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Transmuted Generalized Rayleigh Distribution

*Faton Merovci**

Department of Mathematics, University of Prishtina "Hasan Prishtina",Prishtinë, 10000, Kosovo

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Abstract: In this article, we generalize the generalized Rayleigh distribution using the quadratic rank transmutation map studied by Shaw et al. [9] to develop a transmuted generalized Rayleigh distribution. We provide a comprehensive description of the mathematical properties of the subject distribution along with its reliability behavior. The usefulness of the transmuted generalized Rayleigh distribution for modeling data is illustrated using real data.

Keywords: Generalized Rayleigh distribution, hazard rate function, reliability function, parameter estimation.

1 Introduction

In many applied sciences such as medicine, engineering and finance, amongst others, modeling and analyzing lifetime data are crucial. Several lifetime distributions have been used to model such kinds of data. The quality of the procedures used in a statistical analysis depends heavily on the assumed probability model or distributions. Because of this, considerable effort has been expended in the development of large classes of standard probability distributions along with relevant statistical methodologies. However, there still remain many important problems where the real data does not follow any of the classical or standard probability models.

Burr [3] introduced twelve different forms of cumulative distribution functions for modeling lifetime data. Among those twelve distribution functions, Burr-Type X and Burr-Type XII received the maximum attention. For more detail about those two distributions see [5]. Recently,Surles and Padgett [10] introduced two-parameter Burr Type X distribution and correctly named as the generalized Rayleigh distribution. In this paper, we also prefer to call the two-parameter Burr Type X distribution as the generalized Rayleigh (GR) distribution. For $\alpha > 0$ and $\beta > 0$, the two-parameter GR distribution has the cumulative distribution function(cdf):

$$F(x,\alpha,\beta) = \left(1-e^{-(\beta x)^2}\right)^{\alpha}, \quad x > 0, \tag{1}$$

and the respective probability density function(pdf) is:

$$f(x,\alpha,\beta) = 2\alpha\beta^2 x e^{-(\beta x)^2}\left(1-e^{-(\beta x)^2}\right)^{\alpha-1}, \quad x > 0. \tag{2}$$

In this article we present a new generalization of the generalized Rayleigh distribution called the transmuted generalized Rayleigh distribution.

Definition 1(Shaw et al.(2009)). *A random variable X is said to have transmuted distribution if its cumulative distribution function(cdf) is given by*

$$G(x) = (1+\lambda)F(x) - \lambda F^2(x), |\lambda| \leq 1. \tag{3}$$

where $F(x)$ is the cdf of the base distribution.

Observe that at $\lambda = 0$ we have the distribution of the base random variable. Aryal et al. [1] studied the transmuted Gumbel distribution and it has been observed that transmuted Gumbel distribution can be used to model climate data. In the present study we will provide mathematical formulations of the transmuted generalized Rayleigh distribution and also some of its properties.

* Corresponding author e-mail: fmerovci@yahoo.com

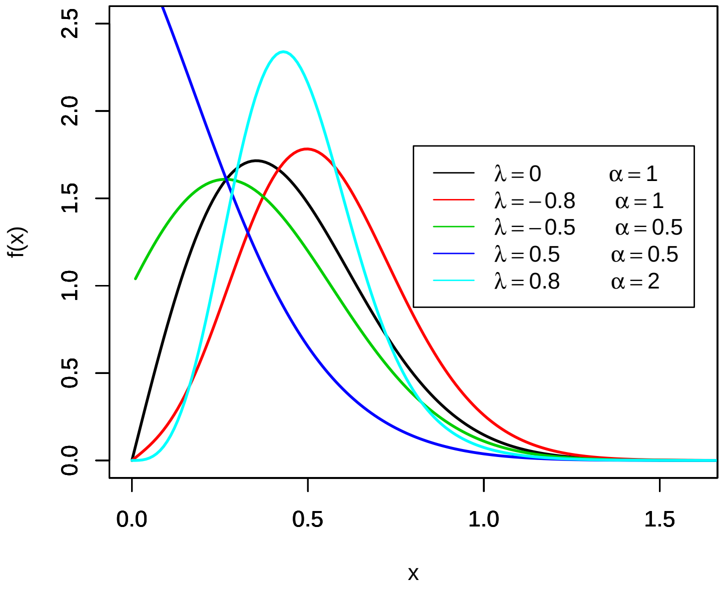

Fig. 1: The pdf's of various transmuted generalized Rayleigh distributions($\beta = 2$).

2 Transmuted generalized Rayleigh Distribution

Definition 2.*The pdf of transmuted generalized Rayleigh distribution is:*

$$g(x;\alpha,\beta,\lambda) = 2\alpha\beta^2 x e^{-(\beta x)^2}\left(1-e^{-(\beta x)^2}\right)^{\alpha-1}\left[1+\lambda-2\lambda\left(1-e^{-(\beta x)^2}\right)^{\alpha}\right] \quad (4)$$

and the respective cdf is:

$$G(x;\alpha,\beta,\lambda) = \left(1-e^{-(\beta x)^2}\right)^{\alpha}\left[1+\lambda-\lambda\left(1-e^{-(\beta x)^2}\right)^{\alpha}\right] \quad (5)$$

Note that the transmuted generalized Rayleigh distribution is an extended model to analyze more complex data. The generalized Rayleigh distribution is clearly a special case for $\lambda = 0$. Figure 1 illustrates some of the possible shapes of the pdf of a transmuted generalized Rayleigh distribution for selected values of the parameters λ, α and β.

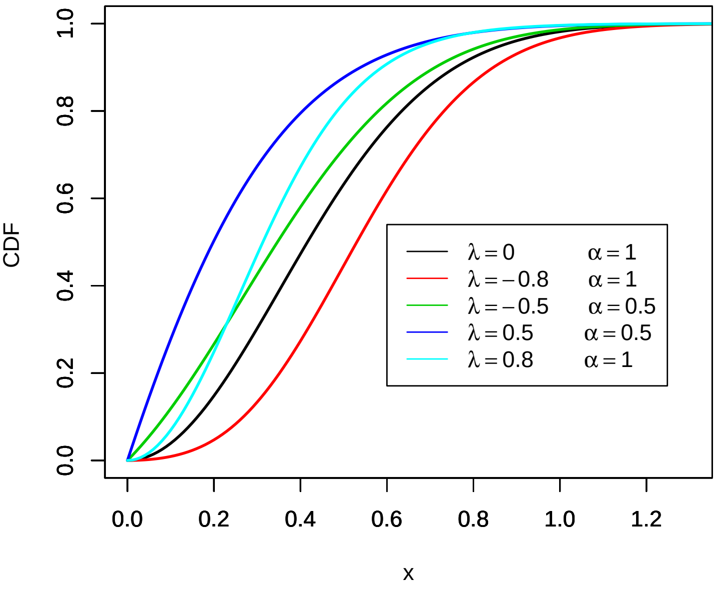

Fig. 2: The cdf's of various transmuted generalized Rayleigh distributions($\beta = 2$).

3 Moments

Theorem 1.*The r^{th} moment $E(X^r)$ of a transmuted generalized Rayleigh distributed random variable X is given as*

$$E(X^r) = \alpha\beta^{-r} r\Gamma\left(\frac{r}{2}\right)\sum_{j=0}^{\infty}\frac{(-1)^j(j+1)^{-\frac{r+2}{2}}}{j!}\left[\frac{(1+\lambda)\Gamma(\alpha)}{2\Gamma(\alpha-j)} - \frac{\lambda\Gamma(2\alpha)}{\Gamma(2\alpha-j)}\right]. \quad (6)$$

Especially we have

$$E(X) = \alpha\beta^{-1}\sqrt{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j(j+1)^{-\frac{3}{2}}}{j!}\left[\frac{(1+\lambda)\Gamma(\alpha)}{2\Gamma(\alpha-j)} - \frac{\lambda\Gamma(2\alpha)}{\Gamma(2\alpha-j)}\right], \quad (7)$$

$$var(X) = \alpha\beta^{-2}\sum_{j=0}^{\infty}\frac{(-1)^j(j+1)^{-2}}{j!}\left[\frac{(1+\lambda)\Gamma(\alpha)}{2\Gamma(\alpha-j)} - \frac{\lambda\Gamma(2\alpha)}{\Gamma(2\alpha-j)}\right] \times\left\{1+\alpha\pi(j+1)^{-1}\left[\frac{(1+\lambda)\Gamma(\alpha)}{2\Gamma(\alpha-j)} - \frac{\lambda\Gamma(2\alpha)}{\Gamma(2\alpha-j)}\right]\right\}. \quad (8)$$

Proof.

$$E(X^r) = \int_0^\infty x^r f(x)dx$$
$$= 2\alpha\beta^2 \int_0^\infty x^{r+1} e^{-(\beta x)^2} \left(1 - e^{-(\beta x)^2}\right)^{\alpha-1} \left[1 + \lambda - 2\lambda \left(1 - e^{-(\beta x)^2}\right)^{\alpha}\right] dx$$
$$= 2\alpha\beta^2(1+\lambda) \int_0^\infty x^{r+1} e^{-(\beta x)^2} \left(1 - e^{-(\beta x)^2}\right)^{\alpha-1} dx$$
$$- 4\alpha\beta^2\lambda \int_0^\infty x^{r+1} e^{-(\beta x)^2} \left(1 - e^{-(\beta x)^2}\right)^{2\alpha-1} dx$$
$$= 2\alpha\beta^2(1+\lambda) \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\alpha)}{\Gamma(\alpha-j)j!} \int_0^\infty x^{r+1} e^{-(j+1)(\beta x)^2} dx$$
$$- 4\alpha\beta^2\lambda \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(2\alpha)}{\Gamma(2\alpha-j)j!} \int_0^\infty x^{r+1} e^{-(j+1)(\beta x)^2} dx$$
$$= \alpha\beta^2(1+\lambda)\frac{r}{2}\Gamma\left(\frac{r}{2}\right) \sum_{j=0}^{\infty} \frac{(-1)^j [(j+1)\beta^2]^{-\frac{r+2}{2}} \Gamma(\alpha)}{\Gamma(\alpha-j)j!}$$
$$- r\alpha\beta^2\lambda\Gamma\left(\frac{r}{2}\right) \sum_{j=0}^{\infty} \frac{(-1)^j [(j+1)\beta^2]^{-\frac{r+2}{2}} \Gamma(2\alpha)}{\Gamma(2\alpha-j)j!}$$
$$= \alpha\beta^{-r} r\Gamma\left(\frac{r}{2}\right) \sum_{j=0}^{\infty} \frac{(-1)^j (j+1)^{-\frac{r+2}{2}}}{j!} \left[\frac{(1+\lambda)\Gamma(\alpha)}{2\Gamma(\alpha-j)} - \frac{\lambda\Gamma(2\alpha)}{\Gamma(2\alpha-j)}\right]$$

Here, we used

$$\int_0^\infty x^{\nu-1} e^{-\mu x^p} dx = \frac{1}{p}\mu^{-\nu/p}\Gamma\left(\frac{\nu}{p}\right), \tag{9}$$

for $p, \nu, \mu > 0$ (see Gradshtein and Ryzhnik(2000), Sec. 3.478), and for $|z| < 1$

$$(1-z)^{b-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j)j!} z^j. \tag{10}$$

By putting $r = 1$, we have:

$$E(X) = \alpha\beta^{-1}\sqrt{\pi} \sum_{j=0}^{\infty} \frac{(-1)^j (j+1)^{-\frac{3}{2}}}{j!} \left[\frac{(1+\lambda)\Gamma(\alpha)}{2\Gamma(\alpha-j)} - \frac{\lambda\Gamma(2\alpha)}{\Gamma(2\alpha-j)}\right]. \tag{11}$$

The second moment is

$$E(X^2) = 2\alpha\beta^{-2} \sum_{j=0}^{\infty} \frac{(-1)^j (j+1)^{-2}}{j!} \left[\frac{(1+\lambda)\Gamma(\alpha)}{2\Gamma(\alpha-j)} - \frac{\lambda\Gamma(2\alpha)}{\Gamma(2\alpha-j)}\right]. \tag{12}$$

and the variance is

$$var(X) = E(X^2) - [E(X)]^2$$
$$= \alpha\beta^{-2} \sum_{j=0}^{\infty} \frac{(-1)^j (j+1)^{-2}}{j!} \left[\frac{(1+\lambda)\Gamma(\alpha)}{2\Gamma(\alpha-j)} - \frac{\lambda\Gamma(2\alpha)}{\Gamma(2\alpha-j)}\right]$$
$$\times \left\{2 + \alpha\pi(j+1)^{-1} \left[\frac{(1+\lambda)\Gamma(\alpha)}{2\Gamma(\alpha-j)} - \frac{\lambda\Gamma(2\alpha)}{\Gamma(2\alpha-j)}\right]\right\}.$$

The skewness and kurtosis measures are:

$$Skewness = \frac{E(X^3)-3E(X^2)\mu+2\mu^3}{\sigma^3} =$$
$$\frac{1}{\sigma^3}\cdot\left\{3\alpha\beta^{-2}\sum_{j=0}^{\infty}\frac{(-1)^j(j+1)^{-2}}{j!}\left[\frac{(1+\lambda)\Gamma(\alpha)}{2\Gamma(\alpha-j)}-\frac{\lambda\Gamma(2\alpha)}{\Gamma(2\alpha-j)}\right]\left[0.5\sqrt{\pi}\beta^{-1}(j+1)^{-0.5}-2\mu\right]+2\mu^3\right\},$$

$$\begin{aligned} Kurtosis &= \frac{E(X^4)-4E(X^3)\mu+6E(X^2)\mu^2-3\mu^4}{\sigma^4} \\ &= \frac{1}{\sigma^4}\cdot\left\{4\alpha\beta^{-2}\sum_{j=0}^{\infty}\frac{(-1)^j(j+1)^{-2}}{j!}\left[\frac{(1+\lambda)\Gamma(\alpha)}{2\Gamma(\alpha-j)}-\frac{\lambda\Gamma(2\alpha)}{\Gamma(2\alpha-j)}\right]\right. \\ &\left.\times\left[\beta^{-2}(j+1)^{-1}-1.5\sqrt{\pi}\beta^{-1}(j+1)^{-0.5}+3\mu^2\right]+3\mu^4\right\}. \end{aligned}$$

Theorem 2.*Let X have a transmuted generalized Rayleigh distribution. Then the moment generating function of X, say $M_X(t)$, is*

$$\begin{aligned} M_X(t) =& 1+\alpha\sum_{i=1}^{\infty}\frac{t^i}{i!} \\ &\times\left\{\beta^{-i}i\Gamma\left(\frac{i}{2}\right)\sum_{j=1}^{\infty}\frac{(-1)^j(j+1)^{-\frac{i+2}{2}}}{j!}\left[\frac{(1+\lambda)\Gamma(\alpha)}{2\Gamma(\alpha-j)}-\frac{\lambda\Gamma(2\alpha)}{\Gamma(2\alpha-j)}\right]\right\}. \end{aligned} \tag{13}$$

Proof.

$$\begin{aligned} M_X(t) &= E(e^{tx}) = \int_0^{\infty} e^{tx} f(x)\,dx \\ &= \int_0^{\infty}\left(1+tx+\frac{t^2x^2}{2!}+\cdots+\frac{t^nx^n}{n!}+\cdots\right)f(x)\,dx \\ &= 1+\sum_{i=1}^{\infty}\frac{t^iE(X^i)}{i!} \\ &= 1+\alpha\sum_{i=1}^{\infty}\frac{t^i}{i!} \\ &\quad\times\left\{\beta^{-i}i\Gamma\left(\frac{i}{2}\right)\sum_{j=0}^{\infty}\frac{(-1)^j(j+1)^{-\frac{i+2}{2}}}{j!}\left[\frac{(1+\lambda)\Gamma(\alpha)}{2\Gamma(\alpha-j)}-\frac{\lambda\Gamma(2\alpha)}{\Gamma(2\alpha-j)}\right]\right\}. \end{aligned}$$

The q^{th} quantile x_q of the transmuted generalized Rayleigh distribution can be obtained from (5) as

$$x_u = \frac{1}{\beta}\left\{-\ln\left[1-\sqrt[\alpha]{\frac{1+\lambda-\sqrt{(1+\lambda)^2-4\lambda u}}{2\lambda}}\right]\right\}^{\frac{1}{2}}. \tag{14}$$

In particular, the distribution median is

$$x_{0.5} = \frac{1}{\beta}\left\{-\ln\left[1-\sqrt[\alpha]{\frac{1+\lambda-\sqrt{1+\lambda^2}}{2\lambda}}\right]\right\}^{\frac{1}{2}}. \tag{15}$$

4 Random Number Generation and Parameters Estimation

Using the method of inversion we can generate random numbers from the transmuted generalized Rayleigh distribution as

$$\left(1-e^{-(\beta x)^2}\right)^{\alpha}\left[1+\lambda-\lambda\left(1-e^{-(\beta x)^2}\right)^{\alpha}\right]=u,$$

where $u\sim U(0,1)$. After simple calculation this yields

$$x=\frac{1}{\beta}\sqrt{-\ln\left[1-\sqrt[\alpha]{\frac{1+\lambda-\sqrt{(1+\lambda)^2-4\lambda u}}{2\lambda}}\right]}. \tag{16}$$

One can use equation (16) to generate random numbers when the parameters α,β and λ are known.

Let $X_1,X_2,\cdots,X_n$ be a sample of size n from a transmuted generalized Rayleigh distribution. Then the likelihood function is given by

$$L=(2\alpha\beta^2)^n e^{-\beta^2\sum_{i=1}^{n}x_i^2}\prod_{i=1}^{n}x_i\left(1-e^{-(\beta x_i)^2}\right)^{\alpha-1}\left[1+\lambda-2\lambda\left(1-e^{-(\beta x_i)^2}\right)^{\alpha}\right] \tag{17}$$

so, the log-likelihood function is:

$$\begin{aligned}LL=\ln L=n(\ln 2+\ln\alpha+2\ln\beta)-\beta^2\sum_{i=1}^{n}x_i^2 \\ +\sum_{i=1}^{n}\ln(x_i)+(\alpha-1)\sum_{i=1}^{n}\ln\left(1-e^{-(\beta x_i)^2}\right) \\ +\sum_{i=1}^{n}\ln\left[1+\lambda-2\lambda\left(1-e^{-(\beta x_i)^2}\right)^{\alpha}\right]\end{aligned} \tag{18}$$

For ease of notation, we will denote, for any function $f(x;y)$, the first partial derivatives by f_x, f_y, and the second partial derivatives by $f_{xx}, f_{yy}, f_{xy}, f_{yx}$.

Now setting

$$LL_\alpha=0,\quad LL_\beta=0\quad \text{and}\quad LL_\lambda=0,$$

we have

$$\frac{n}{\alpha}+\sum_{i=1}^{n}\ln\left(1-e^{-(\beta x_i)^2}\right)-2\lambda\sum_{i=1}^{n}\frac{\left(1-e^{-(\beta x_i)^2}\right)^{\alpha}\ln\left(1-e^{-(\beta x_i)^2}\right)}{\left[1+\lambda-2\lambda\left(1-e^{-(\beta x_i)^2}\right)^{\alpha}\right]}=0, \tag{19}$$

$$\begin{aligned}\frac{2n}{\beta}-2\beta\sum_{i=1}^{n}x_i^2-(\alpha-1)\sum_{i=1}^{n}\frac{e^{-(\beta x_i)^2}\ln\left(1-e^{-(\beta x_i)^2}\right)}{1-e^{-(\beta x_i)^2}} \\ -4\alpha\beta\lambda\sum_{i=1}^{n}\frac{x_i^2e^{-(\beta x_i)^2}\left(1-e^{-(\beta x_i)^2}\right)^{\alpha-1}}{\left[1+\lambda-2\lambda\left(1-e^{-(\beta x_i)^2}\right)^{\alpha}\right]^2}=0\end{aligned} \tag{20}$$

and

$$\sum_{i=1}^{n}\frac{1-2\left(1-e^{-(\beta x_i)^2}\right)^{\alpha}}{1+\lambda-2\lambda\left(1-e^{-(\beta x_i)^2}\right)^{\alpha}}=0. \tag{21}$$

The maximum likelihood estimator $\hat{\theta}=(\hat{\alpha},\hat{\beta},\hat{\lambda})'$ of $\theta=(\alpha,\beta,\lambda)'$ is obtained by solving this nonlinear system of equations. It is usually more convenient to use nonlinear optimization algorithms such as the quasi-Newton algorithm to numerically maximize the log-likelihood function given in (17). Applying the usual large sample approximation, the maximum likelihood estimators of θ can be treated as being approximately trivariate normal with mean θ and variance-covariance matrix equal to the inverse of the expected information matrix. That is,

$$\sqrt{n}(\hat{\theta}-\theta) \to N_3(0, I^{-1}(\hat{\theta}))$$

where $I^{-1}(\hat{\theta})$ is the variance-covariance matrix of the unknown parameters $\theta = (\alpha, \beta, \lambda)$. The elements of the 3×3 matrix I^{-1}, $I_{ij}(\hat{\theta}), i, j = 1,2,3$ can be approximated by $I_{ij}(\hat{\theta})$, where $I_{ij}(\hat{\theta}) = -LL_{\theta_i\theta_j}|_{\theta=\hat{\theta}}$.

From (19)-(21), the second partial derivatives of the log likelihood function are found to be

$$LL_{\alpha\alpha} = -\frac{n}{\alpha^2} - 2\lambda(1+\lambda)\sum_{i=1}^{n} \frac{\left(1-e^{-(\beta x_i)^2}\right)^{\alpha}\left[\ln\left(1-e^{-(\beta x_i)^2}\right)\right]^2}{\left[1+\lambda-2\lambda\left(1-e^{-(\beta x_i)^2}\right)^{\alpha}\right]^2}, \tag{22}$$

$$LL_{\beta\beta} = -\frac{2n}{\beta^2} + 2\beta(\alpha-1)\sum_{i=1}^{n} \frac{x_i^2 e^{-(\beta x_i)^2}\left[\ln\left(1-e^{-(\beta x_i)^2}\right) - e^{-(b x_i)^2}\right.}{\left[1-e^{-(\beta x_i)^2}\right]^2} \tag{23}$$

$$LL_{\lambda\lambda} = -\sum_{i=1}^{n}\left[\frac{1-2\left(1-e^{-(\beta x_i)^2}\right)^{\alpha}}{1+\lambda-2\lambda\left(1-e^{-(\beta x_i)^2}\right)^{\alpha}}\right]^2, \tag{24}$$

$$LL_{\alpha\beta} = 2\beta\sum_{i=1}^{n}\frac{x_i^2 e^{-(\beta x_i)^2}}{1-e^{-(\beta x_i)^2}} - 4\lambda\beta\sum_{i=1}^{n}\frac{x_i^2 e^{-(\beta x_i)^2}\left(1-e^{-(\beta x_i)^2}\right)^{\alpha-1}\cdot A}{\left[1+\lambda-2\lambda\left(1-e^{-(\beta x_i)^2}\right)^{\alpha}\right]^2}, \tag{25}$$

where

$$A = \left[(1+\lambda)\left[\alpha\ln\left(1-e^{-(\beta x_i)^2}\right)+1\right] - 2\lambda\left(1-e^{-(\beta x_i)^2}\right)\right],$$

$$LL_{\alpha\lambda} = -2\sum_{i=1}^{n}\frac{\left(1-e^{-(\beta x_i)^2}\right)^{\alpha}\ln\left(1-e^{-(\beta x_i)^2}\right)}{\left[1+\lambda-2\lambda\left(1-e^{-(\beta x_i)^2}\right)^{\alpha}\right]^2}, \tag{26}$$

and

$$LL_{\lambda\beta} = -4\alpha\beta\sum_{i=1}^{n}\frac{x_i^2 e^{-(\beta x_i)^2}\left(1-e^{-(\beta x_i)^2}\right)^{\alpha-1}}{\left[1+\lambda-2\lambda\left(1-e^{-(\beta x_i)^2}\right)^{\alpha}\right]^2}. \tag{27}$$

Approximate $100(1-\alpha)\%$ two sided confidence intervals for α, β and λ are, respectively, given by

$$\hat{\alpha} \pm z_{\alpha/2}\sqrt{I_{11}^{-1}(\hat{\theta})}, \hat{\beta} \pm z_{\alpha/2}\sqrt{I_{22}^{-1}(\hat{\theta})},$$

and

$$\hat{\lambda} \pm z_{\alpha/2}\sqrt{I_{33}^{-1}(\hat{\theta})},$$

where z_α is the upper $\alpha-$th percentiles of the standard normal distribution. Using R we can easily compute the Hessian matrix and its inverse and hence the values of the standard error and asymptotic confidence intervals.

We can compute the maximized unrestricted and restricted log - likelihoods to construct the likelihood ratio (LR) statistics for testing some transmuted Rayleigh sub-models. For example, we can use LR statistics to check whether the fitted transmuted Rayleigh distribution for a given data set is statistically "superior" to the fitted Rayleigh distribution. In any case, hypothesis tests of the type $H_0 : \Theta = \Theta_0$ versus $H_0 : \Theta \neq \Theta_0$ can be performed using LR statistics. In this case, the LR statistic for testing H_0 versus H_1 is $\omega = 2(L(\hat{\Theta}) - L(\hat{\Theta}_0))$, where $\hat{\Theta}$ and $\hat{\Theta}_0$ are the MLEs under H_1 and H_0. The statistic ω is asymptotically(as $n\to\infty$) distributed as χ_k^2, where k is the dimension of the subset Ω of interest. The LR test rejects H_0 if $\omega > \xi_\gamma$, where ξ_γ denotes the upper $100\gamma\%$ point of the χ_k^2 distribution.

5 Reliability Analysis

The reliability function $R(t)$, which is the probability of an item not failing prior to some time t, is defined by $R(t) = 1 - F(t)$. The reliability function of a transmuted generalized Rayleigh distribution is given by

$$R(t) = 1 - \left(1 - e^{-(\beta t)^2}\right)^{\alpha} \left[1 + \lambda - \lambda \left(1 - e^{-(\beta t)^2}\right)^{\alpha}\right]. \tag{28}$$

The other characteristic of interest of a random variable is the hazard rate function defined by

$$h(t) = \frac{f(t)}{1 - F(t)},$$

which is an important quantity characterizing life phenomenon. It can be loosely interpreted as the conditional probability of failure, given it has survived to time t. The hazard rate function for a transmuted generalized Rayleigh random variable is given by

$$h(t) = \frac{2\alpha\beta^2 t e^{-(\beta t)^2} \left(1 - e^{-(\beta t)^2}\right)^{\alpha-1} \left[1 + \lambda - 2\lambda \left(1 - e^{-(\beta t)^2}\right)^{\alpha}\right]}{1 - \left(1 - e^{-(\beta t)^2}\right)^{\alpha} \left[1 + \lambda - \lambda \left(1 - e^{-(\beta t)^2}\right)^{\alpha}\right]}. \tag{29}$$

Figure 3 illustrates the reliability function of a transmuted generalized Rayleigh distribution for different combinations of parameters α and λ, where $\beta = 2$.

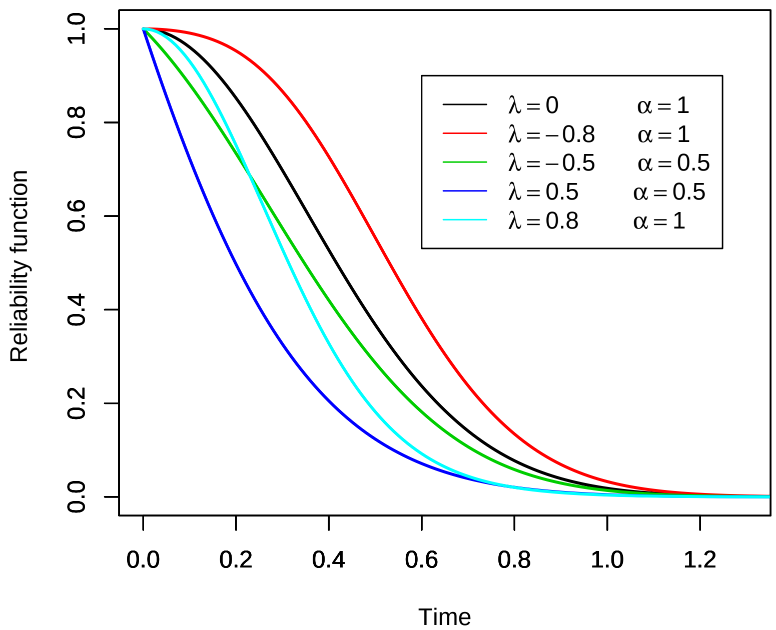

Fig. 3: The reliability function of a transmuted generalized Rayleigh distribution.

6 Order Statistics

In statistics, the k^{th} order statistic of a statistical sample is equal to its k^{th}-smallest value. Together with rank statistics, order statistics are among the most fundamental tools in non-parametric statistics and inference. For a sample of size n, the n^{th} order statistic (or largest order statistic) is the maximum, that is,

$$X_{(n)} = \max\{X_1, X_2, \ldots, X_n\}.$$

The sample range is the difference between the maximum and minimum. It is clearly a function of the order statistics:

$$Range\{X_1, X_2, \ldots, X_n\} = X_{(n)} - X_{(1)}.$$

We know that if $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$ denotes the order statistics of a random sample $X_1, X_2, \ldots, X_n$ from a continuous population with cdf $F_X(x)$ and pdf $f_X(x)$ then the pdf of $X_{(j)}$ is given by

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} f_X(x)[F_X(x)]^{j-1}[1-F_X(x)]^{n-j}, \tag{30}$$

for $j = 1, 2, \ldots, n$. The pdf of the j^{th} order statistic for transmuted generalized Rayleigh distributions is given by

$$\begin{aligned} f_{X_{(j)}}(x) = {} & \frac{2\alpha\beta^2 x e^{-(\beta x)^2} n!}{(j-1)!(n-j)!} \left(1-e^{-(\beta x)^2}\right)^{\alpha j-1} \left[1+\lambda-2\lambda\left(1-e^{-(\beta x)^2}\right)^{\alpha}\right] \\ & \times \left[1+\lambda-\lambda\left(1-e^{-(\beta x)^2}\right)^{\alpha}\right]^{j-1} \\ & \times \left[1-\left(1-e^{-(\beta x)^2}\right)^{\alpha}\left[1+\lambda-\lambda\left(1-e^{-(\beta x)^2}\right)^{\alpha}\right]\right]^{n-j}. \end{aligned} \tag{31}$$

Therefore, the pdf of the largest order statistic $X_{(n)}$ is given by

$$\begin{aligned} f_{X_{(n)}}(x) = {} & 2n\alpha\beta^2 x e^{-(\beta x)^2} \left(1-e^{-(\beta x)^2}\right)^{\alpha n-1} \left[1+\lambda-2\lambda\left(1-e^{-(\beta x)^2}\right)^{\alpha}\right] \\ & \times \left[1+\lambda-\lambda\left(1-e^{-(\beta x)^2}\right)^{\alpha}\right]^{n-1}, \end{aligned} \tag{32}$$

and the pdf of the smallest order statistic $X_{(1)}$ is given by

$$\begin{aligned} f_{X_{(1)}}(x) = {} & 2n\alpha\beta^2 x e^{-(\beta x)^2} \left(1-e^{-(\beta x)^2}\right)^{\alpha-1} \left[1+\lambda-2\lambda\left(1-e^{-(\beta x)^2}\right)^{\alpha}\right] \\ & \times \left[1-\left(1-e^{-(\beta x)^2}\right)^{\alpha}\left[1+\lambda-\lambda\left(1-e^{-(\beta x)^2}\right)^{\alpha}\right]\right]^{n-1}. \end{aligned} \tag{33}$$

7 Application

In this section, we use a real data set to show that the transmuted generalized Rayleigh distribution can be a better model than the generalized Rayleigh, Rayleigh and transmuted Rayleigh distribution.

We work with nicotine measurements made in several brands of cigarettes in 1998. The data have been collected by the Federal Trade Commission which is an independent agency of the US government, whose main mission is the promotion of consumer protection.

The report entitled tar, nicotine, and carbon monoxide of the smoke of 1206 varieties of domestic cigarettes for the year of 1998 at

http://www.ftc.gov/reports/tobacco and consists of the data sets and some information about the source of the data, smokers behaviour and beliefs about nicotine, tar and carbon monoxide contents in cigarettes. The free form data set can be found at http://pw1.netcom.com/ rdavis2/ smoke.html.

The site http://home.att.net/ rdavis2/cigra.html contains $n = 384$ observations. We analysed data about nicotine, measured in milligrams per cigarette, from several cigarette brands. Some summary statistics for the nicotine data are as follows: mean =0.852, median = 0.9, minimum = 0.1 and maximum = 2.

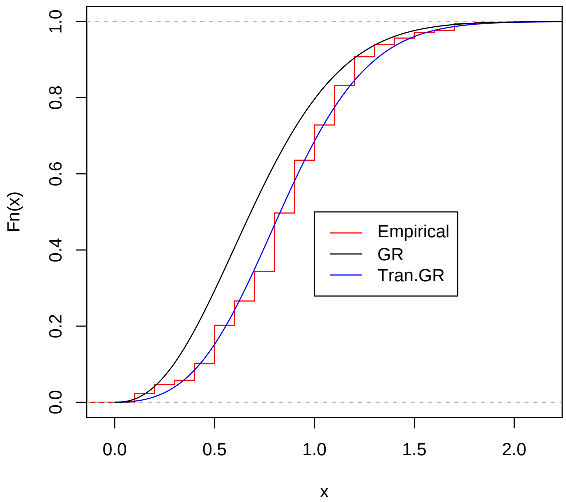

Fig. 4: Empirical, fitted generalized Rayleigh and transmuted generalized Rayleigh cdf of nicotine measurements data.

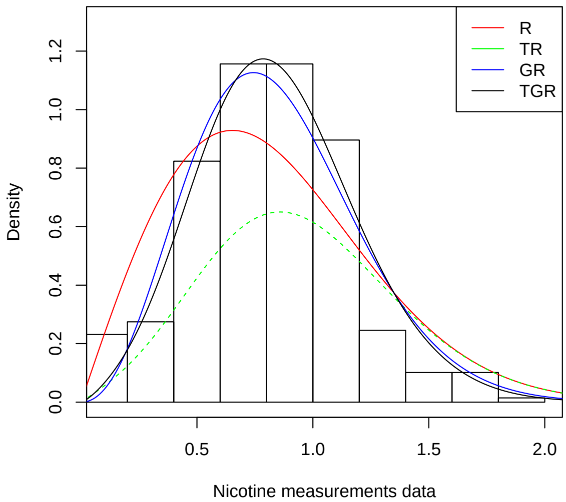

Fig. 5: Estimated densities of the models for nicotine measurements data.

The variance covariance matrix of transmuted generalized $Rayleigh(\hat{\alpha} = 1.173, \hat{\beta} = 1.317, \hat{\lambda} = -0.681)$ is computed as

$$I(\hat{\theta})^{-1} = \begin{pmatrix} 0.199\times10^{-1} & 0.313\times10^{-2} & 0.121\times10^{-1} \\ 0.313\times10^{-2} & 0.155\times10^{-2} & 0.389\times10^{-3} \\ 0.121\times10^{-1} & 0.389\times10^{-3} & 0.145\times10^{-1} \end{pmatrix}$$

Thus, the variances of the MLE of α, β and λ become $Var(\hat{\alpha}) = 0.199\times10^{-1}, Var(\hat{\beta}) = 0.155\times10^{-2}$ and $Var(\hat{\lambda}) = 0.145\times10^{-1}$.

Table 1: Estimated Parameters of the Rayleigh, transmuted Rayleigh, GR and transmuted GR distribution for nicotine measurements data

Model	Parameter Estimates	95% $C.I$	-LL
Transmuted	$\hat{\alpha}=1.173$	$[0.8971, 1.451]$	112.443
Generalized	$\hat{\beta}=1.317$	$[1.239, 1.394]$	
Rayleigh	$\hat{\lambda}=-0.681$	$[-0.917, -0.445]$	
Generalized	$\hat{\alpha}=1.579$	$[1.346, 1.811]$	119.457
Rayleigh	$\hat{\beta}=1.250$	$[1.326, 1.175]$	
Transmuted	$\hat{\sigma}=0.555$	$[0.528, 0.582]$	121.224
Rayleigh	$\hat{\lambda}=-0.7718095$	$[-0.914, -0.629]$	
Rayleigh	$\hat{\sigma}$=0.6475387	$[0.618, 0.687]$	142.3572

Table 2: Criteria for Comparison.

Model	K-S	-2LL	AIC	AICC	BIC
GR	0.281	238.914	242.914	242.949	250.606
TGR	0.122	224.886	230.886	230.956	242.425

The LR statistics to test the hypotheses $H_0 : \lambda = 0$ versus $H_1 : \lambda \neq 0 : \omega = 14.028 > 3.841 = \chi_1^2(\alpha = 0.05)$, so we reject the null hypothesis.

In order to compare the distributions, we consider some other criterion like $K-S$(Kolmogorow Smirnow), $-2\log(L)$,AIC (Akaike Information Criterion), AICC(Akaike Information Criterion Corrected) and BIC(Bayesian information criterion)for the real data set. The best distribution corresponds to lower $K-S$, $-2\log(L)$, AIC, AICC and BIC values:

$$AIC = 2k - 2\log(L), \quad AICC = AIC + \frac{2k(k+1)}{n-k-1},$$

and

$$BIC = k\log(n) - 2\log L.$$

where k is the number of parameters in the statistical model, n the sample size and L is the maximized value of the likelihood function for the estimated model. Also, here for calculating the values of $K-S$ we use the sample estimates of λ and σ. Table 1 shows parameter MLEs to each one of the two fitted distributions, table 2 shows the values of $K-S$, $-2\log(L)$, AIC, AICC and BIC values. The values in table 2 indicate that the transmuted generalized Rayleigh distribution leads to a better fit than the generalized Rayleigh distribution.

8 Conclusion

In this article, we propose a new model: the so-called the transmuted generalized Rayleigh distribution which extends the generalized Rayleigh distribution in the analysis of data with real support. An obvious reason for generalizing a standard distribution is because the generalized form is that it provides greater flexibility in modeling real data. We derive expansions for the expectation, variance, moments and the moment generating function. The estimation of parameters is approached by the method of maximum likelihood, also the information matrix is derived. We consider the likelihood ratio statistic to compare the model with its baseline model. An application of the transmuted generalized Rayleigh distribution to real data show that the new distribution can be used quite effectively to provide better fits than the generalized Rayleigh distribution.

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