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Function of Reachability for Autonomous Systems of Differential Equations

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Abstract: The initial value problems for autonomous systems of differential equations are the subject of this paper. In the phase space of such system is defined the so-called reachable set and a function of reachability is introduced. For each starting point x_0, there is a corresponding function value which is equal to the time necessary to pass from x_0 to the reachable set. Some properties of the function of reachability: continuity, boundedness and more are studied. A generalized model of interaction (competition) of two species, located in the same nutrient medium is considered.

Keywords: initial value problems, autonomous differential equations, reachable sets, function of reachability

1 Introduction

Many dynamic processes change sharply (abruptly) their state as a result of brief (instantaneous) external influences. Such processes are modeled using impulsive differential equations (see [1,3,8,9,12,16,21,22,24,25,27,29,31] and [32]). The determination of the impulsive moments (the exact moments at which the short term external influences take place) is a key element of this type of equations.

The subject is examined in a number of articles and monographs, such as: [5,6,7,14,19,26,28] and [30]. The equations with fixed impulsive moments are studied most completely. The equations with non-fixed moments are divided into several classes. In one of the major classes, the impulsive moments coincide with the moments when the trajectory of the corresponding initial value problem reaches a pre-defined set which is located in the phase space. An important question is to determine the conditions which ensure that the trajectories of the considered equation cross the reachable set. Our paper is devoted to this problem.

Let G be a phase space of an autonomous system of differential equations. Let a set $\Phi \subset G$. If the trajectory of the system considered starts from point $x_0 \in G$ and crosses the set Φ, then x_0 is named a starting point of reachability and Φ is a set of reachability. Some topological properties of the set of all starting points of reachability are studied in [15] and [23]. These studies are developed here. For each starting point, a function of reachability is defined. The functions value is equal to the time necessary to reach the set Φ, starting from x_0. The paper analyses some qualitative properties, such as continuity, boundedness, etc. of the function of reachability. The main limitation of the studied autonomous systems is to have uniformly Lipschitz solutions (see [2], [4], [11], [13] and [18]).

2 Statement of the problem and preliminary remarks

Denote the Euclidean norm and dot product in R^n by $\|.\|$ and $\langle .,. \rangle$, respectively. For the points $a(a_1,a_2,\dots,a_n)$ and $b(b_1,b_2,\dots,b_n)$ in R^n, we have

$$\langle a,b \rangle = a_1b_1 + a_2b_2 + \cdots + a_nb_n;$$

$$\|a\| = \langle a,a \rangle^{\frac{1}{2}} = \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)^{\frac{1}{2}}.$$

The Euclidean distance between nonempty sets A and B, $A,B \subset R^n$, is denoted by

$$\rho(A,B) = inf\left\{\|a-b\|;\ a \in A,\ b \in B\right\}.$$

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An open ball with center $x_0 \in R^n$ and radius $\delta = const > 0$ is denoted by

$$B_\delta(x_0) = \left\{x \in R^n;\ \|x - x_0\| < \delta\right\}.$$

For a neighborhood of the radius δ around the set A is used the notation:

$$B_\delta(A) = \left\{x \in R;\ \rho(x,A) < \delta\right\}.$$

$\overline{A}$ and ∂A are notations for the closure and boundary of the set A.

The length of the curve γ is denoted by $l[\gamma]$. The closed segment with endpoints a and b is denoted by

$$[a,b] = \{c_\lambda \in R^n;\ c_\lambda = (1-\lambda)a + \lambda b,\ 0 \le \lambda \le 1\}.$$

Definition 2.1. The curve γ is said to be p-linear, if

$$(\exists g_0, g_1, \ldots, g_p \in R^n):$$
$$\gamma = [g_0, g_1] \cup [g_1, g_2] \cup \cdots \cup [g_{p-1}, g_p].$$

That is to say, p-linear curve is composed by p sequentially connected line segments.

Definition 2.2. [15] The domain G is said to be p-convex, where p is a natural number, if

$$(\forall g', g'' \in G)\left(\exists \gamma = \cup_{i=1}^{p} [g_{i-1}, g_i] \subset G\right):$$
$$g_0 = g',\ g_p = g''.$$

In other words, any two points of G can be connected by p-linear curve from G. It is clear that each 1-convex domain is convex.

Definition 2.3. [15] The domain G is said to be bounded-connected, if

$$(\exists l_0 = const > 0)(\forall g', g'' \in G)(\exists \gamma \subset G\,;\ g', g'' \in \gamma):$$
$$l[\gamma] \le l_0.$$

Further, we shall use the following theorem, which proof is elementary.

Theorem 2.1. Assume that:

1. The sets A and B are normed spaces. The sequences $\{a_n\} \subset A$ and $\{b_n\} \subset B$.
2. $(\exists C = const > 0) : (\forall n, m \in N)$ $\Rightarrow \|b_n - b_m\| \le \|a_n - a_m\|$.
3. The sequence $\{a_n\}$ is fundamental.

Then:

1. The sequence $\{b_n\}$ is fundamental.
2. If B is a full space, then the sequence $\{b_n\}$ is convergent.

Theorem 2.2. Assume that:

1. The set $G \subset R^n$, $G \ne \emptyset$ and G is a domain. The function $f : G \to R^+$.
2. The function $g \in C[R^+, R^+]$ and g is monotonically increasing in R^+.
3. It is fulfilled

$$(\exists x_0 \in G) : (\exists \delta = \delta(x_0) > 0) : \big(\forall x \in B_\delta(x_0) \cap G\big)$$
$$\Rightarrow \left|f(x) - f(x_0)\right| \le g\big(\min\{f(x), f(x_0)\}\big)\|x - x_0\|.$$

Then:

1. The function f is continuous in x_0.
2. The function f is bounded in $B_\delta(x_0) \cap G$.

Proof. Let $\varepsilon = const > 0$. We choose the constant δ_1 so that

$$0 < \delta_1 < \min\left\{\delta, \frac{\varepsilon}{g\big(f(x_0)\big)}\right\}.$$

Then

$$\big(\forall x \in B_\delta(x_0)\big) \cap G \Rightarrow$$
$$\begin{aligned}\left|f(x) - f(x_0)\right| &\le g\Big(\min\{f(x), f(x_0)\}\Big)\|x - x_0\| \\ &\le \min\Big\{g\big(f(x)\big), g\big(f(x_0)\big)\Big\}\frac{\varepsilon}{g\big(f(x_0)\big)} \\ &\le g\big(f(x_0)\big)\frac{\varepsilon}{g\big(f(x_0)\big)} = \varepsilon.\end{aligned}$$

Hence, the function f is continuous at x_0.

Let x be an arbitrary point in $B_\delta(x_0) \cap G$. Then

$$\begin{aligned}\left|f(x) - f(x_0)\right| &\le g\Big(\min\{f(x), f(x_0)\}\Big)\|x - x_0\| \\ &\le g\big(f(x_0)\big)\|x - x_0\| \le g\big(f(x_0)\big)\delta.\end{aligned}$$

From the last inequality, it follows that

$$f(x) \le f(x_0) + g\big(f(x_0)\big)\delta = const,$$

i.e. f is bounded.

The theorem is proved.

Corollary 2.1. Assume that:

1. The set $G \subset R^n$, $G \ne \emptyset$ and G is a domain. The function $f : G \to R^+$.
2. The function $g \in C[R^+, R^+]$ and g is monotonically decreasing in R^+.
3. It is fulfilled

$$(\exists x_0 \in G) : (\exists \delta = \delta(x_0) > 0) : \big(\forall x \in B_\delta(x_0) \cap G\big)$$
$$\Rightarrow \left|f(x) - f(x_0)\right| \le g\big(max\{f(x), f(x_0)\}\big)\|x - x_0\|.$$

Then:

1. The function f is continuous in x_0.
2. The function f is bounded in $B_\delta(x_0) \cap G$.

Consider the following initial value problem

$$\frac{dx}{dt} = f(x), \quad x(0) = x_0, \tag{1}$$

where:

- The function $f : G \to R^n$;
- The set $G \subset R^n$, $G \ne \emptyset$ and G is a domain (an open and connected set);

- $x_0 \in G$.

The solution of problem (1) is denoted by $x(t;x_0)$. Let $\gamma(\theta,x_0)$ be the trajectory of (1), locked between the points $x(0;x_0)=x_0$ and $x(\theta;x_0)$, where $\theta \in R$. It is satisfied

$$\gamma(\theta,x_0)=\begin{cases} x=x(t;x_0);\ 0\le t\le \theta,\ \text{if } \theta>0;\\ x=x(t;x_0);\ \theta<t\le 0,\ \text{if } \theta<0.\end{cases}$$

In particular

$$\gamma(\infty,x_0)=\big\{x=x(t;x_0);\ 0\le t<\infty\big\}$$

and

$$\gamma(-\infty,x_0)=\big\{x=x(t;x_0);\ -\infty<t\le 0\big\}.$$

Definition 2.4. [23] Assume that:

1. The sets $X_0^+, \Phi \subset G$, $X_0^+ \neq \emptyset$ and $\Phi \neq \emptyset$.
2. For each point $x_0 \in X_0^+$, the solution $x(t;x_0)$ of the initial value problem (1) is defined and unique in the interval $[0,\infty)$.
3. It is valid

$$(\forall x_0 \in X_0^+)\big(\exists \theta=\theta(x_0)>0\big): x(\theta;x_0)\in \Phi.$$

Then, we say that:

1. Φ is a positive reachable set from X_0^+ via system (1);
2. If $X_0^+=G$, then Φ is a totally positive reachable set via system (1);
3. X_0^+ is a positive initial set for system (1);
4. Each point $x_0 \in X_0^+$ is a positive starting point (of reachability) for system (1).

Likewise we define the concepts of:

1. Negative reachable set from the set X_0^- via system (1);
2. Totally negative reachable set via system (1);
3. Negative initial set for system (1);
4. Negative starting point of reachability for system (1).

Note that various configurations are possible for the sets X_0^+ and X_0^-. For example, it is possible $X_0^- \cap X_0^+ = \emptyset$. It is also possible to find a system for which $X_0^- = X_0^+$.

Since the set Φ is a positive and negative reachable, from now on we will name Φ a reachable set for system (1). Furthermore, in the next research, the terminology introduced above will be applied to system (1) and this detail will be omitted. For convenience, the sets of all starting points of positive reachability and all starting points of negative reachability will be denoted by X_0^+ and X_0^-, respectively. Finally, $X_0 = X_0^- \cup X_0^+ \cup \Phi$ is named a starting set.

Definition 2.5. [13]. We say that the solutions of system (1) are uniformly Lipschitz stable, if

$$\begin{gathered}(\exists L=const>0)(\exists \delta_L=const>0):\\ \big(\forall x_{01},x_{02}\in G,\ \|x_{01}-x_{02}\|<\delta_L\big)\\ \Rightarrow \big\|x(t;x_{01})-x(t;x_{02})\big\|<L\|x_{01}-x_{02}\|,\ t>0.\end{gathered}$$

The uniform Lipschitz stability was introduced in 1986 by F. Dannan and S. Elaydi in [11].

We introduce the following conditions:

H1. There exists a constant $C_{Lip}>0$ such that

$$(\forall x',x''\in G)\Rightarrow \big\|f(x')-f(x'')\big\|\le C_{Lip}\|x'-x''\|.$$

H2. There exists a constant $C_f>0$ such that

$$(\forall x\in G)\Rightarrow \big\|f(x)\big\|\le C_f.$$

H3. For each point $x_0\in G$, the solution of initial value problem (1) exists and is unique in R.

H4. The function $\varphi\in C[D,R]$ and $\varphi\in C^1[\Phi,R]$, where the domain $D\subset G$. The reachable set

$$\Phi=\big\{x\in D;\ \varphi(x)=0\big\}\neq\emptyset.$$

There exists a constant $C_{\langle grad\varphi,f\rangle}>0$ such that

$$(\forall x\in\Phi)\Rightarrow\big\langle grad\varphi(x),f(x)\big\rangle\ge C_{\langle grad\varphi,f\rangle}.$$

H5. The set Φ is connected.

H6. The inclusion $\overline{\Phi}\backslash\Phi\subset\partial G$ is satisfied.

H7. There exists a constant C_φ such that

$$(\forall x\in D)\Rightarrow\big|\varphi(x)\big|\le C_\varphi\rho(x,\Phi).$$

The following theorem contains the main results obtained in [15] and [23]. The results of these articles are the fundament on which the current paper is based.

Theorem 2.3. [15,23]. Assume that:

1. The conditions H1, H3 and H4 hold.
2. The set Φ is reachable from the sets X_0^- and X_0^+.

Then:

1. If $x_0\in X_0^-$, then the trajectory $\gamma(\theta,x_0)\subset X_0^-$, where the negative constant θ is determined such that $x(\theta;x_0)\in\Phi$ and $x(t;x_0)\notin\Phi$ for $\theta<t\le 0$.
2. If $x_0\in X_0^+$, then the trajectory $\gamma(\theta,x_0)\subset X_0^+$, where the positive constant θ is chosen such that $x(\theta;x_0)\in\Phi$ and $x(t;x_0)\notin\Phi$ for $0\le t<\theta$.
3. If $x_0\in X_0^-$, then the trajectory $\gamma(\infty,x_0)\subset X_0^-$.
4. If $x_0\in X_0^+$, then the trajectory $\gamma(-\infty,x_0)\subset X_0^+$.
5. The sets $X_0^-\neq\emptyset$ and $X_0^+\neq\emptyset$.
6. It is valid $\Phi\subset\overline{X_0^-}$ and $\Phi\subset\overline{X_0^+}$.
7. The sets X_0^- and X_0^+ are open.
8. The set $X_0=X_0^-\cup X_0^+\cup\Phi$ is open.
9. If in addition condition H5 is satisfied, then the set X_0 is connected.

10. For each point $x_0 \in X_0^-$, we have $\gamma(-\infty, x_0) \subset X_0$.
11. For each point $x_0 \in X_0^+$, we have $\gamma(\infty, x_0) \subset X_0$.
12. For each point $x_0 \in (\partial X_0 \backslash \Phi) \cap G$, it is satisfied $\gamma(\infty, x_0) \subset (\partial X_0 \backslash \Phi) \cap G$.
13. If in addition the conditions H5 and H6 are satisfied, then $\overline{\Phi} \backslash \Phi \subset \partial X_0$.

Definition 2.6. The function $\Theta^+ : X_0^+ \to R^+$, which relates a positive constant $\theta = \Theta(x_0)$ to each point $x_0 \in X_0^+$ such that $x(\theta; x_0) \in \Phi$ and $x(t; x_0) \notin \Phi$ for $0 \le t < \theta$, is called a function of reachability, i.e.

$$(\forall x_0 \in X_0^+)\big(\exists \theta = \Theta^+(x_0) \in R^+\big) :$$

- $x(\theta; x_0) \in \Phi$;
- $\big(\forall t,\ 0 \le t < \theta = \Theta^+(x_0)\big) \Rightarrow x(t; x_0) \notin \Phi$.

The function $\Theta^- : X_0^- \to R^-$ is defined similarly.

3 Main results

Theorem 3.1. Assume that the conditions H1-H7 hold. Then $\Theta^- \in C[X_0^-, R^-]$ and $\Theta^+ \in C[X_0^+, R^+]$.

Proof. We shall prove the second statement of the theorem. Let the constant λ satisfies the inequalities $0 < \lambda < 1$ and the points $x_0^*, x_0 \in X_0^+$. Consider the positive constants $\theta^* = \Theta^+(x_0^*)$ and $\theta = \Theta(x_0)$, i.e. $x(\theta^*; x_0^*) \in \Phi$ and $x(\theta; x_0) \in \Phi$. In other words, $\varphi\big(x(\theta^*; x_0^*)\big) = 0$ and $\varphi\big(x(\theta; x_0)\big) = 0$ is valid. For convenience, assume that $\theta \le \theta^*$. For $t \ge 0$, we have

$$\begin{aligned} x(t; x_0^*) =& x_0^* + \int_0^t f\big(x(\tau; x_0^*)\big) d\tau; \\ x(t; x_0) =& x_0 + \int_0^t f\big(x(\tau; x_0)\big) d\tau, \end{aligned}$$

from which by condition H1, we obtain

$$\begin{aligned} &\big\| x(t; x_0^*) - x(t; x_0) \big\| \\ &\quad \le \| x_0^* - x_0 \| + \int_0^t C_{Lip} \big\| x(\tau; x_0^*) - x(\tau; x_0) \big\| d\tau, \end{aligned}$$

Using Gronwalls inequality, we get the estimate

$$\big\| x(t; x_0^*) - x(t; x_0) \big\| \le \| x_0^* - x_0 \| \exp(C_{Lip} t).$$

From the above inequality for $t = \theta$ we find

$$\big\| x(\theta; x_0^*) - x(\theta; x_0) \big\| \le \| x_0 - x_0^* \| \exp(C_{Lip} \theta). \qquad (2)$$

We extend the functions f and φ continuously over the points of set $\overline{\Phi} \backslash \Phi$. According to condition H4, for each point $x \in \overline{\Phi} \backslash \Phi$, the following inequality is fulfilled

$$\begin{aligned} &\big\langle grad\,\varphi(x), f(x) \big\rangle \\ &= \lim\nolimits_{x* \to x,\ x^* \in \Phi} \big\langle grad\,\varphi(x^*), f(x^*) \big\rangle \ge C_{\langle grad\varphi, f \rangle}. \end{aligned}$$

As $\overline{\Phi}$ is a compact set (closed and bounded) from the last inequality, it follows that

$$\begin{aligned} (\forall \lambda,\ 0 < \lambda < 1)&\big(\exists \delta_{\overline{\Phi}} = \delta_{\overline{\Phi}}(\lambda) > 0\big) : \\ &\big(\forall x \in B_{\delta_{\overline{\Phi}}}(\overline{\Phi}) \cap D\big) \\ &\Rightarrow \big\langle grad\,\varphi(x), f(x) \big\rangle \ge \lambda . C_{\langle grad\varphi, f \rangle}. \end{aligned} \qquad (3)$$

Futher, we assume that

$$\| x_0^* - x_0 \| < \delta_{\overline{\Phi}} . \exp(-C_{Lip} \theta),$$

from which, according to (2) it follows that

$$\big\| x(\theta; x_0^*) - x(\theta; x_0) \big\| \le \delta_{\overline{\Phi}}.$$

Since the point $x(\theta; x_0) \in \Phi$ then from the above inequality, we have

$$\rho\big(x(\theta; x_0^*), \Phi\big) \le \delta_{\overline{\Phi}}.$$

Therefore, the point $x(\theta; x_0^*) \in B_{\delta_{\overline{\Phi}}}(\overline{\Phi}) \cap D$. From (4), we find that

$$\Big\langle grad\,\varphi\big(x(\theta; x_0^*)\big), f\big(x(\theta; x_0^*)\big) \Big\rangle \ge \lambda . C_{\langle grad\varphi, f \rangle}. \qquad (4)$$

By condition H7, we conclude that

$$\Big| \varphi\big(x(\theta; x_0^*)\big) \Big| \le C_\varphi . \rho\big(x(\theta; x_0^*), \Phi\big). \qquad (5)$$

Using that the points $x(\theta; x_0),\ x(\theta^*; x_0^*) \in \Phi$ and the estimates (4) and (5), we find

$$\begin{aligned} &\big\| x(\theta; x_0^*) - x(\theta; x_0) \big\| \\ &= \rho\big(x(\theta; x_0^*), x(\theta; x_0)\big) \\ &\ge \rho\big(x(\theta; x_0^*), \Phi\big) \\ &\ge \frac{1}{C_\varphi} \Big| \varphi\big(x(\theta; x_0^*)\big) \Big| \\ &= \frac{1}{C_\varphi} \Big| \varphi\big(x(\theta^*; x_0^*)\big) - \varphi\big(x(\theta; x_0^*)\big) \Big| \\ &= \frac{1}{C_\varphi} . \frac{d}{dt} \big(\varphi\big(x(\theta'; x_0^*)\big)\big)(\theta^* - \theta) \\ &= \frac{1}{C_\varphi} . \Big\langle grad\,\varphi\big(x(\theta'; x_0^*)\big), f\Big(\varphi\big(x(\theta'; x_0^*)\big)\Big) \Big\rangle (\theta^* - \theta) \\ &\ge \frac{1}{C_\varphi} . \lambda . C_{\langle grad\varphi, f \rangle} (\theta^* - \theta), \end{aligned}$$

where point θ' satisfies the inequalities $\theta < \theta' < \theta^*$. From the above estimate and (2), it follows that

$$\begin{aligned} \theta^* - \theta &\le \frac{C_\phi}{\lambda . C_{\langle grad\varphi, f \rangle}} \big\| x(\theta^*; x_0^*) - x(\theta; x_0) \big\| \\ &\le \frac{C_\phi}{\lambda . C_{\langle grad\varphi, f \rangle}} \| x_0^* - x_0 \| \exp(C_{Lip} \theta) \end{aligned} \qquad (6)$$

Using the definition of function Θ^+, we can rewrite (6) as follows

$$\begin{aligned} & \left|\Theta^+(x_0^*) - \Theta(x_0)\right| \qquad (7) \\ = {} & \Theta^+(x_0^*) - \Theta(x_0) \\ = {} & \theta^* - \theta \\ \leq {} & \frac{C_\phi}{\lambda . C_{\langle grad\varphi, f\rangle}} \|x_0^* - x_0\| \exp\left(C_{Lip} \min\left\{\Theta^+(x_0^*), \Theta^+(x_0)\right\}\right) \\ \leq {} & g\left(\min\left\{\Theta^+(x_0^*), \Theta^+(x_0)\right\}\right) \|x_0^* - x_0\|. \end{aligned}$$

In the previous inequality, the function $g : R^+ \to R^+$ is given analytically

$$g(t) = \frac{C_\phi}{\lambda . C_{\langle grad\varphi, f\rangle}} \exp(C_{Lip} t),\ t \in R^+.$$

It is clear that g is continuous and monotonically increasing. From the inequality (7), applying Theorem 2.2, it follows that Θ^+ is continuous at x_0, from where we deduce that $\Theta^+ \in C[X_0^+, R^+]$.

The theorem is proved.

Theorem 3.2. Assume that:

1. The conditions H1-H7 hold.
2. The solutions of system (1) are uniformly Lipschitz stable.
3. The domain X_0^+ is convex and bounded.

Then the function Θ^+ is bounded in X_0^+.

Proof. For convenience, we divide the proof into several parts.

Part 1. Let x_{00} be a fixed point from X_0^+. Since X_0^+ is bounded then

$$(\exists \delta_{X_0^+} = const > 0) : X_0^+ \subset B_{\delta_{X_0^+}}(x_{00}).$$

It is clear that

$$(\forall x \in X_0^+) \Rightarrow \rho(x_{00}, x) < \delta_G. \qquad (8)$$

Part 2. As in the previous theorem, we obtain

$$\left(\forall x \in B_{\delta_{\overline{\Phi}}} \cap D\right) \Rightarrow \left\langle grad\varphi(x), f(x)\right\rangle \geq C_{\langle grad\varphi, f\rangle}.$$

From condition 2 of the theorem it follows that

$$\begin{gathered} \left(\exists \delta_L = const,\ 0 < \delta_L < \tfrac{\delta_{\overline{\Phi}}}{L}\right) : \\ \left(\forall x_0^*, x_0^{**} \in G,\ \rho(x_0^*, x_0^{**}) < \delta_L\right) \\ \Rightarrow \rho\left(x(t; x_0^*), x(t; x_0^{**})\right) = \left\|x(t; x_0^*) - x(t; x_0^{**})\right\| \\ \leq L\|x_0^* - x_0^{**}\| L . \delta_L < \delta_{\overline{\Phi}}, \quad t \geq 0. \qquad (9) \end{gathered}$$

Part 3. Let x be an arbitrary point from X_0^+. Denote

$$m = \left[\frac{L . \delta_{X_0^+}}{\delta_{\overline{\Phi}}}\right] + 1 \in N,$$

where $[x]$ is the largest integer not exceeding x.

Consider the points:

$$\begin{aligned} x_{01} &= \frac{m-1}{m} x_{00} + \frac{1}{m} x, \\ x_{02} &= \frac{m-2}{m} x_{00} + \frac{2}{m} x, \\ &\vdots \\ x_{0(m-1)} &= \frac{1}{m} x_{00} + \frac{m-1}{m} x, \\ x_{0m} &= x. \end{aligned}$$

All of these points belong to the closed interval $[x_{00}, x] = [x_{00}, x_{0m}]$. Since X_0^+ is a convex domain, then

$$x_{00}, x_{01}, \ldots, x_{0m} \in [x_{00}, x_{0m}] \subset X_0^+. \qquad (10)$$

For $i = 1, 2, \ldots, m$, we have

$$\begin{aligned} & \rho(x_{0(i-1)}, x_{0i}) \\ = {} & \left\|\frac{m-i+1}{m} x_{00} + \frac{i-1}{m} x_{0m} - \frac{m-i}{m} x_{00} - \frac{i}{m} x_{0m}\right\| \\ = {} & \frac{1}{m} \|x_{00} - x_{0m}\| \\ = {} & \frac{1}{m} \rho(x_{00}, x_{0m}) \\ < {} & \frac{1}{m} \delta_{X_0^+} < \delta_L. \qquad (11) \end{aligned}$$

From (9) and (11), it follows that

$$\begin{aligned} & \rho\left(x(t; x_{0(i-1)}), x(t; x_{0i})\right) \\ & \quad < \delta_{\overline{\Phi}},\ t \geq 0,\ i = 1, 2, \ldots, m. \qquad (12) \end{aligned}$$

Part 4. Consider the points $x_{0(i-1)}$ and x_{0i}. From (11), as in (7), we obtain

$$\begin{aligned} & \left|\Theta^+(x_{0(i-1)}) - \Theta^+(x_{0i})\right| \\ \leq {} & \frac{2C_\varphi}{C_{\langle grad\varphi, f\rangle}} \left\| x\left(\min\left\{\Theta^+(x_{0(i-1)}), \Theta^+(x_{0i})\right\}; x_{0(i-1)}\right) - \right. \\ & \left. x\left(\min\left\{\Theta^+(x_{0(i-1)}), \Theta^+(x_{0i})\right\}; x_{0i}\right) \right\|, \end{aligned}$$

from which, using estimate (12), we find

$$\left|\Theta^+(x_{0(i-1)}) - \Theta^+(x_{0i})\right| \leq \frac{2C_\varphi \delta_{\overline{\Phi}}}{C_{\langle grad\varphi, f\rangle}},\ i = 1, 2, \ldots, m.$$

Part 5. Through the above estimate, we find that

$$\begin{aligned} & \left|\Theta^+(x_{00}) - \Theta^+(x)\right| \\ \leq {} & \left|\Theta^+(x_{00}) - \Theta^+(x_{01})\right| + \left|\Theta^+(x_{01}) - \Theta^+(x_{02})\right| + \cdots \\ & + \left|\Theta^+(x_{0(m-1)}) - \Theta^+(x_{0m})\right| \\ \leq {} & \frac{2mC_\varphi \delta_{\overline{\Phi}}}{C_{\langle grad\varphi, f\rangle}}. \end{aligned}$$

Therefore,

$$\Theta^{+}(x) \leq \Theta^{+}(x_{00}) + \frac{2mC_{\varphi}\delta_{\overline{\Phi}}}{C_{\langle grad\,\varphi, f\rangle}} = const,\ x \in X_0^{+}.$$

The theorem is proved.

Corollary 3.1. Assume that:

1. The conditions H1-H7 hold.
2. The solutions of system (1) are uniformly Lipschitz stable.
3. The domain X_0^{+} is k-convex and bounded.

Then the function Θ^{+} is bounded in X_0^{+}.

Corollary 3.2. Assume that:

1. The conditions H1-H7 hold.
2. The solutions of system (1) are uniformly Lipschitz stable.
3. The domain X_0^{+} is bounded-connected.

Then the function Θ^{+} is bounded in X_0^{+}.

Corollary 3.3. Assume that:

1. The conditions H1-H7 hold.
2. The solutions of system (1) are uniformly Lipschitz stable.
3. The domain X_0^{-} is k-convex and bounded.

Then the function Θ^{-} is bounded in X_0^{-}.

Corollary 3.4. Assume that:

1. The conditions H1-H7 hold.
2. The solutions of system (1) are uniformly Lipschitz stable.
3. The domain X_0^{-} is bounded-connected.

Then the function Θ^{-} is bounded in X_0^{-}.

4 Application

Consider the generalized model of interaction (competition) of two species, located in the same nutrient environment. The model was taken from R. Miller [20] and K. Gopalsamy [17]. We have

$$\frac{dm_1(t)}{dt} = a_1\big(m_1(t)\big) - b_1\big(m_1(t), m_2(t)\big); \tag{13}$$

$$\frac{dm_2(t)}{dt} = a_2\big(m_2(t)\big) - b_2\big(m_1(t), m_2(t)\big); \tag{14}$$

$$m_1(0) = m_{01},\ m_2(0) = m_{02}, \tag{15}$$

where:
- $m_1 = m_1(t) > 0$ and $m_2 = m_2(t) > 0$ are biomasses of two species at the moment $t \geq 0$, respectively;
- $a_1, a_2 : R^{+} \to R^{+}$ are two growth rates, respectively;
- the functions $b_1, b_2 : R^{+} \times R^{+} \to R^{+}$ express the intra-species and inter-species competition;
- $m_{01} > 0$ and $m_{02} > 0$ are two species biomasses at the initial moment $t = 0$.

The following conditions are standard.

H8. The functions $a_i \in C^1[R^{+}, R^{+}]$,

$$\frac{d}{dt}a_i(m) > 0 \text{ for } m \in R^{+} \text{ and } a_i(0) = 0,\ i = 1,2.$$

H9. The functions $b_i \in C^1[R^{+} \times R^{+}, R^{+}]$,

$$\frac{\partial}{\partial m_1}b_1(m_1, m_2) > 0,\ \frac{\partial}{\partial m_2}b_1(m_1, m_2) > 0,$$

for

$$(m_1, m_2) \in R^{+} \times R^{+}$$

and

$$b_1(0, m_2) = 0 \text{ for } m_2 \in R^{+}.$$

H10. There exist two positive constants m_1^{*} and m_2^{*} such that

$$a_1(m_1^{*}) - b_1(m_1^{*}, 0) = 0 \text{ and } a_2(m_2^{*}) - b_2(0, m_2^{*}) = 0.$$

H11. It is valid

$$\begin{aligned} (\forall m_2 \in R^{+})\big(\exists m_1^{**} = m_1^{**}(m_2) > 0\big) &: \\ a_1(m_1^{**}) - b_1(m_1^{**}, m_2) &< 0; \\ (\forall m_1 \in R^{+})\big(\exists m_2^{**} = m_2^{**}(m_1) > 0\big) &: \\ a_2(m_2^{**}) - b_2(m_1, m_2^{**}) &< 0. \end{aligned}$$

H12. There exist two positive constants m_1^{st} and m_2^{st} such that

$$a_1(m_1^{st}) - b_1(m_1^{st}, m_2^{st}) = 0$$

and

$$a_2(m_2^{st}) - b_2(m_1^{st}, m_2^{st}) = 0.$$

Note that the conditions above have their explanation in terms of population dynamics. The details are presented in section 3.3 of [17]. The following system is a specific realization of the generalized model (13), (14):

$$\frac{d}{dt}m_1(t) = r_1.m_1(t) - a_{11}.m_1^2(t) - a_{12}.m_1(t).m_2(t); \tag{16}$$

$$\frac{d}{dt}m_2(t) = r_2.m_2(t) - a_{21}.m_1(t).m_2(t) - a_{22}.m_2^2(t), \tag{17}$$

where the constants r_i and a_{ij} are positive, $i = 1,2,\ j = 1,2$.

The system (16), (17) satisfies conditions H8 and H9, which is easily verifiable. According to condition H10, we have

$$\begin{aligned} 0 = a_1(m_1^{*}) - b_1(m_1^{*}, 0) &= r_1.m_1^{*} - a_{11}(m_1^{*})^2 \\ &= m_1^{*}(r_1 - a_{11}.m_1^{*}), \end{aligned}$$

from where we find that $m_1^{*} = r_1/a_{11}$. In the same way $m_2^{*} = r_2/a_{22}$. It is easy to show that for each $m_2 \in R^{+}$, there exists a constant $m_1^{**} = m_1^{**}(m_2) > 0$ such that

$$r_1.m_1^{**} - a_{11}(m_1^{**})^2 - a_{12}.m_1^{**}.m_2.$$

Similarly, there exists a constant $m_2^{**} = m_2^{**}(m_1) > 0$ such that

$$r_2.m_2^{**} - a_{22}(m_2^{**})^2 - a_{21}.m_2^{**}.m_1.$$

Condition H11 is met. We determine the positive constants m_1^{st} and m_2^{st} as the solutions of the next system

$$\left| \begin{array}{l} r_1.m_1^{st} - a_{11}(m_1^{st})^2 - a_{12}.m_1^{st}.m_2^{st} = 0; \\ r_2.m_2^{st} - a_{21}.m_1^{st}.m_2^{st} - a_{22}(m_2^{st})^2 = 0 \end{array} \right.$$

$$\Leftrightarrow \quad \left\| \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right\| . \left\| \begin{array}{c} m_1^{st} \\ m_2^{st} \end{array} \right\| = \left\| \begin{array}{c} r_1 \\ r_2 \end{array} \right\|$$

$$\Leftrightarrow \quad A.m^{st} = r \tag{18}$$

$$\Leftrightarrow \quad m_1^{st} = \frac{r_1 a_{22} - r_2 a_{12}}{a_{11}a_{22} - a_{12}a_{21}}, \qquad m_2^{st} = \frac{r_2 a_{11} - r_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}, \tag{19}$$

where:

$$A = \left| \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right\|, \quad m^{st} = \left\| \begin{array}{c} m_1^{st} \\ m_2^{st} \end{array} \right\|, \quad r = \left\| \begin{array}{c} r_1 \\ r_2 \end{array} \right\|.$$

Without loss of generality, we assume that the inequality $a_{11}a_{22} - a_{12}a_{21} > 0$ holds. From (19), keeping in mind that $m_1^{st} > 0$ and $m_2^{st} > 0$, we obtain the inequalities

$$\frac{a_{12}}{a_{22}} < \frac{r_1}{r_2} < \frac{a_{11}}{a_{21}}.$$

We assume that the above inequalities are fulfilled a-priori.

The asymptotic properties of the solutions of system (16), (17) are successfully explored by the corresponding linearized system. For this purpose we define

$$m_1(t) = m_1^{st} + M_1(t), \quad m_2(t) = m_2^{st} + M_2(t)$$

and from (16) and (17), we obtain

$$\left| \begin{array}{l} \frac{d}{dt}M_1(t) = r_1 m_1^{st} + r_1 M_1(t) - a_{11}(m_1^{st})^2 - 2a_{11}m_1^{st}M_1(t) \\ \qquad -a_{11}\big(M_1(t)\big)^2 - a_{12}m_1^{st}m_2^{st} - a_{12}m_1^{st}M_2(t) \\ \qquad -a_{12}M_1(t)m_2^{st} - a_{12}M_1(t)M_2(t); \\ \frac{d}{dt}M_2(t) = r_2 m_2^{st} + r_2 M_2(t) - a_{21}m_1^{st}m_2^{st} - a_{21}m_1^{st}M_2(t) \\ \qquad -a_{21}M_1(t)m_2^{st} - a_{21}M_1(t)M_2(t) - a_{22}(m_2^{st})^2 \\ \qquad -2a_{22}m_2^{st}M_2(t) - a_{22}\big(M_2(t)\big)^2. \end{array} \right.$$

Using the inequalities (18) and that second order terms are negligible, we obtain the system

$$\left| \begin{array}{l} \frac{d}{dt}M_1(t) = -a_{11}m_1^{st}M_1(t) - a_{12}m_1^{st}M_2(t); \\ \frac{d}{dt}M_2(t) = -a_{21}m_2^{st}M_1(t) - a_{22}m_2^{st}M_2(t) \end{array} \right.$$

$$\Leftrightarrow \frac{d}{dt} \left\| \begin{array}{c} M_1(t) \\ M_2(t) \end{array} \right\| = - \left\| \begin{array}{cc} a_{11}m_1^{st} & a_{12}m_1^{st} \\ a_{21}m_2^{st} & a_{22}m_2^{st} \end{array} \right\| . \left\| \begin{array}{c} M_1(t) \\ M_2(t) \end{array} \right\|$$

$$\Leftrightarrow \frac{dM}{dt} = -A^{st}.M, \tag{20}$$

where:

$$M = \left\| \begin{array}{c} M_1(t) \\ M_2(t) \end{array} \right\|, \quad A^{st} = \left\| \begin{array}{cc} a_{11}m_1^{st} & a_{12}m_1^{st} \\ a_{21}m_2^{st} & a_{22}m_2^{st} \end{array} \right\|.$$

The matrix eigenvalues of system above are the solutions of equation

$$det \left\| \begin{array}{cc} a_{11}m_1^{st} + \lambda & a_{12}m_1^{st} \\ a_{21}m_2^{st} & a_{22}m_2^{st} + \lambda \end{array} \right\| = 0$$

$$\Leftrightarrow \lambda^2 + (a_{11}m_1^{st} + a_{22}m_2^{st})\lambda + (a_{11}a_{22} - a_{12}a_{21})m_1^{st}m_2^{st} = 0$$

$$\Leftrightarrow \lambda^2 + B\lambda + C = 0,$$

where

$$B = a_{11}m_1^{st} + a_{22}m_2^{st} > 0$$

and

$$C = (a_{11}a_{22} - a_{12}a_{21})m_1^{st}m_2^{st} > 0.$$

For the discriminant of the last equation, we get

$$B^2 - 4C = (a_{11}m_1^{st} - a_{22}m_2^{st})^2 + 4a_{12}a_{21}m_1^{st}m_2^{st} > 0.$$

The eigenvalues λ_1 and λ_2 are real and negative. More precisely, we have:

$$\lambda_1 = \frac{1}{2}\Bigg(-a_{11}m_1^{st} - a_{22}m_2^{st} - \Big((a_{11}m_1^{st} - a_{22}m_2^{st})^2 + 4a_{12}a_{21}m_1^{st}m_2^{st} \Big)^{\frac{1}{2}} \Bigg);$$

$$\lambda_2 = \frac{1}{2}\Bigg(-a_{11}m_1^{st} - a_{22}m_2^{st} + \Big((a_{11}m_1^{st} - a_{22}m_2^{st})^2 + 4a_{12}a_{21}m_1^{st}m_2^{st} \Big)^{\frac{1}{2}} \Bigg),$$

$$\lambda_1 < \lambda_2 < 0.$$

We denote the corresponding linearly independent eigenvectors as follows:

$$w_1 = \left\| \begin{array}{c} w_{11} \\ w_{21} \end{array} \right\|, \quad w_2 = \left\| \begin{array}{c} w_{12} \\ w_{22} \end{array} \right\|.$$

Then the fundamental matrix of system (20) has the form:

$$W(t) = \left\| \begin{array}{cc} w_{11}\exp(\lambda_1 t) & w_{12}\exp(\lambda_2 t) \\ w_{21}\exp(\lambda_1 t) & w_{22}\exp(\lambda_2 t) \end{array} \right\|.$$

The solution $M(t; M_0)$ of system (20) with initial condition

$$M(0) = \left\| \begin{array}{c} M_1(0) \\ M_2(0) \end{array} \right\| = \left\| \begin{array}{c} M_{01} \\ M_{02} \end{array} \right\| = M_0$$

can be expressed in the form

$$M(t; M_0) = \left\| \begin{array}{c} M_1(t; M_{01}, M_{02}) \\ M_2(t; M_{01}, M_{02}) \end{array} \right\| = W(t)W^{-1}(0)M_0.$$

Then

$$
\begin{aligned}
&\left\|M(t;M^{*})-M(t;M_0)\right\| \\
&\leq \|W(t)\|.\|W^{-1}(0\|\|M_0^{*}-M_0\| \\
&\leq \exp(\lambda_1 t)\exp(\lambda_2 t)\|W(0)\|.\|W^{-1}(0)\|.\|M_0^{*}-M_0\| \\
&\leq \|M_0^{*}-M_0\|.
\end{aligned}
$$

The above estimation shows that the solutions of system (20) are uniformly stable with Lipschitz constant $L = 1$.

It is easy to demonstrate that system (20) satisfies conditions H1, H2, H3. The constant C_f is defined as follows:

$$
\begin{aligned}
&\left\|f(M_1,M_2)\right\| \\
&\quad = \left\|\left((a_{11}M_1+a_{12}M_2)m_1^{st},(a_{21}M_1+a_{22}M_2)m_2^{st}\right)\right\| \\
&\quad \leq \max\left\{a_{11}m_1^{st}+a_{21}m_2^{st}\ ,\ a_{12}m_1^{st}+a_{22}m_2^{st}\right\} \\
&\quad = C_f
\end{aligned}
$$

Consider the function $\varphi(M_1,M_2) = p - M_1 - M_2$, where

$$
\begin{aligned}
&(M_1,M_2) \in D \\
&= G = \left\{(M_1,M_2);\ M_1 > 0,\ M_2 > 0.\ M_1+M_2 < q\right\}
\end{aligned}
$$

and the constant p which satisfies the inequalities $0 < p < q$. Geometrically the set Φ coincides with the open segment with endpoints $(p,0)$ and $(0,p)$. Condition H5 is verified immediately. Moreover, the function $\varphi \in C^1[D,R]$ and

$$
\begin{aligned}
&\left(\forall (M_1,M_2)\ \in\ \Phi\right) \\
&\qquad \Rightarrow \left\langle grad\varphi(M_1,M_2), f(M_1,M_2)\right\rangle \\
&\qquad = \left\langle(-1,-1),(-a_{11}m_1^{st}M_1 - a_{12}m_2^{st}M_2\ ,\right. \\
&\qquad\qquad \left. -a_{21}m_1^{st}M_1 - a_{22}m_2^{st}M_2)\right\rangle \\
&\qquad = (a_{11}+a_{21})m_1^{st}M_1 + (a_{12}+a_{22})m_2^{st}M_2 \\
&\qquad \geq \min\left\{(a_{11}+a_{21})m_1^{st},(a_{12}+a_{22})m_2^{st}\right\}p \\
&\qquad = C_{\langle grad\varphi, f\rangle} > 0.
\end{aligned}
$$

Therefore, condition H4 is valid.

We have

$$
\begin{aligned}
\overline{\Phi}\backslash\Phi &= \left\{(p,0),(0,p)\right\} \\
&\in \left\{(M_1,0);\ 0 \leq M_1 \leq q\right\} \cup \left\{(0,M_2);\ 0 \leq M_2 \leq q\right\} \\
&\quad \cup \left\{(M_1,M_2) \in R^{+}\times R^{+};\ M_1+M_2 = q\right\} \\
&= \partial G,
\end{aligned}
$$

i.e. condition H6 holds. Finally, it is easy to demonstrate that the following equality holds

$$
\left|\varphi(M_1,M_2)\right| = \rho\left((M_1,M_2),\Phi\right),
$$

which means that condition H7 is met with the constant $C_\varphi = 1$. Obviously the domains

$$
\begin{aligned}
X_0^{-} &= \left\{(M_1,M_2);\ M_1 > 0,\ M_2 > 0,\ M_1+M_2 < p\right\}; \\
X_0^{+} &= \left\{(M_1,M_2);\ M_1 > 0,\ M_2 > 0,\ p < M_1+M_2 < q\right\}
\end{aligned}
$$

are convex.

From Theorem 3.1, it follows that the functions Θ^{-} and Θ^{+} (defined for the model considered above) are continuous in the domains X_0^{-} and X_0^{+}, respectively. One possible interpretation of this fact: Let the initial biomasses in a community of two competing species be close to the corresponding initial values for the biomasses of another pair of species in a similar community. Let the two communities develop under the same conditions and their dynamics are modeled by system (20). Then the biomasses in these two communities will reach a pre-set ratio almost simultaneously.

In brief, from Theorem 3.2 and Corollary 4.4, it follows that under certain conditions, the functions Θ^{-} and Θ^{+} , defined in the domains X_0^{-} and X_0^{+}, respectively, are bounded. The results applied to the modeled system under consideration, we could interpret as follows: Let us consider all communities consisting of two competing species, which differ by the initial species biomasses, and the dynamics of which are described by system (20). Then the set of moments in which the biomasses of each of these communities reach a pre-set ratio, is bounded.

Finally, note that for sufficiently small values of M_1 and M_2, i.e. for sufficiently small values of q, the trajectory of system (16), (17) is similar? to the trajectory of the corresponding linearized system (20). In this case, the conclusions obtained above are also transferable to system (16), (17).

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