Quantum Physics Letters
An International Journal

http://dx.doi.org/10.18576/qpl/060206

Separable Quantum States are Easier to Synthesise

*Dhananjay P. Mehendale**

Department of Physics, Savitribai Phule University of Pune, Pune, India-411007.

Received: 19 Apr. 2017, Revised: 12 Jun. 2017, Accepted: 20 Jun. 2017
Published online: 1 Aug. 2017

Abstract: An important application of Grover's search algorithm [2] in the domain of experimental physics is its use in the synthesis of any selected superposition state [3]. This paper is about showing how one can speedup this synthesis when selected superposition state to be synthesised factorizes into smaller sized factors under the application of factorization algorithm [1]. When selected superposition state is factorable we first factorize this state and carry out the synthesis of its factors in parallel by applying the algorithm for synthesis [3] simultaneously to each factor. Main steps of our *modified algorithm* are as follows: By making use of the factors we construct the corresponding operators needed for the synthesis of these factors as per [3]. We then build the operator called *synthesiser* by taking tensor product of these operators constructed for the synthesis of the factors. We then build a suitable register A, say, whose all the qubits have been initialized to $|0\rangle$. Note that this register A is prepared by taking tensor product of smaller sized registers of suitable lengths chosen from lengths of the computational basis states required to represent the corresponding factors and the first qubit of all these smaller sized registers is ancilla qubit. We then apply the synthesiser on register A and carry out the measurement of all the ancillae qubits. If the measurement finds all the ancillae qubits in state $|0\rangle$ then we have arrived at the desired selected superposition state. We see that the greater the number of factors to the state, the easier it is to synthesise and the task of synthesising an n-qubit state which is completely factorable into n single qubit factors is exponentially easier than the task of synthesising an n-qubit completely entangled state having no factors.

Keywords: Seprable states, Entangled states, Quantum parallelism

1 Introduction

A well known application of Grover's search algorithm [2] in the domain of experimental physics is to synthesise any desired n-qubit quantum state [3] which can be any arbitrary superposition of computational basis states of length n, i.e. containing n qubits. A systematic algorithm for synthesis of any selected superposition developed in [3] consists of performing the following two main steps:

(i)To construct certain operators (matrices of size $2^{(n+1)} \times 2^{(n+1)}$ when the selected state is superposition on computational basis states of length n) using the coefficients of computational basis states present in the superposition state to be synthesised and to carry out certain compositions of these operators and construct the operator called *synthesiser*.

(ii)To operate the synthesiser on the register of length $(n+1)$ whose all qubits are initialized to $|0\rangle$ and whose first qubit is ancilla qubit and then finally to measure the ancilla quibit .

When the measurement finds the ancilla qubit in state $|0\rangle$ the other qubits together form the desired state to be sythesised.

It should noted that this algorithm [3] for systematically manufacturing the desired quantum state does not take into consideration the nature of the desired quantum state, i.e. it does not take into account whether the desired n-qubit quantum state is factorable and has any factors or it is not at all factorable, etc. The aim of this paper is to show the advantage of such consideration. In this paper we begin by applying the factorization algorithm developed in [1] to the desired quantum state to be synthesised and we see that the modified quantum algorithm will very much simplify the task of synthesis for completely separable states but for those quantum states which are completely entangled and have no factors at all our modified algorithm reduces to the original algorithm in [3].

The following definition describing the action of the *product operator* is important to us and so we state it below explicitely:

* Corresponding author e-mail: dhananjay.p.mehendale@gmail.com

Let A and B be the *operators* from vector spaces V and W respectively into a vector space U, say. Then the action of the *product operator* $A \otimes B$ on *product space* $V \otimes W$ is defined by

$$(A \otimes B)(|v\rangle \otimes |w\rangle) = A|v\rangle \otimes B|w\rangle.$$

where $|v\rangle \varepsilon V$ and $|w\rangle \varepsilon W$.

To generalize, let $A_i, i = 1, 2, \cdots, n$ be the *operators* from vector spaces $V_i, i = 1, 2, \cdots, n$ respectively into vector space U, say. Then the action of the *product operator* $A_1 \otimes A_2 \otimes \cdots \otimes A_n$ on the *product space* $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ is defined by

$$(\prod_{i=1}^{\otimes n} A_i)(\prod_{i=1}^{\otimes n} |v_i\rangle) = (\prod_{i=1}^{\otimes n} A_i|v_i\rangle)$$

where $|v_i\rangle \varepsilon V_i$ for all $i = 1, 2, \cdots, n$.

This definition which appears very natural is at the heart of our *modified algorithm* for synthesising the desired quantum state. This important definition allows us to decompose an operation on an entire quantum system into operations on individual components which not only makes the construction of our quantum algorithm much simpler but also causes the exponential rise in its speed when the $n-$qubit state to be synthesised is separable.

With these preliminaries we now proceed with our *modified algorithm* and show how the existence of factors speeds up the synthesis of the desired quantum state. Note that when the desired superposition does not have any factors then our modified algorithm reduces to (or remains identical with) the existing algorithm given in [3] but when the desired superposition to be synthesized is factorable and if it so happens that by applying procedure of full factorization in [1] it factors into say p, factors then the modified algorithm runs in parallel p processes of synthesising these p factors. The detailed discussion below will make it clear why and how much time saving will be achieved in completing the task of synthesising the desired superposition.

2 Algorithm

The desired quantum state, $|\psi\rangle$, that one wishes to synthesize can be any arbitrary superposition and let it be

$$|\psi\rangle = \sum_{i_1, i_2, \dots i_n} a_{i_1 i_2 \dots i_n} |i_1 i_2 \dots i_n\rangle$$

where all $a_{i_1 i_2 \dots i_n}$ belongs to $\mathbb{C}$, the field of complex numbers, and where each of $i_1, i_2 \dots, i_n$ takes values in $\{0, 1\}$. We further assume without any loss of generality (since we can always normalize a state if and when required) that this n-qubit pure quantum state is normalized, i.e. $\sum_{i_1, i_2, \dots i_n} |a_{i_1 i_2 \dots i_n}|^2 = 1$. This expression for $|\psi\rangle$ as sum over computational basis states can contain in all $N = 2^n$ computational basis states, each of length n, namely, $|00\dots0\rangle$, $|00\dots1\rangle$, $\dots$, $|i_1 i_2 \dots i_n\rangle$, $\dots$, $|11\dots1\rangle$.

Some useful definitions are in order:

Definition 1: The factorization of any superposition state, $|\psi\rangle$, obtained by applying factorization algorithm given in [1] is called *full factorization* of that state.

Definition 2: A quantum state, $|\psi\rangle$, is called *completely entangled* if it does not factorizes into nontrivial factors under its full factorization.

Definition 3: A quantum state, $|\psi\rangle$, is called *completely separable or* if it factorizes into tensor product of n single qubit factors under its full factorization.

Definition 4: The operator which is either a single operator when the quantum state to be synthesised has no factors, or made up of the tensor product of some p operators when the quantum state to be synthesized has some p factors, is called *Synthesiser* if when this operator is operated on the register of suitable length whose all qubits are initialized to zero leads to desired quantum state to be synthesised when all the ancillae qubits are found in state $|0\rangle$ when measured.

Our goal is to show the substantial advantage of full factorization using [1] in speeding up the process of synthesising the desired quantum state when the desired quantum state has p factors. The idea behind achieving this enormous advantage is in making use of quantum parallelism by simultaneously running p processes in parallel of synthesizing the p factors.

We now proceed with the steps of our *modified algorithm*:

(i) Using [1] we carry out full factorization of the quantum state, $|\psi\rangle$, desired to be synthesized. Suppose this full factorization leads us to

$$|\psi\rangle = \prod_{i=1}^{\otimes p} |\phi_{k_i}\rangle$$

where

$$|\phi_{k_i}\rangle = \sum_{j_1, j_2, \dots j_{k_i}} a^i_{j_1 j_2 \dots j_{k_i}} |j_1 j_2 \dots j_{k_i}\rangle$$

Case (i) $p = 1$.

In this case $|\psi\rangle$ is a completely entangled state and has no factors and for this case the modified algorithm remains identical with the existing algorithm in [3].

Case (ii) $p > 1$.

In this case $|\psi\rangle$ is not a completely entangled state and we see that we arrive at its two or more than two factors by carrying out full factorization using [1].

Thus the given quantum state factors into tensor product of p factors, namely, $|\phi_{k_1}\rangle, |\phi_{k_2}\rangle, \dots, |\phi_{k_i}\rangle, \dots, |\phi_{k_p}\rangle$ such that each factor in itself is a normalized quantum state.

Suppose states $|\phi_{k_i}\rangle, i = 1, 2, \dots, p$ are respectively superpositions of computational basis states of lengths k_i such that $\sum_{i=1}^{p} k_i = n$.

Now our job is to make use of these p quantum states, $|\phi_{k_i}\rangle, i = 1, 2, \dots, p$ to prepare p *suitable* operators,

$O_{k_i}, i = 1,2,\ldots,p$ corresponding to these p factors using [3] such that these operators are prepared using the the coefficients of the corresponding computational basis states that compose together these factors, $|\phi_{k_i}\rangle, i = 1,2,\ldots,p$, of the desired quantum state to be manufactured. Note that these operators are needed in the synthesis of the corresponding factors, for example, if the operator O_{k_i} will be operated on state $|0\rangle|00\ldots0\rangle_{k_i}$ where the first qubit $|0\rangle$ corresponds to ancilla qubit and $|00\ldots0\rangle_{k_i}$ corresponds to a ket vector of length k_i and further if the measurement of the first qubit (ancilla) will be carried out after operating with this operator, O_{k_i}, and if the ancilla qubit will be found to be in the state $|0\rangle$ then the remaining qubits will be in the state we wish to synthesize, i.e. $O_{k_i}|0\rangle|00\ldots0\rangle_{k_i}$ produce the factor state $|0\rangle|\phi_{k_i}\rangle$. Thus, we have prepared the operators, $O_{k_i}, i = 1,2,\ldots,p$, which are needed in the synthesis of the corresponding factors, $|\phi_{k_i}\rangle, i = 1,2,\ldots,p$.

(ii) We now build the operator, O, needed to synthesise the originally given quantum state $|\psi\rangle$ as follows:

$$O = \left(\prod_{i=1}^{\otimes p} O_{k_i}\right)$$

Thus, the operator, O, is the tensor product of the operators, $O_{k_i}, i = 1,2,\ldots,p$, which are needed in the synthesis of the corresponding factors, $|\phi_{k_i}\rangle, i = 1,2,\ldots,p$.

(iii) We now build a *suitable* quantum register, $|R\rangle$, as follows:

$$|R\rangle = \left(\prod_{i=1}^{\otimes p} |0\rangle|00\ldots0\rangle_{k_i}\right)$$

where $|0\rangle$ in each $|0\rangle|00\ldots0\rangle_{k_i}, i = 1,2,\ldots,p$, corresponds to ancilla qubit and $|00\ldots0\rangle_{k_i}, i = 1,2,\ldots,p$, corresponds to ket vector of length $k_i, i = 1,2,\ldots,p$ whose all qubits are initialized to $|0\rangle$.

(iv) We operate the operator O on the register $|R\rangle$, i.e. in other words we evaluate $O|R\rangle$. Thus, using the definition of the action of the operator O which is the tensor product operators O_{k_i} we have

$$O|R\rangle = \left(\prod_{i=1}^{\otimes p} O_{k_i}\right)\left(\prod_{i=1}^{\otimes p} |0\rangle|00\ldots0\rangle_{k_i}\right) = \left(\prod_{i=1}^{\otimes p} O_{k_i}|0\rangle|00\ldots0\rangle_{k_i}\right)$$

(v) We then measure together all the ancillae qubits, in $O|R\rangle$. If we will find all these ancillae qubits in state $|0\rangle$ then the remaining qubits together will be in the state we wish to synthesize, i.e. the superposition will be projected into state

$$|\Psi\rangle = \prod_{i=1}^{\otimes p}(|0\rangle|\phi_{k_i}\rangle).$$

where

$$|\phi_{k_i}\rangle = \sum_{j_1,j_2,\ldots j_{k_i}} a^i_{j_1 j_2\ldots j_{k_i}} |j_1 j_2 \ldots j_{k_i}\rangle.$$

It is clear to see that the quantum state $|\Psi\rangle$ is actually the desired quantum state $|\psi\rangle$ in disguise.

3 Remarks

(1)*The case of completely separable quantum state:* When given n-qubit quantum state $|\psi\rangle$ to be synthesized is completely separable into n 1-qubit factors, i.e. when $|\psi\rangle = \prod_{i=1}^{\otimes n}|\phi_i\rangle$, where $|\phi_i\rangle$ are n 1-qubit factors then the synthesiser, O, will be tensor product of n operators, $O_i, i = 1,2,\cdots n$ and each of these operators, O_i, is constructed using corresponding factor among the n factors. These operators have representation in terms of $2^2 \times 2^2$ matrices and they are required to be raised to power $m = \frac{\pi}{4}\sqrt{2^2}$ when the quantum state to be synthesized is normalized. We require to operate this operator O on state $\prod^{\otimes(n)}|0\rangle|0\rangle$ and measure all accillae

(2)*The case of completely entangled quantum state:* When given n-qubit normalized quantum state to be synthesized is completely entangled then no simplification is possible and we need to carry out the algorithm given in [3] as it is. The synthesizer (operator) in this case is directly constucted using the the coefficients of the desired quantum state to be synthesised and has representation in terms of the composition of certain $2^{(n+1)} \times 2^{(n+1)}$ matrices and one of these matrices is required to be raised to power $m = \frac{\pi}{4}\sqrt{2^n}$. To synthesize the desired normalized quantum state we require to operate in this case the synthesiser on state $|0\rangle|00\ldots0\rangle_n$ where first qubit is ancilla. We then measure the ancilla qubit and if it will be in state $|0\rangle$ then it will produce the state $|0\rangle|\psi\rangle$.

(3)*Speeding up the synthesis of desired quantum state:* It is clear that if desired quantum state factors then it becomes easier to synthesize it by carrying out the synthesis of its factors simultaneously with much less effort (in terms of sizes and powers of the matrices involved therein). Suppose a state to be synthesized has following factorization:

$$|\psi\rangle = \prod_{i=1}^{\otimes p}|\phi_{k_i}\rangle.$$

where $k_l = \max\{k_i, i = 1,2,\ldots,p\}$. So, to build the operator, O_{k_l}, the size of the matrices involved and the power to which these matrices are required to be raised will be largest. So, in the synthesis of $|\psi\rangle$ through parallel processing as

$$|\Psi\rangle = \prod_{i=1}^{\otimes p}|0\rangle|\phi_{k_i}\rangle$$

the entire synthesis through parallel processing will complete in the time required for synthesis of $|0\rangle|\phi_{k_l}\rangle$.

4 Examples

Example 1:

Synthesise the following quantum state:

$$|\psi\rangle = \frac{1}{87}[4|0000\rangle - 6|0001\rangle + 10i|0010\rangle - 14i|0011\rangle$$

$$-6|0100\rangle+9|0101\rangle-15i|0110\rangle+21i|0111\rangle$$

$$+10i|1000\rangle-15i|1001\rangle-25|1010\rangle+35|1011\rangle$$

$$-14i|1100\rangle+21i|1101\rangle+35|1110\rangle-49|1111\rangle]$$

Solution:

We follow the steps of Grover's algorithm for synthesis as described in [4].

(1.1) Firstly we directly proceed as per [3], [4] without seeking the factorization of $|\psi\rangle$ to build the operator O, called synthesizer:

(1.2) We introduce an ancilla qubit prepared in the state $|0\rangle$ and thus prepare register containing in all 5 qubits, all initialized to state $|0\rangle$, namely, $|0\rangle|0000\rangle$.

(1.3) We define operator

$$U_1 = I \otimes H \otimes H \otimes H \otimes H.$$

U_1 is a matrix of size $2^5 \times 2^5 = 32 \times 32$.

(1.4) We define operator

$$U_2 : U_2|0\rangle|i_1i_2i_3i_4\rangle$$

$$\rightarrow c_{i_1i_2i_3i_4}|0\rangle|i_1i_2i_3i_4\rangle$$

$$+\sqrt{1-|c_{i_1i_2i_3i_4}|^2}|1\rangle|i_1i_2i_3i_4\rangle$$

plus the remaining orthonormal columns. i_j takes values in the set $\{0,1\}$ and $j\varepsilon\{1,2,3,4\}$. Also, $c_{i_1i_2i_3i_4}$ are the coefficients of the respective states $|i_1i_2i_3i_4\rangle$ in $|\psi\rangle$ given above. U_2 is a matrix of size 32×32.

(1.5) We define operator $I_t = diag(-1,-1,\ldots-1,+1,+1,\ldots+1)$, i.e. a sequence of (-1)s 16 in number followed by a sequence of (+1)s again 16 in number along the diagonal of matrix. I_t is a matrix of size 32×32.

(1.6) We define operator $I_s = diag(-1,+1,+1,\ldots+1)$, i.e. a sequence of only one (-1) followed by a sequence of (+1)s in all 31 in number along the diagonal of matrix. I_s is a matrix of size 32×32.

(1.7) We define operator $U = U_2.U_1$. Note that U will be a matrix of size 32×32.

(1.8) We define operator $Q = -(I_s.U^{-1}.I_t.U)$. Note that Q will be a matrix of size 32×32.

(1.9) We define $O = U.Q^m$, where $m = \frac{\pi}{4}\sqrt{2^4}$ and compute $O|0\rangle|0000\rangle$

(1.10) We measure the ancilla (i.e. the first qubit). If ancilla is found in state $|0\rangle$, the remaining qubits will be in the state to be synthesized.

It is important to note that this procedure involves construction of *large* sized matrices and taking their *large* powers.

Now, we proceed with the factorization using [1] of given state, $|\psi\rangle$, and see that when this state factorizes then into two or more factors then how our modified algorithm helps to simplify and speedup the synthesis.

(2.1) We apply factorization algorithm [1] to the state $|\psi\rangle$ as given above. It can be seen that it factorizes into two identical factors, i.e. we get $|\psi\rangle = |\Theta\rangle \otimes |\Theta\rangle$, where

$$|\Theta\rangle = \frac{1}{\sqrt{87}}[2|00\rangle-3|01\rangle+5i|10\rangle-7i|11\rangle].$$

(2.2) The full factorization of $|\psi\rangle$ produces two factors so we need to build (as per modified algorithm above) two operators using these factors and since these factors happen to be identical in the present case these operators will be identical. Thus, we get the the *product* operator or *synthesiser* to synthesise $|\psi\rangle$ as $O = T \otimes T$ and (as per our modified algorithm) we introduce two ancillae qubits both prepared in the state $|0\rangle$ and thus we prepare register containing in all 6 qubits, namely, $|R\rangle = |0\rangle|00\rangle|0\rangle|00\rangle$. We then compute

$$O|R\rangle = (T \otimes T)(|0\rangle|00\rangle \otimes |0\rangle|00\rangle)$$

$$= T|0\rangle|00\rangle \otimes T|0\rangle|00\rangle$$

and measure both the acillae qubits simultaneously. If and when we found them in state $|0\rangle$ the other qubits will form the desired state to be synthesised.

We now proceed to build the required operator T using the factor state $|\Theta\rangle$.

(2.3) We define operator

$$V_1 = I \otimes H \otimes H.$$

V_1 will be a matrix of size $2^3 \times 2^3 = 8 \times 8$.

(2.4) We define operator

$$V_2 : V_2|0\rangle|i_1i_2\rangle \rightarrow d_{i_1i_2}|0\rangle|i_1i_2\rangle + \sqrt{1-|d_{i_1i_2}|^2}|1\rangle|i_1i_2\rangle$$

plus the remaining orthonormal columns. i_j takes values in the set $\{0,1\}$ and $j\varepsilon\{1,2\}$. Also, $d_{i_1i_2}$ are the coefficients of the respective states $|i_1i_2\rangle$ in $|\Theta\rangle$ given above. V_2 will be a matrix of size 8×8.

(2.5) We define operator $J_t = diag(-1,-1,\ldots-1,+1,+1,\ldots+1)$, i.e. a sequence of (-1)s, 4 in number, followed by a sequence of (+1)s again 4 in number along the diagonal of matrix. J_t will be a matrix of size 8×8.

(2.6) We define operator $J_s = diag(-1,+1,+1,\ldots+1)$, i.e. a sequence of only one (-1) followed by a sequence of (+1)s in all 7 in number along the diagonal of matrix. J_s will be a matrix of size 8×8.

(2.7) We define operator $V = V_2.V_1$. Note that V will be a matrix of size 8×8.

(2.8) We define operator $S = -(J_s.V^{-1}.J_t.V)$. Note that S will be a matrix of size 8×8.

(2.9) We define $T = V.S^m$, where $m = \frac{\pi}{4}\sqrt{2^2}$ and compute $(T|0\rangle|00\rangle) \otimes (T|0\rangle|00\rangle)$

(2.10) We measure both the ancillae (i.e. the first qubit and fourth qubit together). If both the ancillae are found in state $|0\rangle$, the remaining qubits will be in the state to be synthesized.

For the present example the synthesis as per modified algorithm produces the state: $|\Phi\rangle = (|0\rangle|\Theta\rangle \otimes (|0\rangle|\Theta\rangle$, which is the desired state $|\psi\rangle$ in disguise, with significantly less efforts.

Example 2:

Synthesise the following two qubit state

$$|\psi\rangle = \frac{1}{\sqrt{3}}|00\rangle - \frac{1}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{6}}|10\rangle - \frac{1}{\sqrt{6}}|11\rangle.$$

Solution:

Instead of directly applying the steps of Grover's algorithm as given in [4] we first apply the factorization algorithm in [1] to check whether this state factorizes into two linear factors and if yes then we apply our *modified algorithm* to the factored state and synthesise this factored state with much reduced efforts.

One can easily check that the above given state factorizes into two single qubit factors as follows:

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$

where

$$|\psi_1\rangle = \sqrt{\frac{2}{3}}|0\rangle + \sqrt{\frac{1}{3}}|1\rangle$$

and

$$|\psi_2\rangle = \sqrt{\frac{1}{2}}|0\rangle - \sqrt{\frac{1}{2}}|1\rangle.$$

To synthesise $|\psi\rangle$

(i)We create the synthesiser, $O = T_1 \otimes T_2$, where operators T_1, T_2 are synthesisers for $|\psi_1\rangle$, and $|\psi_2\rangle$ respectively.

(ii)We create register $|R\rangle = |0\rangle|0\rangle \otimes |0\rangle|0\rangle$, where first and third qubits are ancillae qubits.

(iii)We evaluate

$$O|R\rangle = (T_1 \otimes T_2)(|0\rangle|0\rangle \otimes |0\rangle|0\rangle) = T_1|0\rangle|0\rangle \otimes T_2|0\rangle|0\rangle.$$

(iv)We measure both the ancilla qubits and when both ancillae qubits are in state $|0\rangle$ the remaining qubits will be comprising the desired state $|\psi\rangle$ to be synthesised.

Firstly, we proceed to build the operator T_1, the synthesiser for $|\psi_1\rangle$ by following the steps of the algorithm for synthesis given in [4].

(3.1)$U_1 = I \otimes H$, therefore

$$U_1 = \begin{bmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 & 0 \\ \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ 0 & 0 & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \end{bmatrix}$$

(3.2)$U_2 : U_2|0\rangle|i\rangle \rightarrow c_i|0\rangle|i\rangle + \sqrt{1-|c_i|^2}|1\rangle|i\rangle$ plus remaining *orthonormal* columns, where c_i are the coefficients of computational basis states in $|\psi_1\rangle$, therefore

$$U_2 = \begin{bmatrix} \sqrt{\frac{2}{3}} & 0 & -\sqrt{\frac{1}{3}} & 0 \\ 0 & \sqrt{\frac{1}{3}} & 0 & -\sqrt{\frac{2}{3}} \\ \sqrt{\frac{1}{3}} & 0 & \sqrt{\frac{2}{3}} & 0 \\ 0 & \sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{1}{3}} \end{bmatrix}$$

(3.3)We define operator $I_t = diag(-1,-1,+1,+1)$, i.e. a sequence of (-1)s, 2 in number, followed by a sequence of (+1)s again 2 in number along the diagonal of the matrix. I_t will be a matrix of size 4×4. Thus,

$$I_t = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(3.4)We define operator $I_s = diag(-1,+1,+1,+1)$, i.e. a sequence of only one (-1) followed by a sequence of (+1)s in all 3 in number along the diagonal of matrix. I_s will be a matrix of size 4×4. Thus,

$$I_s = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(3.5)We define operator $U = U_2.U_1$. Note that U will be a matrix of size 4×4. Thus,

$$U = \begin{bmatrix} 0.5774 & 0.5774 & -0.4082 & -0.4082 \\ 0.4082 & -0.4082 & -0.5774 & 0.5774 \\ 0.4082 & 0.4082 & 0.5774 & 0.5774 \\ 0.5774 & -0.5774 & 0.4082 & -0.4082 \end{bmatrix}$$

(3.6)We define operator $S = -(I_s.U^{-1}.I_t.U)$. Note that S will be a matrix of size 4×4. Thus,

$$S = \begin{bmatrix} 0 & -0.3333 & 0.9428 & 0 \\ 0.3333 & 0 & 0 & -0.9428 \\ -0.9428 & -0.0000 & 0.0000 & -0.3333 \\ -0.0000 & -0.9428 & -0.3333 & 0.0000 \end{bmatrix}$$

(3.7)We define $T_1 = U.S^m$, where $m = \frac{\pi}{4}\sqrt{2^1}$. Thus,

$$T_1 = \begin{bmatrix} 0.5774 & 0.1925 & 0.6804 & -0.4082 \\ 0.4082 & -0.6804 & 0.1925 & 0.5774 \\ -0.4082 & -0.6804 & 0.1925 & -0.5774 \\ -0.5774 & 0.1925 & 0.6804 & 0.4082 \end{bmatrix}$$

(3.8)We proceed exactly on similar lines and build the operator T_2, the synthesiser for $|\psi_2\rangle$ using this time the corresponding coefficients of the computational basis states in $|\psi_2\rangle$. Thus,

$$T_2 = \begin{bmatrix} 0.5000 & 0.5000 & 0.5000 & -0.5000 \\ -0.5000 & 0.5000 & -0.5000 & -0.5000 \\ -0.5000 & -0.5000 & 0.5000 & -0.5000 \\ -0.5000 & 0.5000 & 0.5000 & 0.5000 \end{bmatrix}$$

(3.9)We evaluate
$O|R\rangle = (T_1 \otimes T_2)(|0\rangle|0\rangle \otimes |0\rangle|0\rangle) = (T_1|0\rangle|0\rangle) \otimes (T_2|0\rangle|0\rangle)$.
Thus, we get

$$|\Phi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle,$$

where

$$|\phi_1\rangle = 0.5774|00\rangle + 0.4082|01\rangle - 0.4082|10\rangle - 0.5774|11\rangle,$$

and

$$|\phi_2\rangle = 0.5000|00\rangle - 0.5000|01\rangle - 0.5000|10\rangle - 0.5000|11\rangle.$$

We measure the *first qubit (ancilla qubit)* of $|\phi_1\rangle$. It will be found to be $|0\rangle$ with probability $|0.5774|^2 + |0.4082|^2 = 0.5000$. In this case the state $|\phi_1\rangle$ is projected into state

$$\frac{0.5774}{\sqrt{0.5000}}|00\rangle + \frac{0.4082}{\sqrt{0.5000}}|01\rangle = |0\rangle|\psi_1\rangle$$

which is equal to $|\psi_1\rangle$ ignoring the ancilla qubit. Similarly $|\phi_2\rangle$ leads to $|\psi_2\rangle$.

(3.10)Essentially, we measure both the ancillae (i.e. the first qubit and third qubit together) in $|\Phi\rangle$. When both the ancillae are found in state $|0\rangle$, the remaining qubits will be in the state to be synthesized, i.e. we get

$$|\Psi\rangle = (|0\rangle|\psi_1\rangle) \otimes (|0\rangle|\psi_2\rangle).$$

Clearly, $|\Psi\rangle$ is the desired state $|\psi\rangle$ to be synthesised in disguise.

It is important to note that when the state to be synthesized has factors then the process of synthesis involves construction of *smaller* sized matrices and raising them to *smaller* powers and the *parallel processing* in the *modified algorithm* consisting of simultaneously synthesising the factors in parallel significantly speeds up of the process of synthesis.

Acknowledgement

I thank Dr. M. R. Modak, S. P. College, Pune-411030, India, for carefully reading the mannuscript and making useful comments and suggestions.

References

[1] D. P. Mehendale, P. S. Joag, Quant. Phys. Lett. 6, No. 1, pp 73-77, (2017).
[2] Grover L.K. Proceedings of the 28th Annual ACM Symposium on the Theory of Computing,ACM Press, New York, pp 212-219, (1996).
[3] Grover L.K. Phys. Rev. Lett. Volume 85, Issue 6, pp 1334-1337, (2000).
[4] Colin P. Williams, Explorations in Quantum Computing, second edition, Springer-Verlag London Limited, pp 256-260, (2011).

Dhananjay P. Mehendale, Prof. Dhananjay P. Mehendale served as associate professor in S. P. College, affiliated to Savitribai Phule University of Pune, Pune, India-411007. The subjects of his interest and study are Physics, Mathematics, and Engineering and students of science and engineering have successfully completed their project work under his supervision. He taught various courses in Physics, Electronic Science, and Computer Science departments, and has done research in various areas of these subjects. He worked on various science and engineering projects and one of his projects won award in the national science projects competition organized by Department of Science and Technology. His research interest these days is topics in Quantum Computation and Quantum Information.