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A-Statistical Convergence of a Class of Integral Operators

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Abstract: Starting from a general sequence of linear and positive operators of summation integral type, we associate its r-th order generalization. This construction involves high order derivatives of a signal and it looses the positivity property. Considering that the initial approximation process is A-statistically pointwise convergent, we prove that the property is inherited by the new sequence. The study is developed for smooth functions defined both on an unbounded interval and on a compact interval.

Keywords: Positive linear operator, statistical convergence, approximation process.

1. Introduction

Following the investigations of P. P. Korovkin [8], it is known that the positive linear operators (PLOs) have a low rate of convergence and they failing to respond to the smoothness of the function that approximates. The disadvantage of the positive linear approximating sequences is definitely determined by the fact that they don't react to the improvement of the smoothness of functions they are generated from. To overcome this fact, Kirov and Popova [7] proposed a generalization of the r-th order, $r \in \mathbb{N}$. For a given PLO, this generalization is obtained by the action of the operator not directly on the signal f, but on its r-th degree Taylor polynomial. The new operator keeps the linearity property but loose the positivity.

On the other hand, a current subject in Approximation Theory is the approximation of continuous functions by PLOs using the statistical convergence, the first research on this topic being done by Gadjiev and Orhan [6].

In this note, starting from a general class of summation integral PLOs, we indicate its r-generalization. Under the assumption that the initial sequence is A-statistically pointwise convergent, we study how this property is inherited by the new sequence. Also, some examples are delivered. This approach is also interesting from the following perspective: the approximation property of the first sequence can be proved by using a Bohman-Korovkin type criterion. Since the new sequence is not longer positive, this criterion can not be applied to highlight its approximation property. So, we need to use a different technique.

2. The operators L_n and $L_{n,r}$

Let J be a given interval on the real line. To approximate continuous functions f on J, we use a sequence $(l_n)_{n\geq 1}$ of PLOs defined by

$$(l_n f)(x) = \sum_{k\in J_n} \lambda_{n,k}(x) f(x_{n,k}), \quad x \in J, \qquad (1)$$

where, for each $n \in \mathbb{N}$, $I_n \subset \mathbb{N}$ is a set of indices, $\lambda_{n,k}$, $k \in I_n$, are non-negative functions on the space $C(J)$ and $(x_{n,k})_{k\in I_n}$ is a mesh of nodes on J. We assume that l_n reproduces every constant function, this meaning

$$\sum_{k\in I_n} \lambda_{n,k}(x) = 1, \quad x \in J. \qquad (2)$$

In order to generalize l_n defined by (1) to a summation-integral operator L_n, inspired by Durrmeyer technique [4], we use a non-negative family $\omega_{n,k}$, $k \in I_n$, of functions belonging to Lebesgue space $L^1(J)$ and normalized by

$$\int_J \omega_{n,k}(t)dt = 1, \quad k \in I_n. \qquad (3)$$

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Further on, we define L_n as

$$(L_n f)(x) = \sum_{k\in I_n} \lambda_{n,k}(x) \int_J \omega_{n,k}(t) f(t)dt, \quad (4)$$

$x \in J$, $f \in \mathcal{F}(J)$, where $\mathcal{F}(J)$ contains all functions $f \in \mathbb{R}^J$ for which the right hand side in (4) is well defined.

As usual, by $C^r(J)$, $r = 0, 1, 2, \ldots$, we denote the space of all real valued functions defined on the interval J with a continuous derivative of order r on J. Also, e_j stands for the monomial of j-th degree. Let f belong to $C^r(J)$ such that $e_s f^{(s)} \in \mathcal{F}(J)$ for $s = 0, 1, \ldots, r$, and let $T_r f(x; \cdot)$ be the r-th degree Taylor polynomial associated to the function f at the point $x \in J$. We define the linear operators

$$(L_{n,r} f)(x) = L_n(T_r f; x)$$
$$= \sum_{k\in I_n} \lambda_{n,k}(x) \sum_{s=0}^{r} \frac{1}{s!} \int_J \omega_{n,k}(t) f^{(s)}(t)(x-t)^s dt, \quad (5)$$

$x \in J$. Clearly, $L_{n,0} = L_n$, $n \in \mathbb{N}$.

We mention that the general class defined by (4) includes those considered in the literature under the name of "modified operators" being integral analogue in Durrmeyer sense of some classical discrete PLOs.

Examples. 1° If $J = [0, 1]$, $I_n = \{0, 1, \ldots, n\}$,

$$\lambda_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad \omega_{n,k}(t) = (n+1)\lambda_{n,k}(t),$$

then L_n becomes the Bernstein-Durrmeyer operator M_n studied by Derrienic [2].

2° If $J = [0, 1]$, $I_n = \mathbb{N}$,

$$\lambda_{n,k}(x) = \binom{n+k}{k} x^k (1-x)^{n+1},$$

$$\omega_{n,k}(t) = \frac{(n+k+1)(n+k+2)}{n+1} \lambda_{n,k}(t),$$

then L_n becomes the modified operator of Meyer-König and Zeller studied in [1].

3° Choose $J = [0, \infty)$, $I = \mathbb{N}$ and

$$\lambda_{n,k}(x) = e^{-nx}(nx)^k/k!, \quad \omega_{n,k}(t) = n\lambda_{n,k}(t).$$

L_n becomes the Szász-Durrmeyer operator S_n defined by Mazhar and Totik [9].

4° Choose $J = [0, \infty)$, $I_n = \mathbb{N}$ and

$$\lambda_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k},$$

$$\omega_{n,k}(t) = (n-1)\lambda_{n,k}(t).$$

This time L_n reduced to the Baskakov-Durrmeyer operator V_n, see [10].

The generalizations of r-th order of all four classes of operators mentioned above are described by (5).

Throughout the paper we need the following test function namely φ_x. For each $x \in J$, we define φ_x as follows

$$\varphi_x(t) = |x - t|, \quad t \in J. \quad (6)$$

At this point we briefly recall some basic facts with regard to the notion of statistical convergence. This concept, originally appeared in Steinhaus [11] and Fast [5] papers, is based on the notion of the density of subsets of $\mathbb{N}$ and it can be viewed as a regular method of summability of sequences. The density of a set $K \subset \mathbb{N}$ is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_K(k),$$

provided the limit exists, where χ_K is the characteristic function of K. Actually, the sum of the right hand side represents the cardinality of the set $\{k \le n : k \in K\}$. A sequence $x = (x_k)_{k\ge1}$ is statistically convergent to a real number L, denoted $st - \lim_k x_k = L$, if, for every $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : |x_k - L| \ge \varepsilon\}) = 0$$

holds. Closely related to this notion is A-statistical convergence, where $A = (a_{n,k})_{n,k\in\mathbb{N}}$ is an infinite summability matrix. For the above given sequence x, the A-transform of x, denoted by $Ax = ((Ax)_n)$, is defined as follows

$$(Ax)_n = \sum_{k=1}^{\infty} a_{n,k} x_k, \quad n \in \mathbb{N},$$

provided the series convergences for each n. Suppose that A is non-negative regular summability matrix, regular meaning that any convergent sequence is A-summable to its limit. The sequence x is A-statistically convergent to the real number L if, for every $\varepsilon > 0$, one has

$$\lim_{n\to\infty} \sum_{k\in I(\varepsilon)} a_{n,k} = 0,$$

where $I(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \ge \varepsilon\}$. This limit is denoted by $st_A - \lim_k x_k = L$. In the particular case $A = C_1$, the Cesàro matrix of first order, A-statistical convergence reduces to statistical convergence, see, e.g., [5]. Also, if A is the identity matrix, then A-statistical convergence coincides with the ordinary convergence.

3. Results

Given $M > 0$ and $0 < \alpha \le 1$, we shall denote by $Lip_M \alpha$ the subset of all Hölder continuous functions f on J with exponent α and constant M, i.e.

$$|f(x) - f(y)| \le M|x - y|^\alpha$$

for every $(x, y) \in J \times J$.

At first, we focus to the general case when the interval J is unbounded.

Theorem 1. *Let $A = (a_{n,k})_{(n,k)\in\mathbb{N}\times\mathbb{N}}$ be a non-negative regular summability matrix. Let $r \in \mathbb{N}$ be fixed, $\alpha \in (0,1]$ and $M > 0$. Let the operators L_n and $L_{n,r}$, $n \in \mathbb{N}$, be defined by (4) and (5), respectively, where J is unbounded. If $x \in J$ and $\varphi_x^{r+\alpha} \in \mathcal{F}(J)$ such that*

$$st_A - \lim_n (L_n \varphi_x^{r+\alpha})(x) = 0, \tag{7}$$

then

$$st_A - \lim_n |f(x) - (L_{n,r}f)(x)| = 0 \tag{8}$$

holds for any function $f \in C^r(J)$ with the properties $e_s f^{(s)} \in \mathcal{F}(J)$, $s = 0, 1, \ldots, r$ and $f^{(r)} \in Lip_M\alpha$. Here φ_x is given at (6).

Proof. For an arbitrary fixed $x \in J$ and for any $t \in J$, we can write

$$\begin{aligned} -|f(x) - (T_r f)(t;x)| &\le f(x) - (T_r f)(t;x) \\ &\le |f(x) - (T_r f)(t;x)|. \end{aligned}$$

The operator L_n is linear and positive, consequently it is monotone. In the above, applying L_n, one gets

$$\begin{aligned} -L_n(|f(x) - T_r f|;x) &\le f(x)(L_n e_0)(x) - L_n(T_r f;x) \\ &\le L_n(|f(x) - T_r f|;x). \end{aligned}$$

In view of (2) and (3) we deduce $L_n e_0 = e_0$ and the above inequalities imply

$$|f(x) - (L_{n,r}f)(x)| \le L_n(|f(x) - T_r f|;x). \tag{9}$$

We also call Taylor's formula with integral form of the remainder.

Taking into account that f belongs to $C^r(J)$, for all t and x in J, we have

$$f(x) = (T_{r-1}f)(t;x) + \int_t^x \frac{(x-u)^{r-1}}{(r-1)!} f^{(r)}(u)du. \tag{10}$$

Since

$$\begin{aligned} (T_{r-1}f)(t;x) &= (T_r f)(t;x) \\ &\quad - \frac{(x-t)^r}{(r-1)!} f^{(r)}(t) \int_0^1 (1-u)^{r-1} du \end{aligned}$$

and

$$\begin{aligned} &\int_t^x (x-u)^{r-1} f^{(r)}(u)du \\ &= \int_0^1 (x-t)^r (1-u)^{r-1} f^{(r)}(t + u(x-t))du, \end{aligned}$$

the relation (10) can be rewritten as follows

$$\begin{aligned} &f(x) - (T_r f)(t;x) \\ &= \frac{(x-t)^r}{(r-1)!} \int_0^1 (1-u)^{r-1} (f^{(r)}(t+u(x-t)) - f^{(r)}(t))du. \end{aligned}$$

Since $f^{(r)} \in Lip_M\alpha$, we obtain

$$\begin{aligned} |f(x) - (T_r f)(t;x)| &\le M \frac{|x-t|^{r+\alpha}}{(r-1)!} B(r, \alpha+1) \\ &= \frac{M}{(\alpha+1)^{(r)}} \varphi_x^{r+\alpha}(t), \end{aligned}$$

where B indicates Beta function, φ_x is defined by (6) and $a^{(h)}$ represents the Pochhammer symbol for rising factorial, $a^{(h)} = \prod_{j=0}^{h-1}(a+j)$, $h \in \mathbb{N}$.

Returning at (9), we deduce

$$|f(x) - (L_{n,r}f)(x)| \le \frac{M}{(\alpha+1)^{(r)}} (L_n \varphi_x^{r+\alpha})(x). \tag{11}$$

Now, for an arbitrary fixed $\varepsilon > 0$, we define the following two sets

$$\begin{aligned} S_1(x,\varepsilon) &= \{n \in \mathbb{N} : \ |f(x) - (L_{n,r}f)(x)| \ge \varepsilon\}, \\ S_2(x,\varepsilon) &= \left\{n \in \mathbb{N} : \ (L_n \varphi_x^{r+\alpha})(x) \ge \frac{\varepsilon(\alpha+1)^{(r)}}{M}\right\}. \end{aligned}$$

On the basis of (11), we have $S_1(x,\varepsilon) \subset S_2(x,\varepsilon)$. Consequently, for any $j \in \mathbb{N}$ we can write

$$\sum_{n\in S_1(x,\varepsilon)} a_{j,n} \le \sum_{n\in S_2(x,\varepsilon)} a_{j,n}.$$

Because ε is arbitrary, the relation (7) leads us to the identity

$$\lim_{j\to\infty} \sum_{n\in S_1(x,\varepsilon)} a_{j,n} = 0$$

which ensures the required relation (8). The proof is completed. □

Remark 2. Even if (8) is achieved when one single condition is fulfilled, namely (7), in practice it is difficult to work with fractional powers as $r+\alpha \in (r, r+1]$. Therefore, we replace (7) with two more friendly conditions involving only natural powers of the test-function φ_x. Let $x \in J$. For each $t \in J$ with the property $|x-t| \le 1$, one has $\varphi_x^{r+\alpha}(t) \le \varphi_x^r(t)$. For those $t \in J$ satisfying $|x-t| > 1$, one has $\varphi_x^{r+\alpha}(t) \le \varphi_x^{r+1}(t)$. Consequently, $\varphi_x^{r+\alpha} \le \varphi_x^r + \varphi_x^{r+1}$ on the interval J. Knowing that L_n is monotone and appealing to the properties of $st_A - \lim_n$, we deduce the following.

If $\varphi_x^k \in \mathcal{F}(J)$ verifies

$$st_A - \lim_n (L_n \varphi_x^k)(x) = 0 \text{ for } k = r \text{ and } k = r+1, \tag{12}$$

then (7) takes place.

Remark 3. Let K be a compact interval, $K \subset J$. We consider the space $C(K)$ endowed with usual sup-norm

$\|\cdot\|_{C(K)}$, $\|h\|_{C(K)} = \sup_{x\in K} |h(x)|$. If $L_n(C(K)) \subset C(K)$, $n \in \mathbb{N}$, relation (11) implies

$$\|L_{n,r}f - f\|_{C(K)} \leq \frac{M}{(\alpha+1)^{(r)}} \|L_n \varphi_x^{r+\alpha}\|_{C(K)}.$$

Consequently, the A-statistical pointwise convergence in (7) and (8) can be replaced by A-statistical uniform convergence, where A is a non-negative regular summability matrix. In short, we state: if

$$st_A - \lim_n \|L_n \varphi_x^{r+\alpha}\|_{C(K)} = 0, \quad (13)$$

then

$$st_A - \lim \|f - L_{n,r}f\|_{C(K)} = 0. \quad (14)$$

Finally, we analyze the particular case $J = [a, b]$ and $L_n : C([a, b]) \to B([a, b])$. Here $B([a, b])$ stands for the space of all real valued bounded functions defined on $[a, b]$, endowed with the uniform norm.

At this point we recall the result established in [6; Theorem 1].

Theorem 4. *If the sequence of positive linear operators $\Lambda_n : C([a, b]) \to B([a, b])$ satisfies the conditions*

$$st - \lim_n \|\Lambda_n e_j - e_j\|_{C([a,b])} = 0, \quad (15)$$

where $j \in \{0, 1, 2\}$, then, for any function $f \in C([a, b])$, we have

$$st - \lim_n \|\Lambda_n f - f\|_{C([a,b])} = 0. \quad (16)$$

Examining the proof of the above theorem given by the authors, we notice that the statement is also true for A-statistical convergence, where A is a non-negative regular summability matrix.

Due to (2) and (3), for our operators L_n the first requirement in (15) is obviously satisfied. Thus, the conditions imposed in (15) are reduced to two, for $j = 1$ and $j = 2$. If for our operators L_n, $n \in \mathbb{N}$, we require to take place these two conditions, then, on the basis of (16), we deduce $st_A - \lim_n \|L\varphi_x^{r+\alpha}\|_{C([a,b])} = 0$, this being exactly the hypothesis formulated in (13). We underline, in a more general framework, the proof of this relation was given by O. Duman and C. Orhan [3; Lemma 3.4]. Since (13) holds, will result that (15) is true. This way, we have proved the following result.

Theorem 5. *Let $A = (a_{n,k})_{(n,k)\in\mathbb{N}\times\mathbb{N}}$ be a non-negative regular summability matrix. Let $r \in \mathbb{N}$ be fixed, $\alpha \in (0, 1]$, $M > 0$. We consider the operators $L_n : C([a, b]) \to B([a, b])$ defined by (4) and their generalizations $L_{n,r}$ of r-th order defined by (5). If*

$$st_A - \lim_n \|L_n e_j - e_j\|_{C([a,b])} = 0 \ \text{for } j = 1 \text{ and } j = 2,$$

then

$$st_A - \lim_n \|L_{n,r}f - f\|_{C([a,b])} = 0$$

holds for any function $f \in C^r([a, b])$ with the property $f^{(r)} \in Lip_M \alpha$.

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