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On the Numerical Solutions of Heston Partial Differential Equation

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Abstract: In this paper, some schemes are developed to study numerical solution of the Heston partial differential equation with an initial and boundary conditions, by the variational iteration method (VIM). The numerical solutions obtained by the variational iteration method are compared with those of Adomian decomposition method (ADM) and Homotopy perturbation method (HPM). The results show that the variational iteration method is much easier, more convenient, and more stable and efficient than Adomian decomposition method.

Keywords: Heston partial differential equation, Variational iterative method (VIM), Adomian decomposition method (ADM), Homotopy perturbation method (HPM)

1 Introduction

Generally, real-world physical problems and financial problems are modeled as partial, integral and integro differential equations. Since finding the solution of these equations is too complicated, in recent years a lot of attention has been devoted by researchers to find the analytical and numerical solution of these equations. The Adomian decomposition method (ADM) is introduced by the American engineer, G. Adomian (1923–1996). It is based on the canonical form of the equations, considering the solution as a power series, and nonlinear operator of a series of Adomian polynomials [1-3]. The Homotopy perturbation method (HPM), He (1999), is an alternative useful non-perturbative method which has been used to tackle nonlinear problems successfully, by many scientists and researchers and the variational iteration method (VIM) plays an important role in recent researches in this field. This method is proposed by the Chinese mathematician He [4-7] as a modification of a general Lagrange multiplier method [8-9]. It has been shown that this procedure is a powerful tool for solving various kinds of problems. In this paper we discuss the applicability of these numerical schemes for European option valuation of Heston stochastic volatility model.

The paper is organized as follows: Section 2 describes Heston stochastic volatility model. Section 3 discusses the main point of VIM, ADM, and HPM methods. Sections 4-6 explain how to apply these methods for European option pricing under Heston model. Section 7 summarizes the most important results and concludes the paper.

2 Heston stochastic volatility model

Suppose that under a risk-neutral measure a stock price is governed by [10-12]

$$\frac{dS(t)}{S(t)} = rdt + \sqrt{V(t)}d\tilde{W}_1(t)$$
$$\frac{dV(t)}{dt} = (a - bV(t))dt + \sigma\sqrt{V(t)}d\tilde{W}_2(t)$$

where r is the interest rate, the parameters a, b, and $\sigma$are positive constant, and $\tilde{W}_1(t)$and $\tilde{W}_2(t)$ are correlated Brownian motions under the risk-neutral measure with the correlation coefficient $\rho \in (-1,1)$, i.e.

$$d\tilde{W}_1(t)d\tilde{W}_2(t) = \rho dt.$$

The risk-neutral price of a call expiring at time $t \leq T$in the Heston stochastic volatility model is as follows

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$$c(t,S(t),V(t)) = \tilde{E}\left[e^{-r(T-t)}(S(T)-k)^{+}\right], \quad 0 \le t \le T. \tag{1}$$

The Heston partial differential equation (PDE) for the fair values of European style options forms a time-dependent convection-diffusion-reaction equation with mixed spatial derivative terms as,

$$\frac{\partial c}{\partial t} + rs\frac{\partial c}{\partial s} + (a-bv)\frac{\partial c}{\partial v} + \frac{1}{2}s^2 v\frac{\partial^2 c}{\partial s^2} + \rho\sigma sv\frac{\partial^2 c}{\partial s\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 c}{\partial v^2} - rc = 0 \tag{2}$$

On the unbounded two- dimensional spatial domain $s > 0, v > 0$ with $0 < t \le T$. $c(s,v,t)$ denotes the fair value of a European- style option if at the time $\tau = T-t$,the asset price and its variance be equal to s, vrespectively. The equation (2) usually appears with initial and boundary value conditions which are determined by the specific option under consideration. Initial and boundary conditions in European call options equation is denoted by,

$$\begin{aligned} &c(s,v,t) = \max(0, s-K), \\ &c(0,v,t) = 0 \end{aligned} \tag{3}$$

With a given strike price,$K > 0$.

3 The methods

In what follows we will illustrate briefly the main point of each the methods, where details can be found in [13-17].

3.1 Variational iteration method

To show the basic concepts of VIM, consider the following general non-linear partial differential equation [18-20],

$$\begin{aligned} &Lu(x,t) + Ru(x,t) + Nu(x,t) = 0, \\ &u(x,0) = f(x), \end{aligned} \tag{4}$$

Where,$L = \frac{\partial}{\partial t}$,$R$ is a linear operator and $Nu(x,t)$ is the nonlinear term. $Ru(x,t)$ and $Nu(x,t)$don't have partial derivatives with respect to t.
According to the variational iteration method [27, 28], an iteration formulation can be constructed in the following way

$$U_{n+1}(x,t) = U_n(x,t) + \int_0^t \lambda \left\{ LU_n + \overbrace{RU_n}^{\sim} + \overbrace{NU_n}^{\sim} \right\} d\tau, \tag{5}$$

Where λ is a general Lagrange multiplier, which can be identified optimally via variational theory, $\overbrace{RU_n}^{\sim}$ and $\overbrace{NU_n}^{\sim}$ are considered as restricted variations, i.e.$\delta\overbrace{RU_n}^{\sim} = 0$, $\delta\overbrace{NU_n}^{\sim} = 0$, and its stationary conditions can be obtained as:

$$\begin{aligned} &1 + \lambda|_{\tau=t} = 0, \\ &\lambda' = 0. \end{aligned}$$

The Lagrange multiplier, therefore, can be identified as $\lambda = -1$, and the following variational iteration formula can be obtained as

$$U_{n+1} = U_n - \int_0^t \{\, L(U_n) + R(U_n) + N(U_n\,)\,)\, d\tau, \tag{6}$$

The second term on the right is called the correction term. Eq. (6) can be solved iteratively using $U_0(x)$ as the initial approximation, with possible unknowns.

3.2 Adomian Decomposition method

The Adomian decomposition method is a technique for solving functional equation in the following canonical form [21-22]:

$$u = f + N(u). \tag{7}$$

The solution u is considered as the summation of a series, say:

$$u = \sum_{n=0}^{\infty} u_n.$$

And $N(u)$ as the summation of the following series,

$$N(u) = \sum_{n=0}^{\infty} A_n(u_0, .., u_n). \tag{8}$$

Where A_n is called Adomian polynomials, has been introduced by Adomian, as the following:

$$A_n(u_0, .., u_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\sum_{i=0}^{n} u_i\lambda^i)]\,|_{\lambda=0}. \tag{9}$$

Where λ,is an auxiliary parameter and for functional equations, with several variables, the following extension of (9) can be used.

$$A_n(u_{10}, .., u_{1n}, u_{20}, .., u_{2n}, .., u_{m0}, .., u_{mn}) = \frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\sum_{n=0}^{\infty} u_{1n}\lambda_n, .., \sum_{n=0}^{\infty} u_{mn}\lambda_n)]\,|_{\lambda=0}. \tag{10}$$

where $N(u_1, .., u_n)$ is a functional depending on n variables, each of them is an unknown function which are considered as the summation of series say,

$$u_j = \sum_{n=0}^{\infty} u_{jn}\lambda_n, \quad j = 0, 1, ..., n. \tag{11}$$

3.3 Homotopy perturbation method

In this method the solution is considered as the summation of an infinite series, which usually converges rapidly to the exact solution. To illustrate the basic concepts of this method, consider the following nonlinear differential equations.

$$A(u)-f(r)=0,\quad r\in\Omega. \tag{12}$$

With boundary conditions

$B(u,\frac{\partial u}{\partial n})=0,\quad r\in\Gamma . B(u,\frac{\partial u}{\partial n})=0,\quad r\in\Gamma .$

Where A is a general differential operator, B is a boundary operator, $f(r)$is a known analytic operator, and Γ is the boundary of the domain Ω.

Generally speaking the operator A can be divided into two parts L, and N, where L is linear, and N is a nonlinear operator Eq.(12), therefore, can be rewritten as follow

$$L(u)+N(u)-f(r)=0. \tag{13}$$

Let$^{'}$s construct the Homotopy $V(r,p):\Omega\times[0,1]\rightarrow\Re$, which satisfies:

$$H(v,p)=(1-p)[L(v)-L(u_0)]+p[A(v)-f(r)]=0,r\in\Omega. \tag{14}$$

where u_0 is an initial approximation of the solution of Eq. (12) satisfies boundary condition.

4 Methods for Solving Heston PDE

4.1 VIM to solve Heston PDE

In this part, extends variational iteration method is used to find approximate solution Eq. (2), according to the VIM, we can write the iteration formula as follows:

$$c_{n+1}(s,v,t)=c_n(s,v,t)+\int_0^t\lambda(\tau)\left[\frac{\partial c}{\partial\tau}-rs\frac{\partial c}{\partial s}-(a-bv)\frac{\partial c}{\partial v}-\frac{1}{2}s^2v\frac{\partial^2 c}{\partial s^2}-\rho\delta sv\frac{\partial^2 c}{\partial s\partial v}-\frac{1}{2}\delta^2v\frac{\partial^2 c}{\partial v^2}+rc\right]d\tau, \tag{15}$$

Imposing the stationary condition leads to

$$\delta c_{n+1}(s,v,t)=\delta c_n(s,v,t)+\delta\int_0^t\lambda(\tau)\left[\frac{\partial c}{\partial\tau}-rs\frac{\partial c}{\partial s}-(a-bv)\frac{\partial c}{\partial v}-\frac{1}{2}s^2v\frac{\partial^2 c}{\partial s^2}-\rho\delta sv\frac{\partial^2 c}{\partial s\partial v}-\frac{1}{2}\delta^2v\frac{\partial^2 c}{\partial v^2}+rc\right]d\tau. \tag{16}$$

Or

$$\delta c_{n+1}(s,v,t)=\delta c_n(s,v,t)+\delta\int_0^t\lambda(\tau)\left(\frac{\partial c}{\partial\tau}\right)d\tau. \tag{17}$$

Thus, we have

$$\delta c_{n+1}(s,v,t)=\delta c_n(s,v,t)+\delta\lambda c_n(s,v,t)|_{\tau=t}-\int_0^t\delta\lambda' c_n(s,v,t)d\tau=0. \tag{18}$$

Hence we have the following stationary conditions:

$$\begin{aligned}\lambda'(\tau)&=0|_{\tau=t},\\ 1+\lambda(\tau)&=0|_{\tau=t},\end{aligned} \tag{19}$$

which yields to

$$\lambda(\mathrm{t})=-1. \tag{20}$$

Therefore, we obtain the following iteration formula:

$$c_{n+1}(s,v,t)=c_n(s,v,t)-\int_0^t\left(\frac{\partial c}{\partial\tau}-rs\frac{\partial c}{\partial s}-(a-bv)\frac{\partial c}{\partial v}-\frac{1}{2}s^2v\frac{\partial^2 c}{\partial s^2}-\rho\delta sv\frac{\partial^2 c}{\partial s\partial v}-\frac{1}{2}\delta^2v\frac{\partial^2 c}{\partial v^2}+rc\right)d\tau. \tag{21}$$

4.2 ADM for solve Heston PDE

To solve equation (1a), by ADM, well addressed in [21-22], let's take the following canonical form of the equation, Lets the Eq. (2) have applied and if we used $c(s,v,0)=\max(0,s-K)$as the initial condition by using ADM structure explained we construct:

$$c(s,v,t)=u_0(s,v,0)-\int_0^t-\left(rs\frac{\partial c}{\partial s}+(a-bv)\frac{\partial c}{\partial v}+\frac{1}{2}s^2v\frac{\partial^2 c}{\partial s^2}+\rho\delta sv\frac{\partial^2 c}{\partial s\partial v}+\frac{1}{2}\delta^2v\frac{\partial^2 c}{\partial v^2}-rc\right)dt. \tag{22}$$

Let's take the solution as a series, say $c=c_0+c_1\lambda+c_2\lambda^2+\ldots$ and following an alternate algorithm for Adomian polynomials [13] we get

$$\begin{aligned}A_0&=rs\frac{\partial c_0}{\partial s}+(a-bv)\frac{\partial c_0}{\partial v}+\frac{1}{2}s^2v\frac{\partial^2 c_0}{\partial s^2}+\rho\delta sv\frac{\partial^2 c_0}{\partial s\partial v}+\frac{1}{2}\delta^2v\frac{\partial^2 c_0}{\partial v^2}-rc_0,\\ A_1&=rs\frac{\partial c_1}{\partial s}+(a-bv)\frac{\partial c_1}{\partial v}+\frac{1}{2}s^2v\frac{\partial^2 c_1}{\partial s^2}+\rho\delta sv\frac{\partial^2 c_1}{\partial s\partial v}+\frac{1}{2}\delta^2v\frac{\partial^2 c_1}{\partial v^2}-rc_1,\\ A_3&=rs\frac{\partial c_2}{\partial s}+(a-bv)\frac{\partial c_2}{\partial v}+\frac{1}{2}s^2v\frac{\partial^2 c_2}{\partial s^2}+\rho\delta sv\frac{\partial^2 c_2}{\partial s\partial v}+\frac{1}{2}\delta^2v\frac{\partial^2 c_2}{\partial v^2}-rc_2,\\ &\vdots\end{aligned} \tag{23}$$

and

$$\begin{aligned} c_1(s,v,t) &= -\int_o^t A_0(c_0)\,dt,\\ c_2(s,v,t) &= -\int_o^t A_1(c_0,c_1)\,dt,\\ c_3(s,v,t) &= -\int_o^t A_1(c_0,c_1,c_2)\,dt,\\ &\vdots \end{aligned} \tag{24}$$

All of c_n are calculable, and $C = \sum_{n=0}^{\infty} c_n$.

4.3 HPM for solve Heston PDE

In this section, homotopy perturbation method is used to find approximate the model of Hesston partial differential equation Eq. (2), according to the HPM, we have,

$$H(s,v,t) := (1-p)(L(c)-L(v_0)) + p(\frac{\partial c}{\partial t} - rs\frac{\partial c}{\partial s} - (a-bv)\frac{\partial c}{\partial v} - \frac{1}{2}s^2v\frac{\partial^2 c}{\partial s^2} - \rho\delta sv\frac{\partial^2 c}{\partial s\partial v} - \frac{1}{2}\delta^2 v\frac{\partial^2 u}{\partial v^2} + ru) = 0. \tag{25}$$

Consider $v_0(s,v,0) = 2s^2t^2$ as an initial approximation that satisfies in initial condition. Substituting solution series, into Eq. (25) and equating the terms with identical powers of p,leads to:

$$\begin{aligned} p^0 &:= \tfrac{\partial c_0}{\partial t} - \tfrac{\partial v_0}{\partial t} = 0,\\ p^1 &:= \tfrac{\partial c_1}{\partial t} - rs\tfrac{\partial c_0}{\partial s} - (a-bv)\tfrac{\partial c_0}{\partial v} - \tfrac{1}{2}s^2v\tfrac{\partial^2 c_0}{\partial s^2} - \rho\delta sv\tfrac{\partial^2 c_0}{\partial s\partial v} - \tfrac{1}{2}\delta^2 v\tfrac{\partial^2 c_0}{\partial v^2} + rc_0 + \tfrac{\partial v_0}{\partial t} = 0,\\ p^2 &:= \tfrac{\partial c_2}{\partial t} - rs\tfrac{\partial c_1}{\partial s} - (a-bv)\tfrac{\partial c_1}{\partial v} - \tfrac{1}{2}s^2v\tfrac{\partial^2 c_1}{\partial s^2} - \rho\delta sv\tfrac{\partial^2 c_1}{\partial s\partial v} - \tfrac{1}{2}\delta^2 v\tfrac{\partial^2 c_1}{\partial v^2} + rc_1 = 0,\\ &\vdots \end{aligned} \tag{26}$$

The HPM uses the homotopy parameter p as an expansion parameter to obtain

$$C = \sum_{n=0}^{\infty} p^n c_n, \tag{27}$$

When p ? 1 Eq. (27) corresponds to the original one and will gained the approximate solution of Eq. (2).

Theorem1. If a functional $v(u(x))$ which has a variation achieves a maximum or a minimum at$u = u_0(x)$, where $u(x)$ is an interior point of the domain of definition of the functional, then at $u = u_0(x), \delta v = 0$.

Theorem 2. (Banach's fixed point theorem). Assume that X be a Banach space and

$$A : X \to X$$

Is a nonlinear mapping, and suppose that

$$\left\|A[u] - A[\bar{u}]\right\| \leq \gamma \left\|u - \bar{u}\right\|, \quad u, \bar{u} \in X,$$

For some constant $\gamma < 1$.then A has a unique fixed point. Furthermore, the sequence

$$u_{n+1} = A(u_n),$$

with an arbitrary choice of $u_0 \in X$, converges to the fixed point of A and

$$\|u_k - u_1\| \leq \|u_1 - u_0\| \sum_{j=l-1}^{k-2} \gamma^j.$$

According to Theorem 2, for the nonlinear mapping

$$A[u] = u(x,t) + \int_0^t \lambda F(u, \frac{\partial u}{\partial \tau}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial \tau^2}, \frac{\partial^2 u}{\partial x \partial \tau})\,d\tau, \tag{28}$$

a sufficient condition for convergence of the variational iteration method is strictly contraction of A. Furthermore, the sequence (28) converges to the fixed point of A which also is the solution of the partial differential.

Consider the sequence (28) in the following form:

$$u_{n+1}(x,t) - u_n(x,t) = \int_0^t \lambda F(u, \frac{\partial u}{\partial \tau}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial \tau^2}, \frac{\partial^2 u}{\partial x \partial \tau})\,d\tau, \tag{29}$$

It is clear that the optimal value of λ must be chosen such that extremities the residual functional

$$\int_0^t \lambda F(u, \frac{\partial u}{\partial \tau}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial \tau^2}, \frac{\partial^2 u}{\partial x \partial \tau})\,d\tau,$$

which is equivalent to the extermination of A. But in Theorem 1, the necessary condition for minimization is given.

5 Numerical results

In this section we will discuss numerical example. We tested the performance of ADM, HPM, and VIM on Heston stochastic volatility model.

5.1 Example

$$\frac{\partial c}{\partial t} + rs\frac{\partial c}{\partial s} + (0.16 - 0.055v)\frac{\partial c}{\partial v} + \frac{1}{2}s^2v\frac{\partial^2 c}{\partial s^2} + (-00.45)sv\frac{\partial^2 c}{\partial s\partial v} + \frac{1}{2}(0.81)v\frac{\partial^2 c}{\partial v^2} - rc = 0 \tag{30}$$

$T = 15,\ K = 100,\ a = 0.16,\ b = 0.055,\ \delta = 0.9,\ \rho = -0.5$

Table 1: numerical result of ADM,HPM,VIM

$C(t,s,v)$	C_{ADM}	C_{HPM}	C_{VIM}
C(1,10,0.1)	413.2583333	413.2583333	209.5038332
C(2,50,0.2)	81406.34666	81406.34666	82474.33729
C(4,70,0.3)	1.734519680E6	1.734519680E6	1.147814970E6
C(6,90,0.4)	1.335877440E7	1.335877440E7	1.950175983E7
C(8,120,.05)	7.485848701E7	7.485848702E7	7.796505675E7
C(10,150,06)	2.937374855E8	2.937374856E8	2.317375113E8
C(14,200,08)	2.198733198E9	2.198733197E9	2.576414314E9

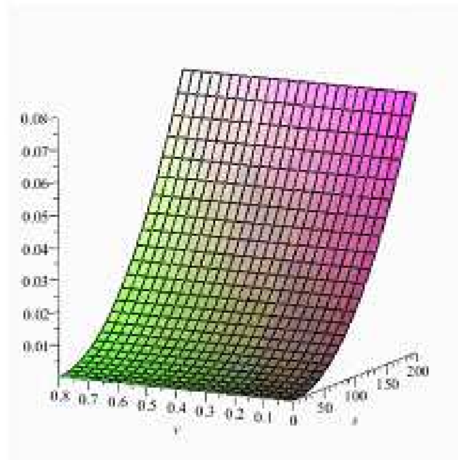

Fig. 2: Solution of European option pricing, Eq. (2), using HPM iterative method

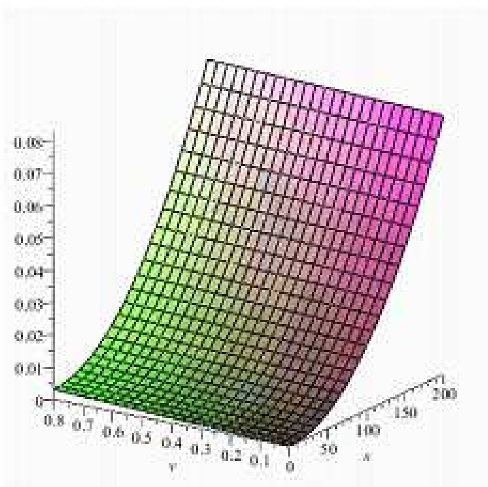

Fig. 1: Solution of European option pricing, Eq. (2), using VIM iterative method

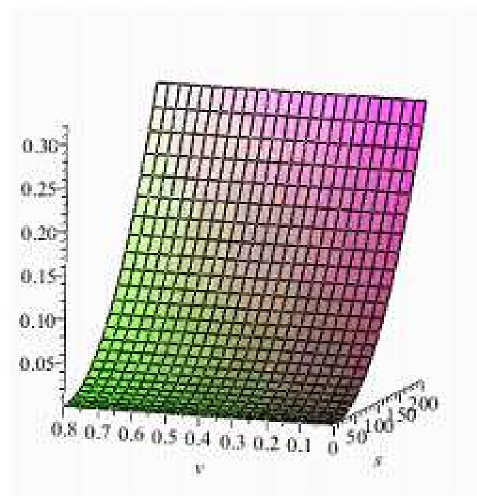

Fig. 3: Solution of European option pricing, Eq. (2), using HPM iterative method

6 Conclusion

The iterative methods have been shown to solve effectively, easily and accurately a large class of nonlinear problems, these methods have been successfully employed to obtain the approximate solution to analytical solution of the Hesston partial differential equation. These techniques are very powerful tools for solving various partial differential equations; Of course one of the main advantages of the variational iteration method over decomposition procedure of Adomian is that the former method provides the solution of the problem without calculating Adomian's polynomials. However the numerical results in last table proved that these method goals in gaining of approximation solution are closed to exact ones.

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