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# A Fractional Calculus Approach to Study Newton's Law of Cooling

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**Abstract:** This paper deals with the application of a novel variable-order and constant-order fractional derivatives in the Newton's law of cooling. The variable-order fractional derivative can be set as a smooth function, bounded on $(0,1]$, while the constant-order fractional derivative can be set as a fractional equation, bounded on $(0,1]$. We solved analytically the fractional equations using the Laplace transform. Numerical simulations were performed for different values of fractional order. The integer-order classical model is recovered when the order of the fractional derivative is equal to 1. Based upon the results obtained, the efficiency rates of the fractional-order operators with non-singular kernel are higher than that of the existing fractional model with singular kernel.

**Keywords:** Fractional operator, Atangana$-$Baleanu fractional derivative Mittag$-$Leffler kernel, Caputo$-$Fabrizio derivative, Caputo generalised fractional derivative, Laplace transform, $\rho-$Laplace transform, modeling problems.

## 1 Introduction

Fractional calculus is a subject of interest for the last two decades. A physical interpretation of equations with fractional derivatives with respect to time is connected with the memory effects. The fractional derivative includes an integral operator of which kernel function is a memory function that involves non-local interaction. It is very useful tool for describing the evolution of system with memory, which typically are dissipative and complex[1,2,3].

The fractional derivatives with power law kernel have disadvantage that their kernel has a singularity, this kernel includes memory effects and therefore these definitions cannot accurately describe these effects [4]. Due to this inconvenience, Caputo and Fabrizio [5] present a new definition of fractional operator with exponential kernel. Properties and applications of this fractional operator are given in [6,7]. Atangana and Baleanu [8] introduced fractional operators with Mittag-Leffler kernel. This kernel considered is non local, non singular and have all benefits of the fractional operators with power law kernel and exponential kernel. Fractional derivative operators without singular kernel have been applied to different systems in [9,10]. Alkahtani *et al.* [11] introduced fractional variable order derivative with non singular kernel. Interesting applications of fractional variable order derivative with non singular kernel can be found in [12,13,14]. The generalized Caputo fractional derivative is introduced by Katungampola [15]. Sene and Gómez-Aguilar [16] has studied the analytical solutions of the electric circuits described by Caputo generalized fractional derivatives. Sene [17] have obtained an analytic solution and numerical solution of certain generalized fractional diffusion equations. Recently, Bhangale and Kachhia studied fractional electromagnetic waves in plasma and dielectric media with Caputo generalized fractional derivative [18].

Newton's law of cooling states that the rate of change of temperature of the body is proportional to the difference between the temperature of the body and that of the surrounding medium [19]

$$\frac{dT(t)}{dt} = \lambda(T(t) - T_e),\ T(0) = T_0, \tag{1.1}$$

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where $T(t)$ is the temperature of the body at any time t, $T_e$ is the environment temperature, $T_0$ is the initial temperature and $\lambda$ is the cooling coefficient (or convective) defined as

$$\lambda = \frac{\alpha A}{mC}, \tag{1.2}$$

where $\alpha$ is the heat transfer coefficient for convection, $A$ is the heat transfer surface area, $m$ is the mass of the body, $C$ is the specific heat. The coefficient $\lambda$ is measured in inverse unity of time, $s^{-1}$. Equation (1.1) predicts that the difference between the initial temperature $T_0$ and surrounding medium temperature $T_e$ drops exponentially.

$$T(t) = T_e + (T_0 - T_e)exp(\lambda t). \tag{1.3}$$

Newton's law of cooling is appeared in a many situations in applied science such as, in materials science, high temperature superconductivity and atmospheric physics [20].

### *1.1 Overview of fractional calculus*

**Definition 1***A real valued function $f(t), t > 0$ is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exist a real number $\rho(> \mu)$ such that $f(t) = t^\rho f_1(t)$, where $f_1 \in C[0,\infty)$, and is said to be in the space $C_\mu^m$ if $f^m \in C_\mu, m \in \mathbb{N} \cup \{0\}$.*

**Definition 2***The Caputo fractional derivative of order $\alpha$ of $f$, $f \in C_{-1}^m, m \in \mathbb{N} \cup \{0\}$ defined as [21]*

$$^{C}D_t^\alpha(f(t)) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \int\limits_0^t \dfrac{f^m(\tau)}{(t-\tau)^{\alpha+1-m}} \, d\tau, & \\ \dfrac{d^m f(t)}{dt^m}, & if\ \alpha = m, \end{cases} \tag{1.4}$$

*where $m-1 < \alpha < m$, $m \in \mathbb{N}$ and $\frac{d^m f(t)}{dt^m}$ is the m-th derivative of the function $f(t)$ with respect to $t$.*

**Definition 3***The Caputo-Fabrizio Fractional derivative operator with order $\nu > 0$ is defined as follows [5]:*

$$^{CFC}D_t^\nu(f(t)) = \frac{M(\nu)}{m-\nu} \int_0^t \frac{d^m}{dt^m}(f(\tau)) exp\left(-\frac{\nu(t-\tau)}{1-\nu}\right) d\tau, \ m-1 < \nu < m, \tag{1.5}$$

*where $M(\nu)$ is a normalization function such that $M(0) = M(1) = 1$.*

The Laplace transform for the Caputo - Fabrizio fractional derivative (1.5) is given by

$$L[^{CFC}D_t^\nu(f(t))] = \frac{s^{m+1}L[f(t)] - s^m f(0) - s^{m-1}f'(0) - \ldots - f^{(m)}(0)}{s + \nu(1-s)}. \tag{1.6}$$

**Definition 4***The Atangana-Baleanu fractional derivative with order $(\nu > 0)$ is defined as follows [8]:*

$$^{ABC}D_t^\nu(f(t)) = \frac{M(\nu)}{1-\nu} \int_0^t \frac{d^n}{dx^n}(f(\tau)) \, E_\nu \left[-\nu \frac{(t-\tau)^\nu}{n-\nu}\right] d\tau, \ m-1 < \nu < m, \tag{1.7}$$

*where $M(\nu)$ is a normalization function such that $M(0) = M(1) = 1$ and Mittag-Leffler function $E_\nu(z)$ is defined as [22]*

$$E_\nu(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\nu k + 1)}, \nu \in C, \Re(\nu) > 0. \tag{1.8}$$

The Laplace transform for the Atangana - Baleanu fractional derivative (1.7) is given by [23]

$$L[^{ABC}D_t^\nu(f(t))] = \frac{M(\nu)}{1-\nu}\left[\frac{s^\nu L[f(t)] - s^{\nu-1}f(0)}{s^\nu + \frac{\nu}{1-\nu}}\right], \ 0 < \nu < 1. \tag{1.9}$$

**Definition 5** *The generalized Mittag-Leffler function $E^{\lambda}_{\delta,\nu}(z)$ is defined as follows [24]*

$$E^{\lambda}_{\delta,\nu}(z)=\sum_{n=0}^{\infty}\frac{(\lambda)_n z^n}{\Gamma(\delta n+\nu)n!},\ (\delta,\nu,\lambda\in\mathbb{C},\Re(\delta)>0,\Re(\nu)>0,\Re(\lambda)>0),$$

*where $(\lambda)_n$ is the Pochhammer symbol*

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)},\ (\lambda)_0=1,\ (\lambda)_n=\prod_{k=1}^{n}(\lambda+k-1),n\geq 1.$$

**Definition 6** *The generalized fractional integral of order $\nu$ of a continuous function $f:[0,+\infty]\to\mathbb{R}$ is defined in [25] as*

$$(I^{\nu,\rho}f)(t)=\frac{1}{\Gamma(\nu)}\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\nu-1}\frac{f(s)ds}{s^{1-\rho}},\tag{1.10}$$

*where $\Gamma(\cdot)$ denotes the gamma function, $\rho>0$, $t>0$ and $0<\nu<1$.*

**Definition 7** *The Caputo generalized fractional derivative of order $\alpha$ of a continuous function $f:[0,+\infty]\to\mathbb{R}$ is defined in [25] as*

$$\left({}^{GC}D^{\nu,\rho}f\right)(t)=\frac{1}{\Gamma(1-\nu)}\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{-\nu}\gamma f(s)\frac{ds}{s^{1-\rho}},\tag{1.11}$$

*where $\rho>0,t>0,\gamma=t^{1-\rho}\frac{d}{dt}$ and $0<\nu<1$.*

**Definition 8** *The $\rho-$Laplace transform of a continuous function $f:[0,+\infty]\to\mathbb{R}$ is defined in [25] as*

$$L_\rho\{f(t)\}(s)=\int_0^\infty e^{-s\frac{t^\rho}{\rho}}f(t)\frac{dt}{t^{1-\rho}}.\tag{1.12}$$

The $\rho-$Laplace transform of the Caputo generalized fractional derivative of a continuous function $f$ is given in [25] as

$$L_\rho\{(D^{\nu,\rho}f)(t)\}=s^\nu L_\rho\{f(t)\}-\sum_{k=0}^{n-1}s^{\nu-k-1}(I^{\nu,\rho}\gamma^n f)(0).\tag{1.13}$$

**Definition 9** *The generalized Atangana-Koca fractional derivative with order $\nu>0$ is defined as [26]:*

$${}^{AKC}D^\nu_t(f(t))=\frac{1}{g(\nu)}\int_0^t f'(\tau)E^\nu_{\nu,\nu}(-g(\nu)(t-\tau)^\nu)d\tau,\ 0<\nu<1,\tag{1.14}$$

*where the function $g(\nu)$ is defined such that*

$$\lim_{\nu\to 0}\frac{1}{g(\nu)}\int_0^t f'(x)E^\nu_{\nu,\nu}(-g(\nu)(t-x)^\nu)dx=\int_0^t\frac{df(x)}{dt}=f(t)-f(0).$$

The Laplace transform for the Atangana Koca fractional derivative (1.14) is given by [26]

$$L[{}^{AK}D^\nu_t f(t)]=\frac{1}{g(\nu)}(sL[f(t)]-f(0))\frac{s^{\nu n-1}}{(1-g(\nu))^\nu}.\tag{1.15}$$

**Definition 10** *For $\alpha,\beta\in\mathbb{C}$ with $\Re(\alpha)>0$ and $\Re(\beta)>0$, the two parameter Mittag-Leffler function is defined as [27]:*

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}.\tag{1.16}$$

Considering $\alpha > 0$ and $\beta > 0$, the Laplace transform of the function $t^{\beta-1}E_{\alpha,\beta}(t^{\alpha})$ is given by [23]:

$$L[t^{\beta-1}E_{\alpha,\beta}(t^{\alpha})] = \frac{s^{\alpha-\beta}}{s^{\alpha}-1}, \ \Re(s) > 1. \tag{1.17}$$

**Definition 11***Let $g(x) \in C^1(a,b), b > a, \ \nu \in [0,1]$ and $f(x)$ a differential function in an open interval $I$ then Atangana-Koca Fractional variable order derivative is defined as [11]:*

$$^{AKV}D_t^{g(x)}(f(t)) = \frac{M(\nu)}{1-\nu}\int_a^t f'(\tau)\ exp(-g(x)(t-\tau))d\tau. \tag{1.18}$$

The Laplace transform for the Atangana-Koca Fractional variable order derivative (1.18) is given by([11])

$$L[^{AKV}D_t^{g(\nu)}(f(t))] = \frac{sL[f(t)]-f(0)}{s+g(\nu)}. \tag{1.19}$$

## 2 Newton's law of cooling with various fractional derivative operators

In this section, we solve Equation (1.1) with various fractional derivative operators using Laplace transform method.

We consider the Newton's law of cooling (1.1) with Caputo fractional derivative [28]

$$^{C}D_t^{\nu}(T(t)) = \lambda(T(t)-T_e). \tag{2.1}$$

The solution of (2.1) is given by [28]

$$T(t) = T_e + (T_0 - T_e)E_{\nu,1}(\lambda t^{\nu}). \tag{2.2}$$

To maintain the fractional differential equation dimensionality, we using the procedure described in [29]. The dimension mismatch can be mathematically corrected considering a parameter $\sigma$ with the dimension of seconds (this parameter is needed on the left-hand side of the equations to maintain a consistent set of units). The auxiliary parameter $\sigma$ is associated with the temporal components in the system (these components change the time constant of the system) [29].

For the fractional derivative with constant order, we have

$$\frac{d}{dt} \rightarrow \frac{1}{\sigma^{1-\nu}}D_t^{\nu}. \tag{2.3}$$

For the fractional derivative with variable order, we have

$$\frac{d}{dt} \rightarrow \frac{1}{\sigma^{1-g(\nu)}}D_t^{g(\nu)}. \tag{2.4}$$

### *2.1 Caputo Fabrizio fractional order derivative*

Let us consider (1.1) via Caputo Fabrizio fractional derivative (1.5) in the following way

$$\left(\frac{1}{\sigma^{1-\nu}}\right){}^{CFC}D_0^{\nu}(T(t)) = \lambda_1(T(t)-T_e), \tag{2.5}$$

where $0 < \nu < 1$.

$$^{CFC}D_0^{\nu}(T(t)) = \lambda(T(t)-T_e).$$

Here we have

$$\lambda = \lambda_1\sigma^{1-\nu}$$

Applying Laplace transform (1.6) to (2.5) and considering $T(0) = T_0$, yields

$$\frac{sL[(T(t))]-T(0)}{s+\nu(1-s)} = \lambda L[T(t)] - \frac{\lambda T_e}{s}.$$

After simplification, we get

$$L[T(t)] = \frac{1}{1-\lambda+\nu\lambda}\left[\frac{T_0}{s-\frac{\nu\lambda}{1-\lambda+\nu\lambda}} - \frac{\lambda T_e(1-\nu)}{s-\frac{\nu\lambda}{1-\lambda+\nu\lambda}} - \frac{\lambda T_e \nu}{s\left(s-\frac{\nu\lambda}{1-\lambda+\nu\lambda}\right)}\right]. \tag{2.6}$$

Applying the inverse Laplace transform in (2.6) and using partial fraction, we get

$$T(t) = \frac{T_0 - \lambda T_e(1-\nu) - T_e}{1-\lambda+\nu\lambda} exp\left(\frac{\lambda\nu t}{1-\lambda+\nu\lambda}\right) + T_e. \tag{2.7}$$

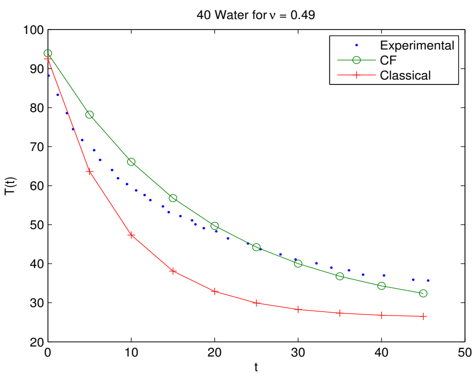


**Fig. 1:** Comparison of CF solution with experimental cooling curve for 40 ml water. The trial with fractional order derivative $\nu = 0.49$.

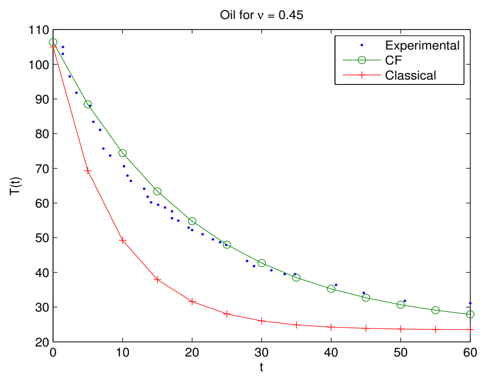


**Fig. 2:** Comparison of CF solution with experimental cooling curve for Oil. The trial with fractional order derivative $\nu = 0.45$.

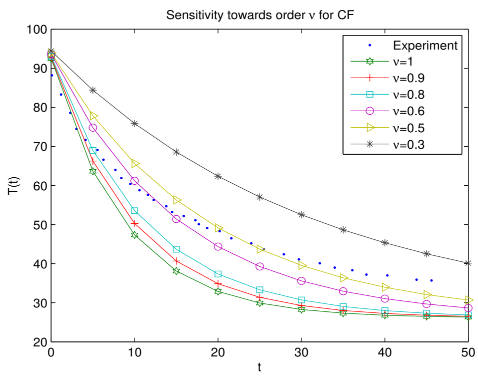


**Fig. 3:** sensitivity towards various $\nu$ for CF

**Table 1:** Comparison of the experimental data with numerical values obtain under CF and Classical derivative for 40ml water.

| t | Experiment | CF | Classical |
|---|---|---|---|
| 0 | 92.7508 | 93.934 | 92.5 |
| 5 | 71.5597 | 78.1971 | 63.6509 |
| 10 | 59.6864 | 66.1111 | 47.336 |
| 15 | 52.7337 | 56.8288 | 38.1095 |
| 20 | 47.8517 | 49.7 | 32.8917 |
| 25 | 44.0043 | 44.225 | 29.9409 |
| 30 | 40.9361 | 40.0202 | 28.2721 |
| 35 | 38.1243 | 36.7908 | 27.3284 |
| 40 | 36.8658 | 34.3106 | 26.7947 |
| 45 | 35.61 | 32.4058 | 26.4929 |
| Error | - | 10.7755 | 36.2225 |

**Table 2:** Comparison of the experimental data with numerical values obtain under CF and Classical derivative for Oil.

| t | Experiment | CF | Classical |
|---|---|---|---|
| 0 | 105.29 | 106.392 | 105 |
| 5 | 82.4516 | 88.4649 | 69.3167 |
| 10 | 68.5161 | 74.4103 | 49.2376 |
| 15 | 59.2258 | 63.3916 | 37.9389 |
| 20 | 51.871 | 54.7531 | 31.5811 |
| 25 | 46.4516 | 47.9805 | 28.0036 |
| 30 | 41.0323 | 42.6709 | 25.9904 |
| 35 | 38.3226 | 38.5082 | 24.8577 |
| 40 | 35.228 | 35.2447 | 24.2202 |
| 45 | 33.6774 | 32.6861 | 23.8615 |
| 50 | 32.129 | 30.6802 | 23.6597 |
| 55 | 30.9677 | 29.1076 | 23.5461 |
| Error | - | 10.45826 | 50.01306 |

Figure-1 and Figure-2 shows the typical experimental cooling curves obtained with 40ml water and oil respectively, and these are compared with the results obtained using the classical integer solution and Caputo-Fabrizio solution. It is clear that the integer order solution fails to fit experimental data, whereas a very good fit is obtained using Caputo-Fabrizio with the fractional order of time derivative $\nu = 0.49$ and 0.45 for 40ml and oil respectively. We have checked the sensitivity of the fit to variations in the $\nu$ value for Caputo-Fabrizio solution. The Table-1 and Table-2 shows the numerical values of the experimental data and the values obtained through simulation for Caputo-Fabrizio fractional derivative. The error norms supports the facts for better performance of the fractional-order models with Caputo-Fabrizio derivative for $\nu = 0.49$ and 0.45 for 40ml water and oil respectively. Figure-3 shows that as $\nu$ decreases the steady state solution is reached at longer times.

## *2.2 Atangana - Baleanu fractional order derivative*

Consider (1.1) via Atangana Baleanu fractional derivative (1.7) in the following way

$$\left(\frac{1}{\sigma^{1-\nu}}\right) {}^{ABC}D_0^{\nu}(T(t)) = \lambda_1(T(t) - T_e), \tag{2.8}$$

where $0 < \nu < 1$.

$${}^{ABC}D_0^{\nu}(T(t)) = \lambda(T(t) - T_e)$$

and

$$\lambda = \lambda_1 \sigma^{1-\nu}.$$

Applying Laplace transform (1.9) to (2.8) and considering $T(0) = T_0$, yields

$$\frac{B(\nu)}{(1-\nu)} \frac{s^\nu L[T(t)]}{s^\nu + \frac{\nu}{1-\nu}} - \frac{B(\nu)}{(1-\nu)} \frac{s^{\nu-1} T(0)}{s^\nu + \frac{\nu}{\nu-1}} = \lambda L[T(t)] - \frac{\lambda T_e}{s}.$$

After simplification, we get

$$L[T(t)] = \frac{1}{B(\nu) - \lambda(1-\nu)} \left[ \frac{(B(\nu)T_0 - (1-\nu)\lambda T_e)s^{\nu-1}}{s^\nu - \frac{\lambda\nu}{B(\nu)\lambda(1-\nu)}} - \frac{\lambda T_e \nu s^{-1}}{s^\nu - \frac{\lambda\nu}{B(\nu)-\lambda(1-\nu)}} \right]. \tag{2.9}$$

Applying the inverse Laplace transform in (2.9) and using partial fraction, we get

$$T(t) = \frac{1}{B(\nu) - \lambda(1-\nu)} \left[ (B(\nu)T_0 - (1-\nu)\lambda T_e) E_{\nu,1}(\Phi) - \lambda T_e \nu t^\nu E_{\nu,\nu+1}(\Phi) \right], \tag{2.10}$$

where $\Phi = \frac{\lambda \nu t^\nu}{B(\nu)-\lambda(1-\nu)}$.

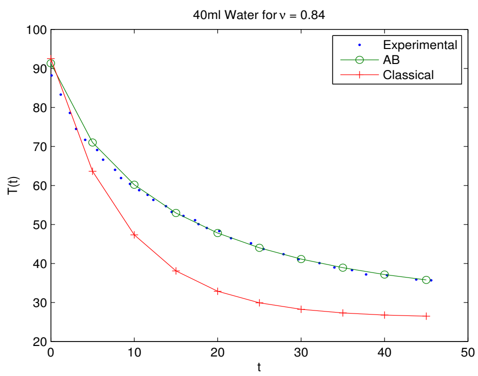


**Fig. 4:** Comparison of AB solution with experimental cooling curve for 40 ml water.The trial with fractional order derivative $\nu = 0.84$.

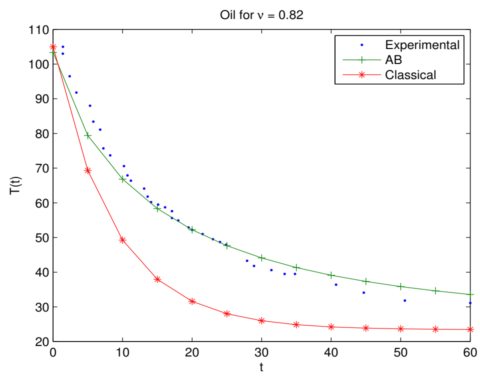


**Fig. 5:** Comparison of AB solution with experimental cooling curve for Oil. The trial with fractional order derivative $\nu = 0.82$.

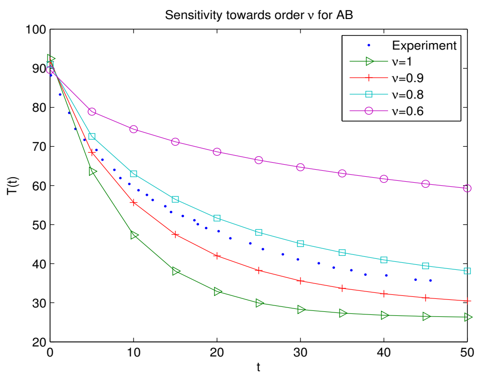


**Fig. 6:** sensitivity towards various $\nu$ for AB

**Table 3:** Comparison of the experimental data with numerical values obtain under AB and Classical derivative for 40ml water.

| t | Experiment | AB | Classical |
|---|---|---|---|
| 0 | 92.7508 | 91.31055 | 92.5 |
| 5 | 71.5597 | 71.017 | 63.6509 |
| 10 | 59.6864 | 60.1872 | 47.336 |
| 15 | 52.7337 | 52.949 | 38.1095 |
| 20 | 47.8517 | 47.8056 | 32.8917 |
| 25 | 44.0043 | 44.0144 | 29.9409 |
| 30 | 40.9361 | 41.1451 | 28.2721 |
| 35 | 38.1243 | 38.9274 | 27.3284 |
| 40 | 36.8658 | 37.183 | 26.7947 |
| 45 | 35.61 | 35.7897 | 26.4929 |
| Error | - | 1.859433 | 36.2225 |

**Table 4:** Comparison of the experimental data with numerical values obtain under AB and Classical derivative for Oil.

| t | Experiment | AB | Classical |
|---|---|---|---|
| 0 | 105.29 | 103.3451 | 105 |
| 5 | 82.4516 | 79.4058 | 69.3167 |
| 10 | 68.5161 | 66.8311 | 49.2376 |
| 15 | 59.2258 | 58.3414 | 37.9389 |
| 20 | 51.871 | 52.221 | 31.5811 |
| 25 | 46.4516 | 47.6365 | 28.0036 |
| 30 | 41.0323 | 44.1084 | 25.9904 |
| 35 | 38.3226 | 41.3356 | 24.8577 |
| 40 | 35.228 | 39.1184 | 24.2202 |
| 45 | 33.6774 | 37.3191 | 23.8615 |
| 50 | 32.129 | 35.8401 | 23.6597 |
| 55 | 30.9677 | 34.6103 | 23.5461 |
| Error | - | 9.601502 | 50.01306 |

Figure-4 and Figure-5 shows the typical experimental cooling curves obtained with 40ml water and oil respectively, and these are compared with the results obtained using the classical integer solution and Atangana-Baleanu solution. It is clear that the integer order solution fails to fit experimental data, whereas a very good fit is obtained using Atangana-Baleanu with the fractional order of time derivative $\nu = 0.84$ and $\nu = 0.82$ for 40ml water and oil respectively. We have checked in Figure-6 the sensitivity of the fit to variations in the $\nu$ value for Atangana-Baleanu solution. The Table-3 and Table-4 shows the numerical values of the experimental data and the values obtained through simulation for Atangana-Baleanu fractional derivative. The error norms supports the facts for better performance of the Atangana-Baleanu fractional-order models with $\nu = 0.84$ and $\nu = 0.82$.

### *2.3 Caputo generalized fractional order derivative*

Consider (1.1) via Generalized Caputo fractional derivative (7) in the following way

$$\left(\frac{1}{\sigma^{1-\nu}}\right) {}^{GC}D_0^{\nu}(T(t)) = \lambda_1(T(t) - T_e), \tag{2.11}$$

where $0 < \nu < 1$.

$${}^{GC}D_0^{\nu}(T(t)) = \lambda(T(t) - T_e),$$

where,

$$\lambda = \lambda_1 \sigma^{1-\nu}$$

Applying $\rho$−Laplace transform (1.13) to (2.11) and considering $T(0) = T_0$, yields

$$s^{\nu} L_{\rho}[T(t)] - s^{\nu-1} T_0 = \lambda L_{\rho}(T(t) - T_e).$$

After simplification, we get

$$L_{\rho}(T(t)) = (T_0 - T_e)\frac{s^{\nu-1}}{s^{\nu} - \lambda} + \frac{T_e}{s}. \quad (2.12)$$

As derived in [16], (2.12) gives

$$T(t) = (T_0 - T_e) E_{\nu}\left(\lambda \left(\frac{t^{\rho}}{\rho}\right)^{\nu}\right) + T_e. \quad (2.13)$$

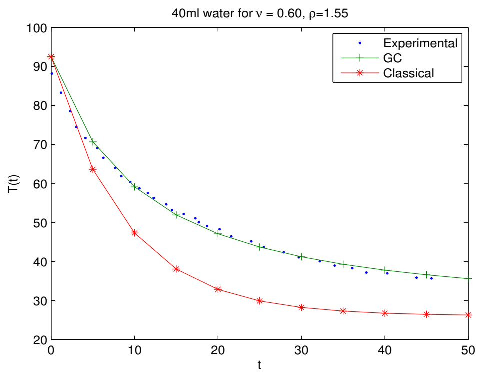


**Fig. 7:** Comparison of GC solution with experimental cooling curve for 40 ml water. The trial with fractional order derivative $\nu = 0.60, \rho = 1.55$.

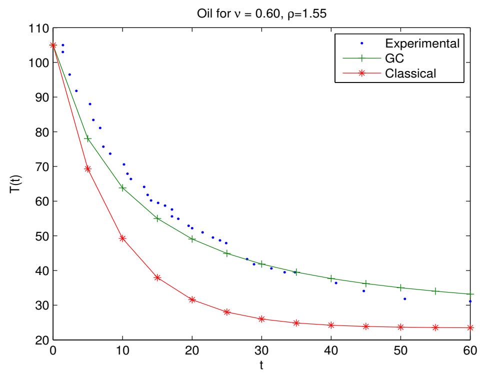


**Fig. 8:** Comparison of GC solution with experimental cooling curve for Oil. The trial with fractional order derivative $\nu = 0.60, \rho = 1.55$.

**Table 5:** Comparison of the experimental data with numerical values obtain under Caputo generalized fractional derivative and Classical derivative for 40ml water.

| t | Experiment | GC | Classical |
|---|---|---|---|
| 0 | 92.7508 | 92.5 | 92.5 |
| 5 | 71.5597 | 70.7029 | 63.6509 |
| 10 | 59.6864 | 59.1521 | 47.336 |
| 15 | 52.7337 | 51.9656 | 38.1095 |
| 20 | 47.8517 | 47.1504 | 32.8917 |
| 25 | 44.0043 | 43.747 | 29.9409 |
| 30 | 40.9361 | 41.2392 | 28.2721 |
| 35 | 38.1243 | 39.3281 | 27.3284 |
| 40 | 36.8658 | 37.8307 | 26.7947 |
| 45 | 35.61 | 36.6301 | 26.4929 |
| Error | - | 2.396482 | 36.2225 |

**Table 6:** Comparison of the experimental data with numerical values obtain under Caputo generalized fractional derivative and Classical derivative for Oil.

| t | Experiment | GC | Classical |
|---|---|---|---|
| 0 | 105.29 | 105 | 105 |
| 5 | 82.4516 | 78.0403 | 69.3167 |
| 10 | 68.5161 | 63.8129 | 49.2376 |
| 15 | 59.2258 | 54.9865 | 37.9389 |
| 20 | 51.871 | 49.0849 | 31.5811 |
| 25 | 46.4516 | 44.9202 | 28.0036 |
| 30 | 41.0323 | 41.8549 | 25.9904 |
| 35 | 38.3226 | 39.5209 | 24.8577 |
| 40 | 35.228 | 35.0315 | 24.2202 |
| 45 | 33.6774 | 36.2287 | 23.8615 |
| 50 | 32.129 | 35.0315 | 23.6597 |
| 55 | 30.9677 | 34.0363 | 23.5461 |
| Error | - | 9.8104 | 50.01306 |

Figure-7 and Figure-8 shows the typical experimental cooling curves obtained with 40ml water and oil respectively, and these are compared with the results obtained using the classical integer solution and Caputo generalized solution. It is clear that the integer order solution fails to fit experimental data, whereas a very good fit is obtained using Caputo generalized with the fractional order of time derivative $\nu = 0.60, \rho = 1.55$ for both 40ml water and oil. The Table-5 and Table-6 shows the numerical values of the experimental data and the values obtained through simulation for Caputo generalized fractional derivative. The error norms supports the facts for better performance of the Caputo generalized fractional-order models with $\nu = 0.60, \rho = 1.55$.

## *2.4 Atangana - Koca fractional order derivative*

Consider (1.1) via Atangana Koca fractional derivative (1.14) in the following way

$$\left(\frac{1}{\sigma^{1-\nu}}\right) {}^{AKC}D_0^{\nu}(T(t)) = \lambda_1(T(t) - T_e), \tag{2.14}$$

where $0 < \nu < 1$.

$${}^{AKC}D_0^{\nu}(T(t)) = \lambda(T(t) - T_e).$$

where,

$$\lambda = \lambda_1 \sigma^{1-\nu}.$$

Applying Laplace transform (1.15) to (2.14) and considering $T(0) = T_0$, yields

$$\frac{L[T(t)]s^{-n\nu} - T_0 s^{-n\nu-1}}{g(\nu)(1-g(\nu))^{\nu}} = \lambda L[T(t)] - \frac{\lambda T_e}{s}.$$

After simplification, we get

$$L[(t)] = -\frac{T_0 s^{-1}}{\lambda g(\nu)(1-g(\nu))^{\nu}[s^{n\nu} - \frac{1}{\lambda g(\nu)(1-g(\nu))^{\nu}}]} + \frac{T_e s^{n\nu-1}}{s^{n\nu} - \frac{1}{\lambda g(\nu)(1-g(\nu))^{\nu}}}. \tag{2.15}$$

Applying the inverse Laplace transform in (2.15) and using partial fraction, we get

$$T(t) = -\frac{t^{n\nu}T_0}{\lambda g(\nu)(1-g(\nu))^{\nu}} E_{n\nu,n\nu+1}\left(\frac{t^{n\nu}}{\lambda g(\nu)(1-g(\nu))^{\nu}}\right) + T_e E_{n\nu,1}\left(\frac{t^{n\nu}}{\lambda g(\nu)(1-g(\nu))^{\nu}}\right) \tag{2.16}$$

### *2.5 Atangana - Koca fractional variable order derivative*

Consider (1.1) via Atangana Koca fractional variable order derivative (1.18) in the following way

$$\left(\frac{1}{\sigma^{1-g(\nu)}}\right)^{AKV} D_0^{g(\nu)}(T(t)) = \lambda_1(T(t) - T_e), \tag{2.17}$$

$$^{AKV}D_0^{\nu}(T(t)) = \lambda(T(t) - T_e),$$

where,

$$\lambda = \lambda_1 \sigma^{1-g(\nu)}$$

Applying Laplace transform (1.19) to (2.17) and considering $T(0) = T_0$, yields

$$\left(\frac{s}{s+g(\nu)} - \lambda\right) L[T(t)] = \frac{T_0}{s+g(\nu)} - \frac{\lambda T_e}{s}.$$

After simplification, we get

$$L[(t)] = \frac{T_0}{(1-\lambda)\left[s - \frac{\lambda g(\nu)}{1-\lambda}\right]} - \frac{\lambda T_e}{(1-\lambda)\left[s - \frac{\lambda g(\nu)}{1-\lambda}\right]} - \frac{\lambda T_e g(\nu)}{(1-\lambda)s\left[s - \frac{\lambda g(\nu)}{1-\lambda}\right]}. \tag{2.18}$$

Applying the inverse Laplace transform in (2.18) and using partial fraction, we get

$$T(t) = \left(\frac{T_0 - \lambda T_e}{1-\lambda}\right) exp\left(\frac{\lambda g(\nu)t}{1-\lambda}\right) - \frac{\lambda T_e g(\nu)}{1-\lambda} E_{1,2}\left(\frac{\lambda g(\nu)t}{1-\lambda}\right). \tag{2.19}$$

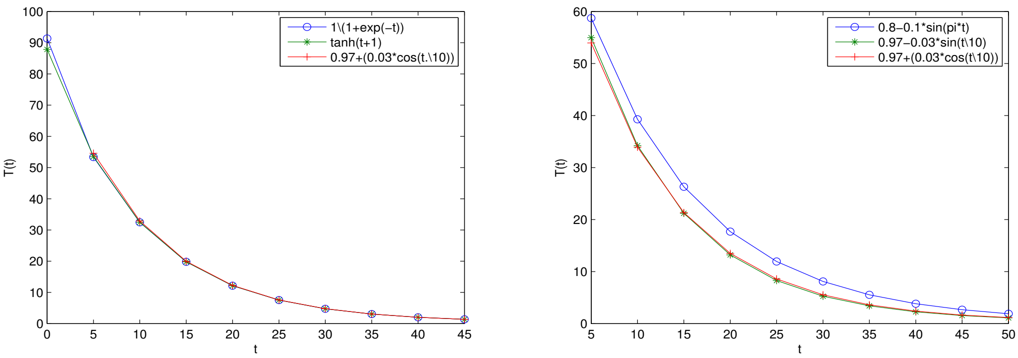


**Fig. 9:** Newton's law of cooling involving the Atangana-Koca fractional derivative with variable order.

Numerical solutions of (2.19) have been depicted in figure-9, for different values of the fractional order $g(\nu)$. The numerical results indicate that the fractional order has an important influence on the temperature and the general solution of the fractional equations depends on the parameter $\nu$ and when the order is a function rather than a constant of arbitrary order $g(\nu)$, respectively. These solutions represent a new family of solutions for the Newton's law of cooling, which allows for the possibility of multiple solutions that are not observed in experiments.

## 3 Conclusion

We aimed to study behaviour of solution of Newton's law of cooling using fractional operator. For this purpose, we compare the various fractional derivative results with experimental results given in [28] for Newton's law of cooling. The experimental cooling curves obtained with 40ml water and oil and these are compared with the results obtained using constant and variable order fractional derivative. To test the accuracy of the model with various fractional derivative an experiment was carried out. With a Borosil beaker filled with 40ml of water and oil heated to a specific temperature. Its cooling data were then noted with time, the results obtained can be viewed in [28].

Based on experimental data, our aim is to test several fractional differential equations with various fractional derivative operators and see which ones better describe the problems using method given in [30].

We also calculated the error norms which supports the facts for better performance of operators. In the solution obtain using Caputo Fabrizio observed that an order $\nu = 0.49$ and $\nu = 0.45$ are best fit compared to experiment for 40ml water and oil respectively. Also in the solution obtain using Atangana Baleanu the best fit noted for an order $\nu = 0.84$ and $\nu = 0.82$ for 40ml water and oil respectively. In the solution using Caputo generalised fractional derivative best fit as a minimum error observed for an order $\nu = 0.60, \rho = 1.55$ for both 40ml water and oil.

Among all derivatives Caputo Fabrizio, Atangana Baleanu and Caputo generalised fractional derivative, we observed the Atangana Baleanu fractional derivative is best fit for these experiments as it has minimum error norm. The Mittag-Leffler and exponential decay laws can capture both Gaussian and non-Gaussian aspects due to their associated density distribution. The model based on the classical differentiation cannot capture such phenomena due to its Markovian property.

The experimental data presented a distribution similar to non-Gaussian distribution. The comparison of experimental data and mathematical models in particular those with non-local differential operators are in good agreement. This can be explained with the fact that the power-law kernel possess density distribution that is non-Gaussian as the per the probability distribution which in statistics does not have good statistical properties. We also obtained the solution of Newton's law of cooling for variable orders. It may help in future research work.

**Conflicts of Interests**

The authors declare that they have no conflicts of interests.

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