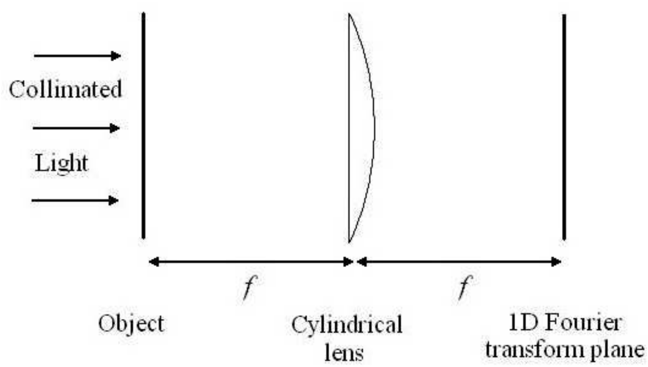

Figure 3.1: The astigmatic processor used in this work. A cylindrical lens is illuminated with a collimated beam.

Figs. 3.2a and 3.2b show the gray level plots of the numerical values obtained for the bilinear signals given in Eqs. (3.4) and (2.6), respectively, for the signal given in Eq. (3.3). These plots were taken like the objects to be used in the astigmatic optical processor.

4 Experimental Process

Fig. 3.1 is an schematic diagram of a 1D Fourier transform processor, the astigmatic optical processor used for detecting the HF and the squared modulus of the WF. A convergent cylindric lens is illuminated by a coherent collimated light, He-Ne laser beam having a

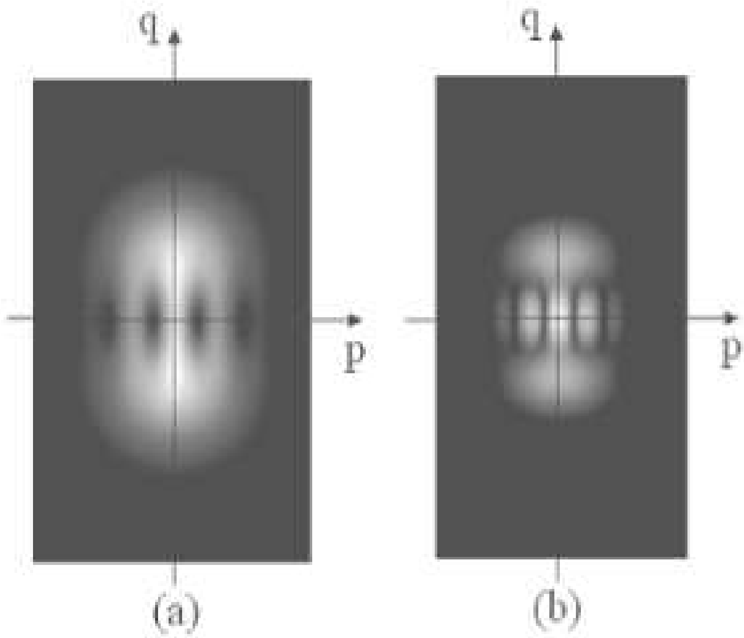

Figure 3.2: Distributions in gray levels of (a) the Husimi function, and (b) squared modulus of the Wigner function.

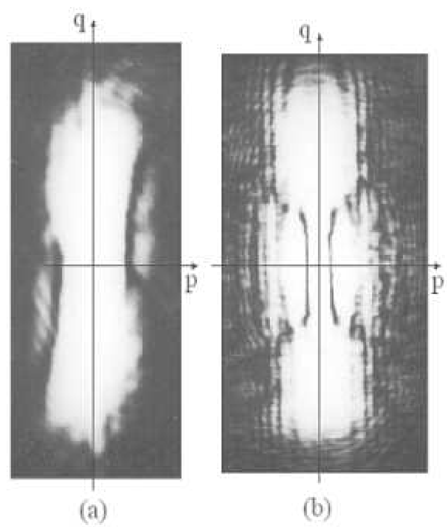

Figure 3.3: Images obtained in laboratory. (a) For Husimi function, and (b) for the squared modulus of the Wigner function. These have to be compared against those given in Fig. 3.2.

wavelength of 632.8 nm. In front of the lens is collocated the object, a transparency of Fig. 2.1a for the Husimi function case and Fig. 2.1b for the Wigner function case. These transparencies were photo-reduced 170 times from its original size. The negatives were made in a Technical Pan photographic film from Kodak and developed using a D-19 dilution. In order to avoid the vignetting effects, the transparencies were put in contact with the cylindrical lens [11]. Because of the photographic process some noise sources are present. The printing process and the non linearity response of the film reduce and change the gray tones of the photoreduction. In the optical reproduction, another sources of errors are present as the spatial coherence of the light source and the aberrations of the lens that produces strong speckle.

Fig. 3.2 shows the results obtained for the Husimi and Wigner functions, Fig. 3.2a and 3.2b respectively. A mechanical misalignment of the photoreduction film produces the effect that can be seen in Fig. 3.3a were the secondary lobes, i.e., the maxima of intensity are moved down and up from their nominal positions. A good alignment produces a good reconstruction, as can be seen in Fig. 3.3b.

5 Conclusions

We have detected in gray levels the Husimi function and the squared modulus of the Wigner function. The function used as a signal to be recorded is the superposition of two Gaussian functions displaced from each other a quantity $2q_0$ and having two different standard variances. They are compared against those obtained using theoretical calculation, which were plotted numerically. In spite of many sources of error are present during the recording, the results given are good enough for recognizing the HF and the WF of the signal. Finally, we have introduce in classical optics a commonly used distribution function in quantum mechanics, namely, the Husimi Q-function.

References

[1] E. P. Wigner, On the quantum correction for thermodynamic equiligrium, *Phys. Rev.* **40** (1932), 749–759.

[2] C. J. Román-Moreno, R. Ortega-Martnez and C. Flores-Arvizo, The Wigner function in paraxial Optics II. Optical diffraction pattern representation, *Rev. Mex. Fs.*, **49** (2003), 290–295.

[3] M. J. Bastiaans, The Wigner distribution function applied to optical signals and systems, *Opt. Comm.* **25** (1978), 26–30.

[4] R. J. Glauber, Coherent and incoherent states of the radiation field, *Phys. Rev.* A **131** (1963), 2766–2788.

[5] E. C. G. Sudarshan, Equivalence of semiclassical and quantum mechanical descriptions of statistical light beams, *Phys. Rev. Lett.* **10** (1963), 277–279.

[6] K. Husimi, Some formal properties of the density matrix, *Proc. Phys. Math. Soc. Japan*, **22** (1940), 264–314.

[7] U. Leonhardt, *Measuring the Quantum State of Light*, Cambridge Univ. Press, 28, 1997.

[8] H. Moya-Cessa, Decoherence in atomfield interactions: a treatment using superoperator techniques, *Phys. Rep.* **432** (2006), 1–41.

[9] V. Buzek, H. Moya-Cessa, S. J. D. Phoenix and P. L. Knight, Schrödinger cat states in the resonant Jaynes-Cummings model: collapse and revival of oscillations of the photon number distribution, *Phys. Rev. A* **45** (1992), 8190–8230.

[10] G. Arfken, *Mathematical Methods for Physicists*, Academic Press, Inc., 1985.

[11] J. W. Goodman, *Introduction to Fourier Optics*, McGraw-Hill, first edition, 88, 1968.