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Exact Soliton Solutions for Conformable Fractional Four-Wave Interaction Equations Using Ansatz Method

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Abstract: In this paper, the conformable time-fractional derivative of order $\alpha \in (0,1]$ is considered, instead of the classical time derivative for $\alpha = 1$, in view of the Lax-pair operator that leads to a fractional nonlinear evolution system of four-wave-interaction-equations (4-WIEs). The resulted system is then solved by an ansatz contains tan and secant hyperbolic functions with complex coefficients. A systematic steps are introduced to obtain a general form of exact soliton solutions for the resulted system in (1+1) one spatial and one temporal dimensions. We showed that the obtained solutions could be modified to represent solutions of a similar system but in (2+1) two spatial and one temporal dimensions too. In fact, our suggested ansatz can be used to obtain exact soliton solutions for fractional N-wave-interaction-equations (N-WIEs) in one or more spatial dimensions for N greater than or equal to four. Eventually, some numerical examples are stated with 3D graphs to give a better understanding of the behavior of the soliton waves while the interaction is turned on.

Keywords: Conformable derivative, Lax-pair operator, four wave interaction equation, soliton solution, ansatz method.

1 Introduction

The system of 4-WIEs is a mathematical model for many physical phenomena, including the surface gravity waves [1], the freak waves which are waves with small depth [2], the four wave mixing phenomenon, which occurs when two or more light waves or laser beams interact to produce a new signal wave with specific length and frequency, where the interaction takes place in some physical medium, such as the optical fiber medium [3,4,5], and the study of some processes in nonlinear optics such as: (i) Raman scattering process which is used to produce new frequencies by interacting laser intense light with some material [6], (ii) Raman induced Kerr effect spectroscopy process which is used to stimulate the molecular vibrations [7], and (iii) the process of generating new solitons by interacting four optics waves, which has many applications such as super continuum generation [8]. Many more fields of applications for the 4-WIEs could be found with their references in literature as in [9].

The definition of conformable derivative together with its properties and many proved theorems were introduced in the past decades, the motivation of constructing this definition was to find some physical applications of it [10,11,12,13,14,15]. In fact, the conformable derivative concept was introduced after many concepts of fractional derivative appeared, such as the fractional derivative definitions introduced by Caputo, Riemann-Liouville, Caputo-Fabrizio, Atangana-Baleanu, and Grunwald-Letnikov. More definitions of fractional derivatives together with their properties and useful applications could be found in [16,17,18,19,20,21,22,23,24,25].

The present paper is considered as a new application on the conformable derivative concept. This application is the system of conformable 4-WIEs, in which the order of the time derivative is $\alpha \in (0,1]$. Anyhow, there are many ways used to derive the mathematical model which represents the system of 4-WIEs for $\alpha = 1$, each way starts from some point

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of view. For example, the model is derived in [26], when Zakharov and Shabat tried to generalize the Inverse Scattering Transform (IST) method. The model is also derived in [27], when Sung and Fokas tried to look for the solvability of a partial differential equation represents the time-independent part of the Lax pair. The model is also derived in [28], when Gerdjikov et. al. classified the problems related to a simple Lie algebra and solved by the generalized IST method. The Lax-pair is a set of two operators constructed carefully by satisfying some conditions, and when used, they lead to one or more equations, called nonlinear evolution equations, including the famous KDV equation [29], the sine Gorden equation [30], and the N-wave interaction equations for $N \geq 2$ [28]. The resulted nonlinear evolution equations are then solved by many methods as we will see soon.

The rest of the present paper is as follows: In Section 1, we state the basic definition and some needed properties of the conformable derivative. Also, we state the problem of the present paper, which is the conformable time-fractional derivative of the 4-WIEs in (1+1) and (2+1) dimensions. In Section 2, we state the methodology used to solve the problem of the present paper in addition to the obtained sets of solutions. In Section 3, we state the results of the used method, which show the benefits of the used method in obtaining exact solutions, then we discuss some of these results through examples together with their 3D graphs. Eventually, the conclusions are provided in Section 4.

2 Conformable derivative

In the present paper, we state the definition of conformable derivative of order $\alpha \in (0,1]$ for a given function $f(x)$, together with some needed properties. See [31,32,33,34,35,36] for more discussion and applications of fractional calculus.

Definition 1: [37] If $f(x) : \mathbb{R}^{+} \longrightarrow \mathbb{R}$ is a given function, then the conformable derivative of order $\alpha \in (0,1]$ denoted by $D_x^{\alpha} f(x)$ is given by the following limit:

$$D_x^{\alpha} f(x) = \underset{\varepsilon \longrightarrow 0}{Lim} \frac{f\left(x + \varepsilon\, x^{1-\alpha}\right) - f(x)}{\varepsilon}. \tag{1}$$

The conformable derivative has the following properties:
(I) $D_x^{\alpha}(x^n) = n x^{n-\alpha}, \forall n \in \mathbb{R}$.
(II) $D_x^{\alpha}((f \circ g)(x)) = x^{1-\alpha} g'(x) f'(g(x))$. where $f'(x)$ and $g'(x)$ are the classical first derivative of $f(x)$ and $g(x)$.
(III) If we put $g(x) = x$, then by I and II we get the relation: $D_x^{\alpha}(f(x)) = x^{1-\alpha} f'(x)$.

2.1 Conformable 4-WIEs in 1+1 dimensions

In the present section, we considered a simple generalization of the well known nonlinear evolution system of four partial differential equations whose solution represents the wave packets of four waves moving separately in the same direction with different velocities, where the nonlinear interaction between them takes place at $t = 0$ at some interval of the axis of motion called the interaction zone. The generalization comes from considering the conformable time derivative of order $\alpha \in (0,1]$ instead of $\alpha = 1$. The inputs of the model are [30]:
1- The numbers $\{a_j, b_j : j = 1,2\} \in \mathbb{R}^{+}$, chosen such that $a_1 > a_2, b_1 > b_2$.
2- The functions $\{Q_j(x,t) : j = 1,2,3,4\}$, called the potential functions and represent the complex packets of the interacting waves.
3- The matrices $Q(x,t)$, A, and B, represented as follows:

$$Q(x,t) = \begin{pmatrix} 0 & Q_1(x,t) & Q_2(x,t) & Q_3(x,t) & 0 \\ Q_1^*(x,t) & 0 & Q_4(x,t) & 0 & Q_3(x,t) \\ Q_2^*(x,t) & Q_4^*(x,t) & 0 & Q_4(x,t) & -Q_2(x,t) \\ Q_3^*(x,t) & 0 & Q_4^*(x,t) & 0 & Q_1(x,t) \\ 0 & Q_3^*(x,t) & -Q_2^*(x,t) & Q_1^*(x,t) & 0 \end{pmatrix}, \tag{2}$$

where * is for complex conjugate,

$$A = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_2 & 0 \\ 0 & 0 & 0 & 0 & -a_1 \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 & 0 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b_2 & 0 \\ 0 & 0 & 0 & 0 & -b_1 \end{pmatrix}. \tag{3}$$

4- The Lax-pair operator defined by:

$$\mathbb{I}\,[A, D_t^{\alpha} Q(x,t)] - \mathbb{I}\,[B, \partial_x Q(x,t)] - [\,[A, Q(x,t)], [B, Q(x,t)]\,] = 0, \tag{4}$$

where $\mathbb{I} = \sqrt{-1}$, and $[X,Y] = X\,Y - Y\,X$.

The result is a 5×5 matrix, with the property that it equals the transpose of its complex conjugate, and whose elements are the following equations called conformable 4-WIEs:

$$\begin{aligned}
&\mathbb{I}\,(a_1 - a_2)\, D_t^{\alpha} Q_1(x,t) - \mathbb{I}\,(b_1 - b_2)\,\partial_x Q_1(x,t) - k\, Q_2(x,t)\, Q_4^*(x,t) = 0,\\
&\mathbb{I}\, a_1\, D_t^{\alpha} Q_2(x,t) - \mathbb{I}\, b_1\,\partial_x Q_2(x,t) - k\,(Q_1(x,t)\, Q_4(x,t) - Q_3(x,t)\, Q_4^*(x,t)) = 0,\\
&\mathbb{I}\,(a_1 + a_2)\, D_t^{\alpha} Q_3(x,t) - \mathbb{I}\,(b_1 + b_2)\,\partial_x Q_3(x,t) - k\, Q_2(x,t)\, Q_4(x,t) = 0,\\
&\mathbb{I}\, a_2\, D_t^{\alpha} Q_4(x,t) - \mathbb{I}\, b_2\,\partial_x Q_4(x,t) + k\,(Q_3(x,t)\, Q_2^*(x,t) + Q_2(x,t)\, Q_1^*(x,t)) = 0,
\end{aligned} \tag{5}$$

where $k = a_1 b_2 - a_2 b_1$.

3 Conformable 4-WIEs in (2+1) dimensions

The inputs which lead to the system in (5) can be modified to include the (2+1) dimensions (x,y,t) instead of the (1+1) dimensions (x,t) [30]. After inserting the conformable derivative D_t^{α}, for $\alpha \in (0,1]$ instead of the classical first time derivative, and following a similar derivation used to obtain the system in (5), we get the following conformable 4-WIEs in (2+1) dimensions:

$$\begin{aligned}
&2D_t^{\alpha} Q_1(x,y,t) - (\tfrac{a_1-a_2}{b_1-b_2})\partial_x Q_1(x,y,t) - (\tfrac{a_1-a_2}{c_1-c_2})\partial_y Q_1(x,y,t) + \mathbb{I}\, k_1 Q_2(x,y,t) Q_4^*(x,y,t) := 0,\\
&2D_t^{\alpha} Q_2(x,y,t) - (\tfrac{a_1}{b_1})\partial_x Q_2(x,y,t) - (\tfrac{a_1}{c_1})\partial_y Q_2(x,y,t) - \mathbb{I}\, k_2 (Q_1(x,y,t) Q_4(x,y,t) - Q_3(x,y,t) Q_4^*(x,y,t)) = 0,\\
&2D_t^{\alpha} Q_3(x,y,t) - (\tfrac{a_1+a_2}{b_1+b_2})\partial_x Q_3(x,y,t) - (\tfrac{a_1+a_2}{c_1-c_2})\partial_y Q_3(x,y,t) - \mathbb{I}\, k_3 Q_2(x,y,t) Q_4(x,y,t) = 0,\\
&2D_t^{\alpha} Q_4(x,y,t) - (\tfrac{a_2}{b_2})\partial_x Q_4(x,y,t) - (\tfrac{a_2}{c_2})\partial_y Q_4(x,y,t) + \mathbb{I}\, k_4 (Q_3(x,y,t) Q_2^*(x,y,t) + Q_2(x,y,t) Q_1^*(x,y,t)) = 0,
\end{aligned} \tag{6}$$

where the inputs of the model are $\{a_j, b_j, c_j : j = 1,2\} \subset \mathbb{R}^+$, chosen so that $a_1 > a_2, b_1 > b_2, c_1 > c_2$, and the potential functions $\{Q_r(x,y,t) : r = 1,2,3,4\}$ which are the complex wave packets of the interacting waves, where $\{k_r, r = 1,2,3,4\}$ are given by

$$\begin{aligned}
k_1 &= \tfrac{a_1b_2-a_2b_1}{b_1-b_2} + \tfrac{a_1c_2-a_2c_1}{c_1-c_2}, & k_2 &= \tfrac{a_1b_2-a_2b_1}{b_1} + \tfrac{a_1c_2-a_2c_1}{c_1},\\
k_3 &= \tfrac{a_1b_2-a_2b_1}{b_1+b_2} + \tfrac{a_1c_2-a_2c_1}{c_1+c_2}, & k_4 &= \tfrac{a_1b_2-a_2b_1}{b_2} + \tfrac{a_1c_2-a_2c_1}{c_2}.
\end{aligned} \tag{7}$$

4 Methodology of ansatz method

There are many numerical and exact methods used to get solutions for the models in Equations (5) and (6) at $\alpha = 1$. The numerical methods are usually used to get a better understanding of the behaviour of the interacting waves under some restrictions and circumstances, for example, some of these numerical methods are: The shooting method [38], and the finite element method [39]. Even the numerical methods give a good point of view, unfortunately, many of them didn't give the whole picture of what is going on inside the interaction zone, and didn't answer many important questions concerned with the physical applications. Parallel to the numerical methods, researchers used the exact methods, which lead to one or more soliton solutions, for example, some of these exact methods are the inverse scattering transform method [40], the Darboux dressing method [41], and the Generalized Darboux-Manakov-Zakharov (GDMZ) method [42].

Herein, we will use the ansatz method by suggesting a form of the solutions consisting of tan and secant hyperbolic functions with complex coefficients, then plugging this form in Equations (5) and (6), then trying to find these coefficients. The proposed method proved its efficiency in solving the current nonlinear systems of partial differential equations as well as it did in solving the system of three wave interaction equations [43,44].

The ansatz method is simple and direct, to solve the system in (5), we start by introducing the variable ζ defined by the following equation:

$$\zeta(x,t) = m_1 x - \lambda \frac{t^\alpha}{\alpha}, \tag{8}$$

where m_1, λ are constants yet to be found. Consequently, based on equation (8), we have

$$D_t^\alpha Q_j(x,t) = -\lambda Q_j'(\zeta), \qquad \partial_x Q_j(x,t) = m_1 Q_j'(\zeta), \qquad j = 1,2,3,4. \tag{9}$$

Now, if we substitute equation (9) in Equation (5), then we get the following system

$$\begin{aligned}
&Q_1'(\zeta) - \mathbb{I}\,\delta_1\, Q_2(\zeta) Q_4^*(\zeta) = 0,\\
&Q_2'(\zeta) - \mathbb{I}\,\delta_2\, (Q_1(\zeta)Q_4(\zeta) - Q_3(\zeta)Q_4^*(\zeta)) = 0,\\
&Q_3'(\zeta) - \mathbb{I}\,\delta_3\, Q_2(\zeta)Q_4(\zeta) = 0,\\
&Q_4'(\zeta) + \mathbb{I}\,\delta_4\, (Q_3(\zeta)Q_2^*(\zeta) + Q_2(\zeta)Q_1^*(\zeta)) = 0,
\end{aligned} \tag{10}$$

where δ_i are given by

$$\begin{aligned}
\delta_1 &= \frac{k}{\lambda(a_1-a_2)+m_1(b_1-b_2)}, & \delta_2 &= \frac{k}{\lambda a_1+m_1 b_1},\\
\delta_3 &= \frac{k}{\lambda(a_1+a_2)+m_1(b_1+b_2)}, & \delta_4 &= \frac{k}{\lambda a_2+m_1 b_2}.
\end{aligned} \tag{11}$$

From Equation (11), δ_2 and δ_4 can be written in terms of δ_1 and δ_3 as follows

$$\delta_2 = \frac{2\delta_1\delta_3}{\delta_1+\delta_3}, \qquad \delta_4 = \frac{2\delta_1\delta_3}{\delta_1-\delta_3}, \tag{12}$$

Furthermore, m_1 and λ using equation (11) can be given in terms of δ_1 and δ_3 as follows

$$\lambda = \frac{1}{2}\left(\frac{b_1+b_2}{\delta_1} + \frac{b_2-b_1}{\delta_3}\right), \qquad m_1 = \frac{1}{2}\left(-\frac{a_1+a_2}{\delta_1} + \frac{a_1-a_2}{\delta_3}\right), \tag{13}$$

where δ_1 and δ_3 must satisfy the condition $\delta_1 \neq \delta_3 \neq 0$.

While to solve the system in (6), the variable ζ defined in Equation (8) now becomes as follows:

$$\zeta(x,y,t) = m_1 x + m_2 y - \lambda\,\frac{t^\alpha}{\alpha}, \tag{14}$$

where m_1, m_2, λ are constants yet to be found, and α is the order of the fractional derivative, hence, the derivatives in Equation (6) become:

$$\begin{aligned}
&D_t^\alpha\, Q_j(x,y,t) = -\lambda\, Q_j{}'(\zeta), \quad D_x^\alpha\, Q_j(x,y,t) = m_1\, Q_j{}'(\zeta),\\
&D_y^\alpha\, Q_j(x,y,t) = m_2\, Q_j{}'(\zeta), \qquad j = 1,2,3,4.
\end{aligned} \tag{15}$$

If we substitute Equation (15) into Equation (6), then we obtain the following system:

$$\begin{aligned}
&Q_1{}'(\zeta) - \mathbb{I}\,\Delta_1\, Q_2(\zeta)Q_4^*(\zeta) = 0,\\
&Q_2{}'(\zeta) - \mathbb{I}\,\Delta_2\, (Q_1(\zeta)Q_4(\zeta) - Q_3(\zeta)Q_4^*(\zeta)) = 0,\\
&Q_3{}'(\zeta) - \mathbb{I}\,\Delta_3\, Q_2(\zeta)Q_4(\zeta) = 0,\\
&Q_4{}'(\zeta) + \mathbb{I}\,\Delta_4\, (Q_3(\zeta)Q_2^*(\zeta) + Q_2(\zeta)Q_1^*(\zeta)) = 0,
\end{aligned} \tag{16}$$

where Δ_i are given by

$$\begin{aligned}
\Delta_1 &= \frac{k_1}{(2\lambda + m_1(\frac{a_1-a_2}{b_1-b_2}) + m_2(\frac{a_1-a_2}{c_1-c_2}))}, & \Delta_2 &= \frac{-k_2}{(2\lambda + m_1(\frac{a_1}{b_1}) + m_2(\frac{a_1}{c_1}))},\\
\Delta_3 &= \frac{-k_3}{(2\lambda + m_1(\frac{a_1+a_2}{b_1+b_2}) + m_2(\frac{a_1+a_2}{c_1-c_2}))}, & \Delta_4 &= \frac{-k_4}{(2\lambda + m_1(\frac{a_2}{b_2}) + m_2(\frac{a_2}{c_2}))}.
\end{aligned} \tag{17}$$

The system in (16) is the same as the system in (10), except that δ_i become Δ_i, and with small modification in the definition of ζ. This shows that the two systems in (10) and (16) have the same solutions, but the solutions of the first system will be in (1+1) dimensions, while the solutions of the second system will be in (2+1) dimensions.

The next step is to suggest a formula for the solutions $Q_r(\zeta)$, $r = 1,2,3,4$, suppose $Q_r(\zeta)$ have the form:

$$\begin{aligned} Q_1(\zeta) &= \mathbb{A}_0 + \mathbb{A}_1 \tanh(\zeta) + \mathbb{A}_2 \operatorname{sech}(\zeta), \qquad Q_2(\zeta) = \mathbb{B}_0 + \mathbb{B}_1 \tanh(\zeta) + \mathbb{B}_2 \operatorname{sech}(\zeta), \\ Q_3(\zeta) &= \mathbb{C}_0 + \mathbb{C}_1 \tanh(\zeta) + \mathbb{C}_2 \operatorname{sech}(\zeta), \qquad Q_4(\zeta) = \mathbb{D}_0 + \mathbb{D}_1 \tanh(\zeta) + \mathbb{D}_2 \operatorname{sech}(\zeta), \end{aligned} \tag{18}$$

where $\{\mathbb{A}_j, \mathbb{B}_j, \mathbb{C}_j, \mathbb{D}_j : j = 0,1,2\}$ are unknown complex constants yet to be found. If we substitute Equation (18) into Equation (10), and use the identity $\tanh^n(\zeta) = \tanh^{n-2}(\zeta)(1-\operatorname{sech}^2(\zeta))$, then each single equation of the system in (10) becomes a combination of some coefficients multiplied by the functions 1, $\tanh(\zeta)$, $\operatorname{sech}(\zeta)$, $\operatorname{sech}(\zeta)\tanh(\zeta)$, $\operatorname{sech}^2(\zeta)$. Anyhow, if we put the coefficients of these functions respectively equal to zero, then we get:

The coefficients of the first equation of the system in (10):

$$\begin{aligned} \mathbb{I}\delta_1(\mathbb{B}_0\mathbb{D}_0^* + \mathbb{B}_1\mathbb{D}_1^*) &= 0, \qquad \mathbb{I}\delta_1(\mathbb{B}_1\mathbb{D}_0^* + \mathbb{B}_0\mathbb{D}_1^*) = 0, \\ \mathbb{I}\delta_1(\mathbb{B}_2\mathbb{D}_0^* + \mathbb{B}_0\mathbb{D}_2^*) &= 0, \qquad \mathbb{A}_2 + \mathbb{I}\delta_1(\mathbb{B}_2\mathbb{D}_1^* + \mathbb{B}_1\mathbb{D}_2^*) = 0, \\ \mathbb{A}_1 + \mathbb{I}\delta_1(\mathbb{B}_1\mathbb{D}_1^* - \mathbb{B}_2\mathbb{D}_2^*) &= 0. \end{aligned} \tag{19}$$

The coefficients of the second equation of the system in (10):

$$\begin{aligned} \mathbb{I}\delta_2(\mathbb{A}_0\mathbb{D}_0 + \mathbb{A}_1\mathbb{D}_1 - \mathbb{C}_0\mathbb{D}_0^* - \mathbb{C}_1\mathbb{D}_1^*) &= 0, \qquad \mathbb{I}\delta_2(\mathbb{A}_1\mathbb{D}_0 + \mathbb{A}_0\mathbb{D}_1 - \mathbb{C}_1\mathbb{D}_0^* - \mathbb{C}_0\mathbb{D}_1^*) = 0, \\ \mathbb{I}\delta_2(\mathbb{A}_2\mathbb{D}_0 + \mathbb{A}_0\mathbb{D}_2 - \mathbb{C}_2\mathbb{D}_0^* - \mathbb{C}_0\mathbb{D}_2^*) &= 0, \qquad \mathbb{B}_2 + \mathbb{I}\delta_2(\mathbb{A}_2\mathbb{D}_1 + \mathbb{A}_1\mathbb{D}_2 - \mathbb{C}_2\mathbb{D}_1^* - \mathbb{C}_1\mathbb{D}_2^*) = 0, \\ \mathbb{B}_1 + \mathbb{I}\delta_2(\mathbb{A}_1\mathbb{D}_1 - \mathbb{A}_2\mathbb{D}_2 - \mathbb{C}_1\mathbb{D}_1^* + \mathbb{C}_2\mathbb{D}_2^*) &= 0. \end{aligned} \tag{20}$$

The coefficients of the third equation of the system in (10):

$$\begin{aligned} \mathbb{I}\delta_3(\mathbb{B}_0\mathbb{D}_0 + \mathbb{B}_1\mathbb{D}_1) &= 0, \qquad \mathbb{I}\delta_3(\mathbb{B}_1\mathbb{D}_0 + \mathbb{B}_0\mathbb{D}_1) = 0, \\ \mathbb{I}\delta_3(\mathbb{B}_2\mathbb{D}_0 + \mathbb{B}_0\mathbb{D}_2) &= 0, \qquad \mathbb{C}_2 + \mathbb{I}\delta_3(\mathbb{B}_2\mathbb{D}_1 + \mathbb{B}_1\mathbb{D}_2) = 0, \\ \mathbb{C}_1 + \mathbb{I}\delta_3(\mathbb{B}_1\mathbb{D}_1 - \mathbb{B}_2\mathbb{D}_2) &= 0. \end{aligned} \tag{21}$$

The coefficients of the fourth equation of the system in (10):

$$\begin{aligned} \mathbb{I}\delta_4(\mathbb{B}_0\mathbb{A}_0^* + \mathbb{B}_1\mathbb{A}_1^* + \mathbb{C}_0\mathbb{B}_0^* + \mathbb{C}_1\mathbb{B}_1^*) &= 0, \qquad \mathbb{I}\delta_4(\mathbb{B}_1\mathbb{A}_0^* + \mathbb{B}_0\mathbb{A}_1^* + \mathbb{C}_1\mathbb{B}_0^* + \mathbb{C}_0\mathbb{B}_1^*) = 0, \\ \mathbb{I}\delta_4(\mathbb{B}_2\mathbb{A}_0^* + \mathbb{B}_0\mathbb{A}_2^* + \mathbb{C}_2\mathbb{B}_0^* + \mathbb{C}_0\mathbb{B}_2^*) &= 0, \qquad \mathbb{D}_2 - \mathbb{I}\delta_4(\mathbb{B}_2\mathbb{A}_1^* + \mathbb{B}_1\mathbb{A}_2^* + \mathbb{C}_2\mathbb{B}_1^* + \mathbb{C}_1\mathbb{B}_2^*) = 0, \\ \mathbb{D}_1 - \mathbb{I}\delta_4(\mathbb{B}_1\mathbb{A}_1^* - \mathbb{B}_2\mathbb{A}_2^* + \mathbb{C}_1\mathbb{B}_1^* - \mathbb{C}_2\mathbb{B}_2^*) &= 0. \end{aligned} \tag{22}$$

Herein, Equations (19) to (22) are 20 dependent nonlinear algebraic equations with 14 unknowns, namely $\{\delta_1, \delta_3, \mathbb{A}_j, \mathbb{B}_j, \mathbb{C}_j, \mathbb{D}_j : j = 0,1,2\}$, where δ_2, δ_4, λ, m_1 are related with δ_1 and δ_3 by Equations (12) and (13).

5 Sets of solutions

Our aim is to look for the general solution of Equations (19) - (22), then plug the founded values of the unknowns in Equation (18), and use Equation (8) to get the general form of the system in (10). Before we write the steps of constructing a solution for Equations (19) - (22), we need to prove the following claim, which means that the formula of the obtained solutions are valid as we will see.

Claim: If we choose
(1)$\{a_1, a_2, b_1, b_2\} \in \mathbb{R}^+$ such that $a_1 > a_2$, $b_1 > b_2$, and $k = b_2 a_1 - a_2 b_1 > 0$,
(2)$\{\lambda, m_1\} \in \mathbb{R}$ such that $\lambda < 0$ and $m_1 > \frac{-\lambda a_1}{b_1}$,

then the δ_j defined in Equation (11) satisfy the following conditions:
(i) $\{\delta_j : j=1,2,3,4\} \subset \mathbb{R}^+$,
(ii) $\delta_1-\delta_3>0$.

Proof: (i) Since $k=b_2a_1-a_2b_1>0$, we have $\frac{a_2}{b_2}<\frac{a_1}{b_1}$, so $\frac{-\lambda a_2}{b_2}<\frac{-\lambda a_1}{b_1}<m_1$, which means $\delta_2=\frac{k}{\lambda a_1+m_1b_1}>0$, $\delta_4=\frac{k}{\lambda a_2+m_1b_2}>0$, $\delta_3=\frac{k}{m_1(b_1+b_2)+\lambda(a_1+a_2)}>0$.
(ii) by (i) $(\lambda a_2+m_1b_2)>0$, and $(\lambda a_1+m_1b_1)>0$, so $(\lambda a_1+m_1\,b_1)+(\lambda a_2+m_1b_2)>(\lambda a_1+m_1b_1)-(\lambda a_2+m_1b_2)$. Thus $\frac{k}{m_1(b_1+b_2)+\lambda(a_1+a_2)}<\frac{k}{m_1(b_1-b_2)+\lambda(a_1-a_2)}$, and then $0<\delta_3<\delta_1$.

Solution set (1): To get a solution for the algebraic system in Equations (19)-(22), we do the following steps:
(i) choose $\{a_1,a_2,b_1,b_2,k,\lambda,m_1\}$ as in the above claim.
(ii) Put

$$\begin{aligned}
&\zeta(x,t)=m_1x-\lambda\frac{t^\alpha}{\alpha},\ \alpha\in(0,1],\\
&Q_1(\zeta)=(a_{10}+\mathbb{I}\,a_{20})+\frac{\delta_1}{(\delta_1-\delta_3)\sqrt{\delta_2\,\delta_4}}\,(a_{11}+\mathbb{I}\,a_{21})\,\tanh(\zeta),\\
&Q_2(\zeta)=\frac{1}{\sqrt{(\delta_1-\delta_3)\,\delta_4}}\,(b_{12}+\mathbb{I}\,b_{22})\,\text{sech}(\zeta),\\
&Q_3(\zeta)=(c_{10}+\mathbb{I}\,c_{20})+\frac{\delta_3}{(\delta_1-\delta_3)\sqrt{\delta_2\,\delta_4}}\,(c_{11}+\mathbb{I}\,c_{21})\,\tanh(\zeta),\\
&Q_4(\zeta)=\frac{1}{\sqrt{(\delta_1-\delta_3)\,\delta_2}}\,(d_{12}+\mathbb{I}\,d_{22})\,\text{sech}(\zeta).
\end{aligned}\tag{23}$$

(iii) Consider $a_{10}\in\mathbb{R}$, $b_{12},d_{12}\in[-1,1]$ such that $b_{12}\neq\pm d_{12}\neq 0$, and $\{a_{20},a_{11},a_{21},b_{22},c_{10},c_{20},c_{11},c_{21},d_{22}\}$ equal one of the following sets in Equations (24)-(27):

$$\begin{aligned}
&a_{20}=-\frac{a_{10}(b_{12}\sqrt{1-b_{12}^2}+d_{12}\sqrt{1-d_{12}^2})}{b_{12}^2-d_{12}^2}, \qquad a_{11}=\sqrt{1-b_{12}^2}d_{12}-b_{12}\sqrt{1-d_{12}^2},\\
&a_{21}=b_{12}d_{12}+\sqrt{1-b_{12}^2}\sqrt{1-d_{12}^2}, \qquad b_{22}=-\sqrt{1-b_{12}^2}, \qquad c_{10}=-\frac{a_{10}(\sqrt{1-b_{12}^2}d_{12}+b_{12}\sqrt{1-d_{12}^2})}{-\sqrt{1-b_{12}^2}d_{12}+b_{12}\sqrt{1-d_{12}^2}},\\
&c_{20}=\frac{a_{10}(-b_{12}d_{12}+\sqrt{1-b_{12}^2}\sqrt{1-d_{12}^2})}{-\sqrt{1-b_{12}^2}d_{12}+b_{12}\sqrt{1-d_{12}^2}}, \qquad c_{11}=\sqrt{1-b_{12}^2}d_{12}+b_{12}\sqrt{1-d_{12}^2},\\
&c_{21}=b_{12}d_{12}-\sqrt{1-b_{12}^2}\sqrt{1-d_{12}^2}, \qquad d_{22}=-\sqrt{1-d_{12}^2}
\end{aligned}\tag{24}$$

$$\begin{aligned}
&a_{20}=\frac{a_{10}(b_{12}\sqrt{1-b_{12}^2}-d_{12}\sqrt{1-d_{12}^2})}{b_{12}^2-d_{12}^2}, \qquad a_{11}=-\sqrt{1-b_{12}^2}d_{12}-b_{12}\sqrt{1-d_{12}^2},\\
&a_{21}=b_{12}d_{12}-\sqrt{1-b_{12}^2}\sqrt{1-d_{12}^2}, \qquad b_{22}=\sqrt{1-b_{12}^2}, \qquad c_{10}=\frac{a_{10}(\sqrt{1-b_{12}^2}d_{12}-b_{12}\sqrt{1-d_{12}^2})}{\sqrt{1-b_{12}^2}d_{12}+b_{12}\sqrt{1-d_{12}^2}},\\
&c_{20}=-\frac{a_{10}(b_{12}d_{12}+\sqrt{1-b_{12}^2}\sqrt{1-d_{12}^2})}{\sqrt{1-b_{12}^2}d_{12}+b_{12}\sqrt{1-d_{12}^2}}, \qquad c_{11}=-\sqrt{1-b_{12}^2}d_{12}+b_{12}\sqrt{1-d_{12}^2},\\
&c_{21}=b_{12}d_{12}+\sqrt{1-b_{12}^2}\sqrt{1-d_{12}^2}, \qquad d_{22}=-\sqrt{1-d_{12}^2},
\end{aligned}\tag{25}$$

$$
\begin{aligned}
&a_{20}=\frac{a_{10}(-b_{12}\sqrt{1-b_{12}^2}+d_{12}\sqrt{1-d_{12}^2})}{b_{12}^2-d_{12}^2}, \qquad a_{11}=\sqrt{1-b_{12}^2}d_{12}+b_{12}\sqrt{1-d_{12}^2},\\
&a_{21}=b_{12}d_{12}-\sqrt{1-b_{12}^2}\sqrt{1-d_{12}^2}, \qquad b_{22}=-\sqrt{1-b_{12}^2}, \qquad c_{10}=\frac{a_{10}(\sqrt{1-b_{12}^2}d_{12}-b_{12}\sqrt{1-d_{12}^2})}{\sqrt{1-b_{12}^2}d_{12}+b_{12}\sqrt{1-d_{12}^2}},\\
&c_{20}=\frac{a_{10}(b_{12}d_{12}+\sqrt{1-b_{12}^2}\sqrt{1-d_{12}^2})}{\sqrt{1-b_{12}^2}d_{12}+b_{12}\sqrt{1-d_{12}^2}}, \qquad c_{11}=\sqrt{1-b_{12}^2}d_{12}-b_{12}\sqrt{1-d_{12}^2},\\
&c_{21}=b_{12}d_{12}+\sqrt{1-b_{12}^2}\sqrt{1-d_{12}^2}, \qquad d_{22}=\sqrt{1-d_{12}^2}
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
&a_{20}=\frac{a_{10}(b_{12}\sqrt{1-b_{12}^2}+d_{12}\sqrt{1-d_{12}^2})}{b_{12}^2-d_{12}^2}, \qquad a_{11}=-\sqrt{1-b_{12}^2}d_{12}+b_{12}\sqrt{1-d_{12}^2},\\
&a_{21}=b_{12}d_{12}+\sqrt{1-b_{12}^2}\sqrt{1-d_{12}^2}, \qquad b_{22}=\sqrt{1-b_{12}^2}, \qquad c_{10}=-\frac{a_{10}(\sqrt{1-b_{12}^2}d_{12}+b_{12}\sqrt{1-d_{12}^2})}{-\sqrt{1-b_{12}^2}d_{12}+b_{12}\sqrt{1-d_{12}^2}},\\
&c_{20}=\frac{a_{10}(b_{12}d_{12}-\sqrt{1-b_{12}^2}\sqrt{1-d_{12}^2})}{-\sqrt{1-b_{12}^2}d_{12}+b_{12}\sqrt{1-d_{12}^2}}, \qquad c_{11}=-\sqrt{1-b_{12}^2}d_{12}-b_{12}\sqrt{1-d_{12}^2},\\
&c_{21}=b_{12}d_{12}-\sqrt{1-b_{12}^2}\sqrt{1-d_{12}^2}, \qquad d_{22}=\sqrt{1-d_{12}^2}.
\end{aligned}
\tag{27}
$$

Solution set (2): To get another solution for the algebraic system in Equations (19)-(22), we do the following steps:
(i) Do steps (i) and (ii) of solution set (1).
(ii) Consider $a_{10}\in\mathbb{R}, b_{12}\in[-1,1]$ such that $b_{12}\neq 0$, and
$\{a_{20},a_{11},a_{21},b_{22},c_{10},c_{20},c_{11},c_{21},d_{12},d_{22}\}$ equal one of the following sets in Equations (28)-(31):

$$
\begin{aligned}
&a_{20}=-\frac{a_{10}\sqrt{1-b_{12}^2}}{b_{12}}, \qquad a_{11}=-b_{12}, \qquad a_{21}=\sqrt{1-b_{12}^2}, \qquad b_{22}=-\sqrt{1-b_{12}^2},\\
&c_{10}=-a_{10}, \qquad c_{20}=\frac{a_{10}\sqrt{1-b_{12}^2}}{b_{12}}, \qquad c_{11}=b_{12}, \qquad c_{21}=-\sqrt{1-b_{12}^2}, \qquad d_{12}=0, d_{22}=-1,
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
&a_{20}=\frac{a_{10}\sqrt{1-b_{12}^2}}{b_{12}}, \qquad a_{11}=-b_{12}, \qquad a_{21}=-\sqrt{1-b_{12}^2},\\
&b_{22}=\sqrt{1-b_{12}^2}, \qquad c_{10}=-a_{10}, \qquad c_{20}=-\frac{a_{10}\sqrt{1-b_{12}^2}}{b_{12}},\\
&c_{11}=b_{12}, \qquad c_{21}=\sqrt{1-b_{12}^2}, \qquad d_{12}=0, \qquad d_{22}=-1,
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
&a_{20}=-\frac{a_{10}\sqrt{1-b_{12}^2}}{b_{12}}, \qquad a_{11}=b_{12}, \qquad a_{21}=-\sqrt{1-b_{12}^2},\\
&b_{22}=-\sqrt{1-b_{12}^2}, \qquad c_{10}=-a_{10}, \qquad c_{20}=\frac{a_{10}\sqrt{1-b_{12}^2}}{b_{12}},\\
&c_{11}=-b_{12}, \qquad c_{21}=\sqrt{1-b_{12}^2}, \qquad d_{12}=0, \qquad d_{22}=1,
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
&a_{20}=\tfrac{a_{10}\sqrt{1-b_{12}^2}}{b_{12}}, \quad a_{11}=b_{12}, \quad a_{21}=\sqrt{1-b_{12}^2},\\
&b_{22}=\sqrt{1-b_{12}^2}, \quad c_{10}=-a_{10}, \quad c_{20}=-\tfrac{a_{10}\sqrt{1-b_{12}^2}}{b_{12}},\\
&c_{11}=-b_{12}, \quad c_{21}=-\sqrt{1-b_{12}^2}, \quad d_{12}=0, \quad d_{22}=1.
\end{aligned}
\tag{31}
$$

Solution set (3) : To get another solution for the algebraic system in Equations (19)-(22), we do the following steps:
(i) Do steps (i) and (ii) of solution set (1).
(ii) Consider $a_{20}\in\mathbb{R}, d_{12}\in[-1,1]$, and $\{a_{10},a_{11},a_{21},b_{12},b_{22},c_{10},c_{20},c_{11},c_{21},d_{22}\}$ equal one of the following sets in Equations (32)-(35):

$$
\begin{aligned}
&a_{10}=0, \quad a_{11}=0, \quad a_{21}=-1, \quad b_{12}=-d_{12},\\
&b_{22}=\sqrt{1-d_{12}^2}, \quad c_{10}=2a_{20}d_{12}\sqrt{1-d_{12}^2}, \quad c_{20}=a_{20}(-1+2d_{12}^2),\\
&c_{11}=-2d_{12}\sqrt{1-d_{12}^2}, \quad c_{21}=1-2d_{12}^2, \quad d_{22}=-\sqrt{1-d_{12}^2},
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
&a_{10}=0, \quad a_{11}=0, \quad a_{21}=1, \quad b_{12}=d_{12}, \quad b_{22}=-\sqrt{1-d_{12}^2},\\
&c_{10}=2a_{20}d_{12}\sqrt{1-d_{12}^2}, \quad c_{20}=a_{20}(-1+2d_{12}^2), \quad c_{11}=2d_{12}\sqrt{1-d_{12}^2},\\
&c_{21}=-1+2d_{12}^2, \quad d_{22}=-\sqrt{1-d_{12}^2},
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
&a_{10}=0, \quad a_{11}=0, \quad a_{21}=-1, \quad b_{12}=-d_{12}, \quad b_{22}=-\sqrt{1-d_{12}^2},\\
&c_{10}=-2a_{20}d_{12}\sqrt{1-d_{12}^2}, \quad c_{20}=a_{20}(-1+2d_{12}^2), \quad c_{11}=2d_{12}\sqrt{1-d_{12}^2},\\
&c_{21}=1-2d_{12}^2, \quad d_{22}=\sqrt{1-d_{12}^2},
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
&a_{10}=0, \quad a_{11}=0, \quad a_{21}=1, \quad b_{12}=d_{12}, \quad b_{22}=\sqrt{1-d_{12}^2},\\
&c_{10}=-2a_{20}d_{12}\sqrt{1-d_{12}^2}, \quad c_{20}=a_{20}(-1+2d_{12}^2), \quad c_{11}=-2d_{12}\sqrt{1-d_{12}^2},\\
&c_{21}=-1+2d_{12}^2, \quad d_{22}=\sqrt{1-d_{12}^2}.
\end{aligned}
\tag{35}
$$

Solution set (4): To get another solution for the algebraic system in Equations (19)-(22), we do the following steps:
(i) Do steps (i) and (ii) of solution set (1).
(ii) Consider $a_{20}\in\mathbb{R}$, and $\{a_{10},a_{11},a_{21},b_{12},b_{22},c_{10},c_{20},c_{11},c_{21},d_{12},d_{22}\}$ equal one of the following sets in Equations (36)-(39):

$$
\begin{aligned}
&a_{10}=0, \quad a_{11}=0, \quad a_{21}=1, \quad b_{12}=0, \quad b_{22}=-1, \quad c_{10}=0,\\
&c_{20}=-a_{20}, \quad c_{11}=0, \quad c_{21}=-1, \quad d_{12}=0, \quad d_{22}=-1,
\end{aligned}
\tag{36}
$$

$$
\begin{aligned}
&a_{10}=0, \quad a_{11}=0, \quad a_{21}=-1, \quad b_{12}=0, \quad b_{22}=1, \quad c_{10}=0,\\
&c_{20}=-a_{20}, \quad c_{11}=0, \quad c_{21}=1, \quad d_{12}=0, \quad d_{22}=-1,
\end{aligned}
\tag{37}
$$

$$a_{10}=0, \quad a_{11}=0, \quad a_{21}=-1, \quad b_{12}=0, \quad b_{22}=-1, \quad c_{10}=0,$$
$$c_{20}=-a_{20}, \quad c_{11}=0, \quad c_{21}=1, \quad d_{12}=0, \quad d_{22}=1, \tag{38}$$

$$a_{10}=0, \quad a_{11}=0, \quad a_{21}=1, \quad b_{12}=0, \quad b_{22}=1, \quad c_{10}=0,$$
$$c_{20}=-a_{20}, \quad c_{11}=0, \quad c_{21}=-1, \quad d_{12}=0, \quad d_{22}=1. \tag{39}$$

6 Results and discussion

The results obtained by applying the used method on the problem of the present paper, are explained and discussed through the following examples, this shows the robustness of the used method in obtaining exact solutions. Moreover, since the exact solutions are now can be obtained, one can study any physical behavior of the obtained solution, by simply controlling the inputs of his interested problem.

Example 1: Following the steps of solution set (1), if we randomly choose

$$\begin{aligned}
&\{a_1,a_2,b_1,b_2,k,\lambda,m_1,\alpha\}=\{28.7653,9.61206,25.755,10.5996,57.3419,-2,3.23376,0.5\}, \quad \text{and}\\
&\{a_{10},b_{12},d_{12}\}=\{3.6207,0.556191,0.206078\}, \quad \text{then}\\
&\{\delta_1,\delta_2,\delta_3,\delta_4\}=\{5.35778,2.22644,1.40518,3.80947\}, \quad \text{and}\\
&\{a_{20},a_{11},a_{21},b_{22},c_{10},c_{20},c_{11},c_{21},d_{22}\}=\\
&\{-9.00671,-0.372991,0.927835,-0.831055,-6.94566,6.78145,0.715514,-0.698598,-0.978536\}
\end{aligned} \tag{40}$$

then after substituting these values in Equations (8), (23), and (24), the following solution of Equation (5) can be obtained:

$$\begin{aligned}
Q_1(x,t)&=(3.6207-9.00671\,\mathbb{I})-(0.173605-0.431853\,\mathbb{I})\,\tanh(4\,t^{0.5}+3.23376\,x),\\
Q_2(x,t)&=(0.143334-0.214169\,\mathbb{I})\,\text{sech}(4\,t^{0.5}+3.23376\,x),\\
Q_3(x,t)&=(-6.94566+6.78145\,\mathbb{I})+(0.0873435-0.0852785\,\mathbb{I})\,\tanh(4\,t^{0.5}+3.23376\,x),\\
Q_4(x,t)&=(0.069468-0.32986\,\mathbb{I})\,\text{sech}(4\,t^{0.5}+3.23376\,x).
\end{aligned} \tag{41}$$

Figure 1 shows the interaction stages. Since the coefficient of t is positive, the direction of motion is toward the negative t-axis. So, if we wish to reverse the direction of motion then we just multiply the coefficient of t by minus sign. Before the interaction started, the waves moving successively and separately for about 20 time units, when they catch each others, the interaction started and they start mixing, at these moments they united to form one intense signal wave with some length, frequency and phase shift, which is the interesting physical phenomenon of solitons, for example the carried energies could be transferred among them. The interaction lasts for about 15 time units, then they start separating and keep moving till they disappear again because the limits of the above functions go to $0,\pm1$ as x goes to $\pm\infty$.

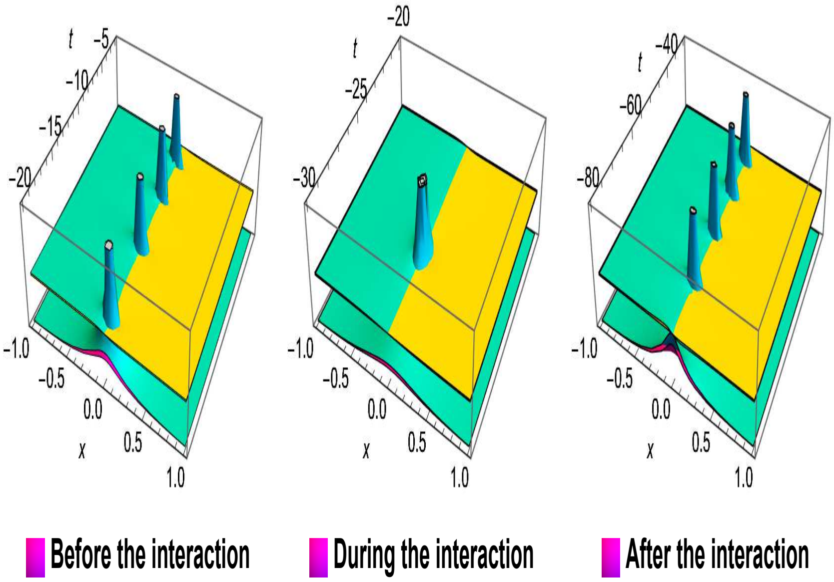

Fig. 1: Plots of interaction of $\{|Q_1(x,t)|, 10|Q_2(x,t)|, |Q_3(x,t)|, 10|Q_4(x,t)|\}$ of Example 1

Example 2: Following the steps of solution set (2), if we randomly choose

$$\begin{aligned}
&\{a_1,a_2,b_1,b_2,k,\lambda,m_1,\alpha\} = \{21.9075, 7.12165, 24.9157, 11.9223, 83.747, -1.76599, 3.55278, 0.5\} \quad \text{and}\\
&\{a_{10},b_{12}\} = \{9.17805, 0.537563\} \quad \text{then}\\
&\{\delta_1,\delta_2,\delta_3,\delta_4\} = \{4.1767, 1.68061, 1.05194, 2.81215\} \quad \text{and}\\
&\{a_{20},a_{11},a_{21},b_{22},c_{10},c_{20},c_{11},c_{21},d_{12},d_{22}\} =\\
&\{-14.3968, -0.537563, 0.843224, -0.843224, -9.17805, 14.3968, 0.537563, -0.843224, 0, -1\},
\end{aligned} \tag{42}$$

then after substituting these values in equations (8), (23), and (28), the following solution of Equation (5) can be obtained:

$$\begin{aligned}
Q_1(x,t) &= (9.17805 - 14.3968\,\mathbb{I}) - (0.330517 - 0.518451\,\mathbb{I})\ \tanh(\ 3.53199\,t^{0.5} + 3.55278\,x),\\
Q_2(x,t) &= (0.181343 - 0.284456\,\mathbb{I})\ \text{sech}(\ 3.53199\,t^{0.5} + 3.55278\,x),\\
Q_3(x,t) &= (-9.17805 + 14.3968\,\mathbb{I}) + (0.0832437 - 0.130577\,\mathbb{I})\ \tanh(\ 3.53199\,t^{0.5} + 3.55278\,x),\\
Q_4(x,t) &= (0. - 0.436374\,\mathbb{I})\ \text{sech}(\ 3.53199\,t^{0.5} + 3.55278\,x).
\end{aligned} \tag{43}$$

For Figure 2, the description of this figure is similar to Figure 1, but with different parameters, which shows that if one wishes to study another physical phenomenon, then he can control and change the arbitrary parameters of the problem and still get another solution.

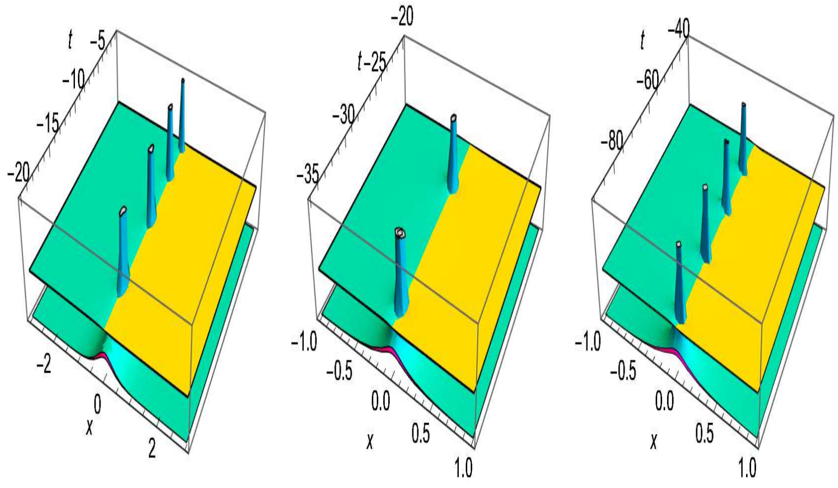

Fig. 2: Plots of interaction of $\{|Q_1(x,t)|, 20|Q_2(x,t)|, |Q_3(x,t)|, 20|Q_4(x,t)|\}$ of Example 2

Example 3: Following the steps of solution set 3, if we randomly choose

$$
\begin{aligned}
&\{a_1,a_2,b_1,b_2,k,\lambda,m_1,\alpha\} = \{11.9496, 4.25957, 14.5568, 7.42032, 26.6642, -3.52818, 3.89627, 0.5\} \quad \text{and,}\\
&\{a_{20}, d_{12}\} = \{4.65269, -0.0279702\}, \quad \text{then,}\\
&\{\delta_1,\delta_2,\delta_3,\delta_4\} = \{39.5764, 1.83174, 0.937567, 1.92063\} \quad \text{and,}\\
&\{a_{10},a_{11},a_{21},b_{12},b_{22},c_{10},c_{20},c_{11},c_{21},d_{22}\} =\\
&\{0, 0, -1, 0.0279702, 0.999609, -0.260171, -4.64541, 0.0559184, 0.998435, -0.999609\},
\end{aligned}
\tag{44}
$$

then after substituting these values in equations (8), (23)), and (32), the following solution of equation (5) can be obtained:

$$
\begin{aligned}
Q_1(x,t) &= (0. + 4.65269\,\mathbb{I}) - (0. + 0.546082\,\mathbb{I})\tanh(\,7.05636\,t^{0.5} + 3.89627\,x),\\
Q_2(x,t) &= (0.00324684 + 0.116037\,\mathbb{I})\text{sech}(\,7.05636\,t^{0.5} + 3.89627\,x),\\
Q_3(x,t) &= (-0.260171 - 4.64541\,\mathbb{I}) + (0.0007234 + 0.0129165\,\mathbb{I})\tanh(\,7.05636\,t^{0.5} + 3.89627\,x),\\
Q_4(x,t) &= (-0.00332469 - 0.118819\,\mathbb{I})\,\text{sech}(\,7.05636\,t^{0.5} + 3.89627\,x).
\end{aligned}
\tag{45}
$$

For Figure 3, the description of the interaction is similar to the previous examples too, but here, the waves become faster and the interaction lasts shorter, this is because the coefficient of time is increased.

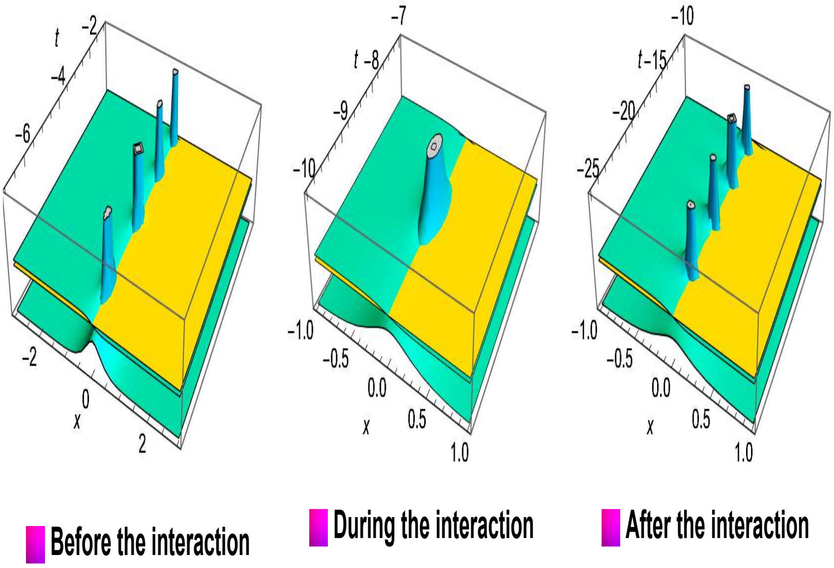

Fig. 3: Plots of interaction of $\{|Q_1(x,t)|, 25|Q_2(x,t)|, |Q_3(x,t)|, 25|Q_4(x,t)|\}$ of Example 3

Example 4: Following the steps of solution set (4), if we randomly choose

$$\begin{aligned}
&\{a_1, a_2, b_1, b_2, k, \lambda, m_1, \alpha\} = \{23.8454, 9.90918, 24.3258, 14.7082, 109.674, -1.97561, 4.9366, 0.5\} \quad \text{and,}\\
&a_{20} = 2.66457, \text{ then}\\
&\{\delta_1, \delta_2, \delta_3, \delta_4\} = \{5.49858, 1.50285, 0.870365, 2.06808\}\\
&\{a_{10}, a_{11}, a_{21}, b_{12}, b_{22}, c_{10}, c_{20}, c_{11}, c_{21}, d_{12}, d_{22}\} = \{0, 0, 1, 0, -1, 0, -2.66457, 0, -1, 0, -1\},
\end{aligned} \tag{46}$$

then after substituting these values in equations (8), (23)), and (36), the following solution of equation (5) can be obtained:

$$\begin{aligned}
Q_1(x,t) &= (0. + 2.66457\,\mathbb{I}) + (0. + 0.6739\,\mathbb{I})\ \tanh(3.95123\,t^{0.5} + 4.9366\,x),\\
Q_2(x,t) &= (0. - 0.323228\,\mathbb{I})\ \mathrm{sech}(3.95123\,t^{0.5} + 4.9366\,x),\\
Q_3(x,t) &= (0. - 2.66457\,\mathbb{I}) - (0. + 0.106671\,\mathbb{I})\ \tanh(3.95123\,t^{0.5} + 4.9366\,x),\\
Q_4(x,t) &= (0. - 0.379172\mathbb{I})\ \mathrm{sech}.(3.95123\,t^{0.5} + 4.9366\,x).
\end{aligned} \tag{47}$$

In Figure 4, the waves moving faster so they take shorter time to start interacting, when they start interacting, a delay in their speeds occurred and the interaction lasts longer.

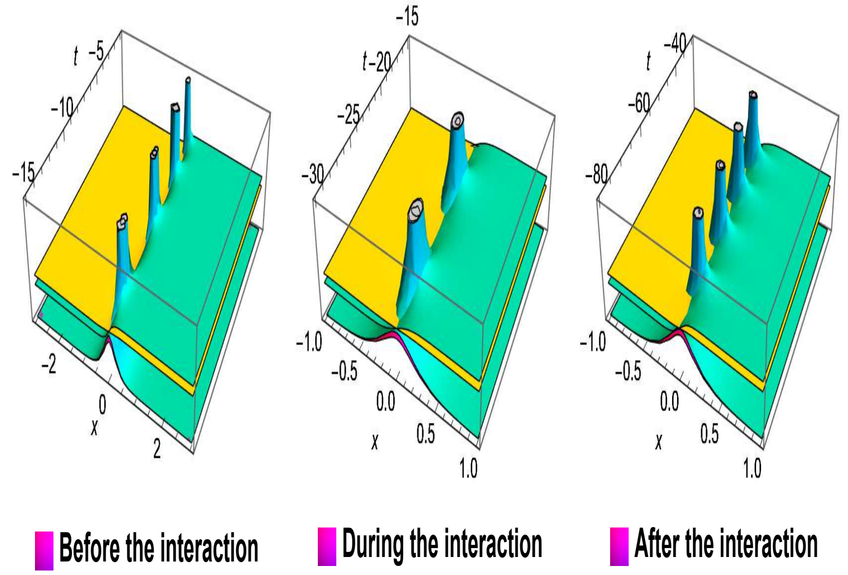

Fig. 4: Plots of interaction of $\{|Q_1(x,t)|, 6|Q_2(x,t)|, |Q_3(x,t)|, 6|Q_4(x,t)|\}$ of Example 4.

Example 5: Following the steps of solution set (1), if we randomly choose

$$\begin{aligned}
&\{a_1,a_2,b_1,b_2,c_1,c_2\} = \{26.7869, 9.97166, 23.8354, 12.1829, 12.711, 2.44439\},\\
&\{\lambda, m_1, m_2\} = \{-2.67798, 4.25088, 1.41024\}, \quad \text{and}\\
&\{a_{10}, b_{12}, d_{12}\} = \{8.62981, -0.797496, -0.766498\}, \quad \text{then}\\
&\{k_1,k_2,k_3,k_4\} = \{1.64108, -1.1005, -1.58125, -17.7886\},\\
&\{\Delta_1,\Delta_2,\Delta_3,\Delta_4\} = \{0.531418, 0.459842, 0.392225, 4.58908\}, \quad \text{and}\\
&\{a_{20},a_{11},a_{21},b_{22},c_{10},c_{20},c_{11},c_{21},d_{22}\} =\\
&\{1.98157, 0.974636, 0.223795, 0.603324, -0.440446, 8.84343, -0.0497432, 0.998762, -0.642247\},
\end{aligned} \tag{48}$$

then after substituting these values in equations, (23), (25), and (14), the following solution of the equation (6) can be obtained:

$$\begin{aligned}
Q_1(x,y,t) &= (8.62981 + 1.98157\,\mathbb{I}) + (2.5615 + 0.588171\,\mathbb{I})\,\tanh\left(\frac{2.67798\,t^\alpha}{\alpha} + .4.25088\,x + 1.41024\,y\right),\\
Q_2(x,y,t) &= (-0.997832 + 0.754883\,\mathbb{I})\,\operatorname{sech}\left(\frac{2.67798\,t^\alpha}{\alpha} + .4.25088\,x + 1.41024\,y\right),\\
Q_3(x,y,t) &= (-0.440446 + 8.84343\,\mathbb{I}) - (0.0964907 - 1.93737\,\mathbb{I})\,\tanh\left(\frac{2.67798\,t^\alpha}{\alpha} + .4.25088\,x + 1.41024\,y\right),\\
Q_4(x,y,t) &= (-3.02969 - 2.53857\,\mathbb{I})\,\operatorname{sech}\left(\frac{2.67798\,t^\alpha}{\alpha} + .4.25088\,x + 1.41024\,y\right).
\end{aligned} \tag{49}$$

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7 Conclusion

We were able to obtain a general form of infinite exact soliton solutions for the 4-WIEs by an ansatz contains tan and secant hyperbolic functions with complex coefficients. A systematic steps toward writing a solution was obtained too. Then generalized the obtained solutions to be solutions for s similar system of 4-WIEs but in (2+1) dimensions. We believe that there are more solutions which could be obtained by other ansatz. We also believe that our used ansatz is useful and could be used to obtain exact soliton solutions for the N-wave interaction equations for higher values of N. Finally, we believe that our obtained solutions have some physical applications, where one can control the inputs of the system to satisfy the requirements of his studied problem.

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