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On Applications of the Fractional Calculus for Some Singular Differential Equations

Okkes Ozturk[1,*] *and Resat Yilmazer*[2]

[1] Department of Mathematics, Faculty of Arts and Sciences, Bitlis Eren University, TR-13000 Bitlis, Turkey
[2] Department of Mathematics, Faculty of Science, Firat University, TR-23119 Elazig, Turkey

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Abstract: Generalized Leibniz rule and some theorems in the calculus of the fractional derivatives and integrals was used to obtain the fractional solutions of the radial Schrödinger equation transformed into a singular differential equation and, these solutions were also exhibited as hypergeometric notations.

Keywords: Fractional calculus, generalized Leibniz rule, fractional calculus theorems, radial Schrödinger equation.

1 Introduction

Claim of the derivatives and integrals with any arbitrary order (*that is*, *fractional calculus*) was born in 1695 and, this new and remarkable subject has intensive work fields such as mathematics, physics, chemistry, biology, medicine, engineering and so on since that day [1,2,3,4,5,6].
In our present article, we analyze this concept for the radial Schrödinger equation and so, we can summarize some scientific studies related to fractional calculus and Schrödinger equation in this section. For instance, Yildirim [7] used Homotopy Perturbation Method for the fractional nonlinear Schrödinger equation. Rida et al. [8] applied the ADM for finding the solution of the generalized fractional nonlinear Schrödinger equation subject to some initial conditions. In [9], Muslih et al. obtained solutions of the fractional Schrödinger equation via Lagrangian and Hamiltonian approach. And, Naber [10] studied on the time fractional Schrödinger equation. Zhao et al. [11] investigated the local fractional Schrödinger equations in the one-dimensional Cantorian system. In his study, the approximations solutions were obtained by using the local fractional series expansion method. Dong developed a space-time fractional Schrödinger equation containing Caputo fractional derivative and the quantum Riesz fractional operator from a space fractional Schrödinger equation in [12]. Baleanu et al. [13] exhibited approximate analytical solutions of the fractional non-linear Schrödinger equations by using the homotopy perturbation method. Jumarie [14] introduced from Lagrangian mechanics fractal in space to space fractal Schrödinger's equation via fractional Taylor's series. In [15], nonlinear Schrödinger equations with steep potential well was investigated. A numerical method for the solution of the time-fractional nonlinear Schrödinger equation in one and two dimensions which appear in quantum mechanics was applied in [16]. In [17], free particle wavefunction of the fractional Schrödinger wave equation was obtained and the wavefunction of the equation was represented in terms of generalized three-dimensional Green's function that involves fractional powers of time as variable t^{α}. Laskin [18] introduced some properties of the fractional Schrödinger equation and proved the Hermiticity of the fractional Hamilton operator and established the parity conservation law for fractional quantum mechanics and, also studied on the relationships between the fractional and standard Schrödinger equations. Yasuk et al. [19] obtained the general solutions of Schrödinger equation for non central potential via Nikiforov Uvarov method. Al-Jaber [20] formulated analytic solution of the free particle radial dependent Schrödinger equation in N-dimensional space by means of homotopy perturbation method. In this paper, we also studied to find the fractional solutions of the Schrödinger's radial equation via generalized Leibniz rule and some fractional calculus theorems.

* Corresponding author e-mail: oozturk27@gmail.com

The paper is organized as follows. In Section 2, we give the essential materials and methods. In Section 3, we state the main results of this paper. In Section 4, we specify the conclusions of this work.

2 Materials and Methods

In this section, we exhibit the essential materials and methods.

Definition 1. *Riemann-Liouville fractional derivative and integral formulas are defined by* [21],

$$ {}_{\alpha}D_x^{\kappa}\chi(x) = [\chi(x)]_{\kappa} = \frac{1}{\Gamma(n-\kappa)}\frac{d^n}{dx^n}\int_{\alpha}^{x}\frac{\chi(y)}{(x-y)^{\kappa+1-n}}dy, \qquad (1)$$

$$(n-1 \leq \kappa < n, n \in \mathbf{N})$$

and

$$ {}_{\alpha}D_x^{-\kappa}\chi(x) = [\chi(x)]_{-\kappa} = \frac{1}{\Gamma(\kappa)}\int_{\alpha}^{x}\frac{\chi(y)}{(x-y)^{1-\kappa}}dy \quad (x > \alpha, \kappa > 0). \qquad (2)$$

Definition 2. *Suppose that $\chi(z)$ is analytic and, branch point of $\chi(z)$ isn't found inside and on Ω, where $\Omega := \{\Omega^-, \Omega^+\}$, Ω^- is a contour along the cut joining the points z and $-\infty + iIm(z)$, which starts from the point at $-\infty$, encircles the point z once counter-clockwise, and returns to the point $-\infty$, and Ω^+ is a contour along the cut joining the points z and $\infty + iIm(z)$, which starts from the point at ∞, encircles the point z once counter-clockwise, and returns to the point at ∞,*

$$\chi_{\kappa}(z) := \frac{\Gamma(\kappa+1)}{2\pi i}\int_{\Omega}\frac{\chi(x)dx}{(x-z)^{\kappa+1}} \quad (\kappa \notin \mathbf{Z}^-)$$

and

$$\chi_{-n}(z) := \lim_{\kappa \to -n}\chi_{\kappa}(z) \quad (n \in \mathbf{Z}^+),$$

where $x \neq z$, $-\pi \leq \arg(x-z) \leq \pi$ for Ω^- and, $0 \leq \arg(x-z) \leq 2\pi$ for Ω^+, and, fractional derivative of $\chi(z)$ with κ order is shown as $\chi_{\kappa}(z) \ (\kappa > 0)$ and, similarly, fractional integral of $\chi(z)$ with $-\kappa$ order is shown as $\chi_{\kappa}(z) \ (\kappa < 0)$, where $|\chi_{\kappa}(z)| < \infty$ and $\kappa \in \mathbf{R}$ [22].

Lemma 1. *Suppose that $\chi(z)$ and $\psi(z)$ are analytic and single-valued functions. Linearity rule is given by*

$$[K\chi(z) + L\psi(z)]_{\kappa} = K\chi_{\kappa}(z) + L\psi_{\kappa}(z), \qquad (3)$$

where K and L are constants and, $\kappa \in \mathbf{R}$, $z \in \mathbf{C}$ [23].

Lemma 2. *Suppose that $\chi(z)$ is an analytic and single-valued function. Then, index rule is defined by*

$$(\chi_{\upsilon})_{\kappa}(z) = (\chi_{\upsilon+\kappa})(z) = (\chi_{\kappa})_{\upsilon}(z), \qquad (4)$$

where $\kappa, \upsilon \in \mathbf{R}$, $z \in \mathbf{C}$ and $\left| \frac{\Gamma(\kappa+\upsilon+1)}{\Gamma(\kappa+1)\Gamma(\upsilon+1)} \right| < \infty$ [23].

Lemma 3. *Suppose that $\chi(z)$ and $\psi(z)$ are analytic and single-valued functions. Generalized Leibniz rule is shown as follows:*

$$[\chi(z)\psi(z)]_{\kappa} = \sum_{n=0}^{\infty}\frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-n)\Gamma(n+1)}\chi_{\kappa-n}(z)\psi_n(z), \qquad (5)$$

where $\kappa \in \mathbf{R}$, $z \in \mathbf{C}$ and $\left| \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-n)\Gamma(n+1)} \right| < \infty$ [24].

Remark. In the fractional calculus, the following equalities are provided [24]:

$$(e^{\alpha z})_{\kappa} = \alpha^{\kappa}e^{\alpha z} \quad (\alpha \neq 0, \kappa \in \mathbf{R}, z \in \mathbf{C}), \qquad (6)$$

$$(e^{-\alpha z})_{\kappa} = e^{-i\pi\kappa}\alpha^{\kappa}e^{-\alpha z} \quad (\alpha \neq 0, \kappa \in \mathbf{R}, z \in \mathbf{C}), \qquad (7)$$

$$(z^{\alpha})_{\kappa} = e^{-i\pi\kappa}\frac{\Gamma(\kappa-\alpha)}{\Gamma(-\alpha)}z^{\alpha-\kappa} \quad \left(\kappa \in \mathbf{R}, z \in \mathbf{C}, \left|\frac{\Gamma(\kappa-\alpha)}{\Gamma(-\alpha)}\right| < \infty\right), \qquad (8)$$

where α is a constant and,

$$\Gamma(\kappa-n) = (-1)^n\frac{\Gamma(\kappa)\Gamma(1-\kappa)}{\Gamma(n+1-\kappa)} \quad (\kappa \in \mathbf{R}, n \in \mathbf{N}) \qquad (9)$$

Theorem 1. *Suppose that $\psi_{-\kappa} \neq 0$ and, $M(z;m)$, $N(z;n)$ are polynomials in z of degrees m,n. And,*

$$M(z;m) = \sum_{i=0}^{m} a_i z^{m-i} = a_0 \prod_{j=1}^{m} (z - z_j) \quad (a_0 \neq 0, m \in \mathbf{N}), \tag{10}$$

and,

$$N(z;n) = \sum_{i=0}^{n} b_i z^{n-i} \quad (b_0 \neq 0, n \in \mathbf{N}). \tag{11}$$

Thus, the nonhomogeneous linear ordinary fractional differintegral equation

$$\begin{aligned} M(z;m)\chi_\upsilon(z) + \left[\sum_{i=1}^{m} \binom{\kappa}{i} M_i(z;m) + \sum_{i=1}^{n} \binom{\kappa}{i-1} N_{i-1}(z;n)\right] \chi_{\upsilon-i}(z) \\ + \binom{\kappa}{i} n! b_0 \chi_{\upsilon-n-1}(z) = \psi(z) \quad (m,n \in \mathbf{N}, \kappa, \upsilon \in \mathbf{R}), \end{aligned} \tag{12}$$

has the following solution:

$$\chi(z) = \left[\left(\frac{\psi_{-\kappa}(z)}{M(z;m)} e^{\sigma(z;m,n)}\right)_{-1} e^{-\sigma(z;m,n)}\right]_{\kappa-\upsilon+1} \quad (z \in \mathbf{C}\backslash\{z_1,...,z_m\}), \tag{13}$$

where

$$\sigma(z;m,n) = \int^z \frac{N(x;n)}{M(x;m)} dx \quad (z \in \mathbf{C}\backslash\{z_1,...,z_m\}). \tag{14}$$

And, the homogeneous linear ordinary fractional differintegral equation

$$\begin{aligned} M(z;m)\chi_\upsilon(z) + \left[\sum_{i=1}^{m} \binom{\kappa}{i} M_i(z;m) + \sum_{i=1}^{n} \binom{\kappa}{i-1} N_{i-1}(z;n)\right] \chi_{\upsilon-i}(z) \\ + \binom{\kappa}{i} n! b_0 \chi_{\upsilon-n-1}(z) = 0 \quad (m,n \in \mathbf{N}, \kappa, \upsilon \in \mathbf{R}), \end{aligned} \tag{15}$$

has the following solution:

$$\chi(z) = K[e^{-\sigma(z;m,n)}]_{\kappa-\upsilon+1}, \tag{16}$$

where K is an arbitrary constant [24].

Theorem 2. *When $\mid \psi_\kappa(z) \mid < \infty \ (\kappa \in \mathbf{R})$ and $\psi_{-\kappa} \neq 0$, then*

$$Az^2\chi_2 + Bz\chi_1 + (Dz^2 + Ez + F)\chi = \psi \quad (A, D \neq 0, z \in \mathbf{C}\backslash\{0\}, \chi = \chi(z)), \tag{17}$$

has the solution as follows:

$$\chi = z^\nu e^{\varepsilon z}\left\{\left[A^{-1} z^{-(\kappa+1)+\frac{2A\nu+B}{A}} e^{2\varepsilon z}\left(z^{-(\nu+1)} e^{-\varepsilon z}\psi\right)_{-\kappa}\right]_{-1} z^{\kappa-\frac{2A\nu+B}{A}} e^{-2\varepsilon z}\right\}_{\kappa-1}, \tag{18}$$

where ν, ε and κ are in the form:

$$\nu = \frac{A - B \pm \sqrt{(A-B)^2 - 4AF}}{2A}, \quad \varepsilon = \pm i\sqrt{\frac{D}{A}}, \tag{19}$$

and,

$$\kappa = \frac{(2A\nu + B)\varepsilon + E}{2A\varepsilon}. \tag{20}$$

Moreover,

$$Az^2\chi_2 + Bz\chi_1 + (Dz^2 + Ez + F)\chi = 0 \quad (A, D \neq 0, z \in \mathbf{C}\backslash\{0\}, \chi = \chi(z)), \tag{21}$$

has the solution in the form:

$$\chi = Kz^\nu e^{\varepsilon z}\left(z^{\kappa-\frac{2A\nu+B}{A}} e^{-2\varepsilon z}\right)_{\kappa-1}. \tag{22}$$

where K is an arbitrary constant [24].

In the next section, we investigate the solutions in the fractional forms and hypergeometric forms for the radial equation by using the generalized Leibniz rule and Theorem 2.

3 Main Results

3.1 On the Generalized Leibniz Rule

In the β-dimensional space, fractional Schrödinger equation's radial component is

$$\chi_2(y)+\frac{\beta-1}{y}\chi_1(y)+\left[\frac{2m}{\hbar^2}\left(E+e^2\frac{\alpha_c}{y^{c-2}}\right)-\frac{\rho(\rho-\beta-2)}{y^2}\right]\chi(y)=0, \tag{23}$$

where constant α_c is $\alpha_c=\frac{\Gamma(c/2)}{2\pi^{c/2}(c-2)\varepsilon_0}$ $(c>2)$, $1\leq\beta\leq 3$ and $0\leq y\leq\infty$.
For Equ. (23), we get some transformations as

$$z=2\alpha y, \quad \chi=y^{\rho}e^{-\alpha y}\psi, \quad a=\frac{me^2\alpha_c}{\hbar^2}$$

where $\alpha^2=-2mE/\hbar^2$. Thus, we obtain the following singular differential equation:

$$z\psi_2+(\lambda-z)\psi_1+\left(\omega z^{3-c}-\frac{\lambda}{2}\right)\psi=0, \tag{24}$$

where $\lambda=2\rho+\beta-1$, $\omega=\frac{a}{2^{3-c}\alpha^{4-c}}$ [24].

Theorem 3.*Suppose that $c=4$ in Equ. (24), and so, we have*

$$z\psi_2+(\lambda-z)\psi_1+\left(\frac{\omega}{z}-\frac{\lambda}{2}\right)\psi=0. \tag{25}$$

Equ. (25) has the following fractional solutions:

$$\psi^{I}(z)=Kz^{\frac{1-\lambda+\xi}{2}}\left[z^{-\left(\frac{1+\xi}{2}\right)}e^z\right]_{-\left(\frac{1-\xi}{2}\right)}, \tag{26}$$

and,

$$\psi^{II}(z)=Lz^{\frac{1-\lambda-\xi}{2}}\left[z^{-\left(\frac{1-\xi}{2}\right)}e^z\right]_{-\left(\frac{1+\xi}{2}\right)}, \tag{27}$$

where $z\in\mathbf{C}$, $\psi\in\{\psi:0\neq|\psi_\kappa|<\infty,\kappa\in\mathbf{R}\}$ and K, L, λ, ξ are constants.

*Proof.*At first, we get $\psi=z^{\tau}\phi$ $(z\neq 0,\phi=\phi(z))$, and so,

$$z\phi_2+(2\tau+\lambda-z)\phi_1+\left[(\tau^2+\tau(\lambda-1)+\omega)z^{-1}-(\tau+\frac{\lambda}{2})\right]\phi=0. \tag{28}$$

If we assume that $\tau^2+\tau(\lambda-1)+\omega=0$ in Equ. (28), we write an equality as $\tau=\frac{1-\lambda\pm\xi}{2}$ where $\xi=\sqrt{(\lambda-1)^2-4\omega}$.
(i.) When $\tau=\frac{1-\lambda+\xi}{2}$, we have

$$z\phi_2+(1+\xi-z)\phi_1-\left(\frac{1+\xi}{2}\right)\phi=0. \tag{29}$$

If we apply Equ. (5) (*Generalized Leinbiz rule*) for all of terms in Equ. (29), so, we obtain

$$z\phi_{2+\kappa}+(\kappa+1+\xi-z)\phi_{1+\kappa}-\left(\kappa+\frac{1+\xi}{2}\right)\phi=0. \tag{30}$$

Now, we assume that $\kappa+\frac{1+\xi}{2}=0$ in Equ. (30), thus, $\kappa=-\left(\frac{1+\xi}{2}\right)$, and in the end, we have

$$\varphi_1+\left[\left(\frac{1+\xi}{2}\right)z^{-1}-1\right]\varphi=0 \quad \left(\varphi=\varphi(z)=\phi_{\left(\frac{1-\xi}{2}\right)}\right). \tag{31}$$

The solution of Equ. (31) is found easily as follows:

$$\varphi(z) = Kz^{-\left(\frac{1+\xi}{2}\right)} e^z,$$

and, by substituting above assumptions, we write

$$\psi(z) = Kz^{\frac{1-\lambda+\xi}{2}} \left[z^{-\left(\frac{1+\xi}{2}\right)} e^z \right]_{-\left(\frac{1-\xi}{2}\right)}. \tag{32}$$

(ii.) By means of similar steps, the second fractional solution is

$$\psi(z) = Lz^{\frac{1-\lambda-\xi}{2}} \left[z^{-\left(\frac{1-\xi}{2}\right)} e^z \right]_{-\left(\frac{1+\xi}{2}\right)}. \tag{33}$$

After, the hypergeometric notations of Equ. (32) and Equ. (33) are presented by the following theorems:

Theorem 4. *Let G be the Gauss hypergeometric function, and suppose that* $\left| \left[z^{-\left(\frac{1+\xi}{2}\right)} \right]_n \right| < \infty \quad (n \in \mathbf{N}, z \neq 0)$. *Thus, function* $\psi(z)$ *in Equ. (32) is written by*

$$\psi(z) = Kz^{-\frac{\lambda}{2}} e^z G\left[\frac{1-\xi}{2}, \frac{1+\xi}{2}; \frac{1}{z} \right] \quad \left(\left| \frac{1}{z} \right| < 1 \right). \tag{34}$$

Proof. At first, we use the generalized Leibniz rule for Equ. (32), and so,

$$\psi(z) = Kz^{\frac{1-\lambda+\xi}{2}} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{1+\xi}{2}\right)}{\Gamma\left(\frac{1+\xi}{2} - n\right) n!} \left[z^{-\left(\frac{1+\xi}{2}\right)} \right]_n (e^z)_{-\left(\frac{1-\xi}{2}\right)}. \tag{35}$$

After, the following form is obtained by means of (6), (8) and (9):

$$\psi(z) = Kz^{-\frac{\lambda}{2}} e^z \sum_{n=0}^{\infty} \frac{\Gamma\left(n + \frac{1-\xi}{2}\right)}{\Gamma\left(\frac{1-\xi}{2}\right)} \frac{\Gamma\left(n + \frac{1+\xi}{2}\right)}{\Gamma\left(\frac{1+\xi}{2}\right)} \frac{1}{n!} \left(\frac{1}{z}\right)^n.$$

In the end, we write

$$\psi(z) = Kz^{-\frac{\lambda}{2}} e^z \sum_{n=0}^{\infty} \left(\frac{1-\xi}{2}\right)_n \left(\frac{1+\xi}{2}\right)_n \frac{1}{n!} \left(\frac{1}{z}\right)^n,$$

and,

$$\psi(z) = Kz^{-\frac{\lambda}{2}} e^z G\left[\frac{1-\xi}{2}, \frac{1+\xi}{2}; \frac{1}{z} \right].$$

Theorem 5. *Let G be the Gauss hypergeometric function, and suppose that* $\left| \left[z^{-\left(\frac{1-\xi}{2}\right)} \right]_n \right| < \infty \quad (n \in \mathbf{N}, z \neq 0)$. *Thus, function* $\psi(z)$ *in Equ. (33) is written by*

$$\psi(z) = Lz^{-\frac{\lambda}{2}} e^z G\left[\frac{1+\xi}{2}, \frac{1-\xi}{2}; \frac{1}{z} \right] \quad \left(\left| \frac{1}{z} \right| < 1 \right). \tag{36}$$

3.2 On the Fractional Calculus Theorems

Under the Coulomb potential, Schrödinger equation's radial component is [25],

$$\chi_2(y) + \frac{2}{y}\chi_1(y) + \left[\frac{2m}{\hbar^2}\left(E + \frac{e^2}{y}\right) - \frac{\rho(\rho+1)}{y^2} \right] \chi(y) = 0. \tag{37}$$

For Equ. (37), we get some transformations as

$$z = 2\alpha y, \quad \chi = z^{-1/2}\psi, \quad b = \frac{me^2}{\alpha\hbar^2},$$

where $\alpha^2 = -2mE/\hbar^2$. So, we obtain the following singular differential equation:

$$z^2\psi_2 + z\psi_1 - \left(\frac{z^2}{4} - bz + \frac{k^2}{4}\right)\psi = 0, \tag{38}$$

where $\rho(\rho+1) = \frac{k^2-1}{4}$.
If we use the Theorem 2 for Equ. (38), we can write

$$A = B = 1, \quad D = -\frac{1}{4}, \quad E = b, \quad F = -\frac{k^2}{4}, \tag{39}$$

and, by applying (19) and (20), we have

$$\nu = \pm\frac{k}{2}, \quad \varepsilon = \pm\frac{1}{2},$$

and,

$$\kappa = \frac{(2\nu+1)\varepsilon + b}{2\varepsilon}.$$

Thus, the fractional solution of Equ. (38) is

$$\psi = Kz^{\nu}e^{\varepsilon z}\left[z^{-(2\nu+1-\kappa)}e^{-2\varepsilon z}\right]_{\kappa-1}. \tag{40}$$

Theorem 6. *Let G be the Gauss hypergeometric function, and suppose that* $\left|\left[z^{-(2\nu+1-\kappa)}\right]_n\right| < \infty \quad (n \in \mathbf{N}, z \neq 0)$. *Thus, function* $\psi(z)$ *in Equ. (40) is written by*

$$\psi(z) = Kz^{-(\varepsilon - \frac{b}{2\varepsilon})}e^{-\varepsilon z}G\left[1-\kappa, 1-\kappa-n+\frac{b}{\varepsilon}; -\frac{1}{2\varepsilon z}\right] \quad \left(\left|-\frac{1}{2\varepsilon z}\right| < 1\right). \tag{41}$$

4 Conclusion

We studied on the Schrödinger equation's radial components in the β -dimensional space and under the Coulomb potential respectively, and we first transformed these equations to the singular differential equations by means of some assumptions. After, the generalized Leibniz rule and some fractional calculus theorems were applied to these singular equations due to find the fractional solutions, and hypergeometric solutions were also obtained. Thus, we exhibited two different solution methods for two different equations. Moreover, we used Theorem 2 to Equ. (25) and, applied generalized Leibniz rule to Equ. (38) in our different studies.

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