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On a Bivariate Fréchet Distribution

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Abstract: The bivariate Fréchet distribution is an important lifetime distribution in survival analysis. In this paper, Farlie-Gumbel-Morgenstern (FGM), Ali-Mikhail-Haq (AMH) copulas and univariate Fréchet distribution are used for creating bivariate distributions which will be called FGM bivariate Fréchet (FGMBF) and AMH Bivariate Fréchet (AMHBF) distributions. The reliability function and hazard function will be obtained for Bivariate Fréchet distributions. Some properties of the FGMBF distribution are obtained such as product moments and moment generation function. Two different estimation methods for the unknown parameters of the two bivariate Fréchet models will be discussed. Asymptotic and bootstrap confidence intervals for the models parameter are also considered. To evaluate the performance of the estimators, a Monte Carlo simulations study is conducted to compare the efficiency between the two models and the preferences between estimation methods. Also, a two real data sets are analyzed to investigate the models and useful results are obtained for illustrative purposes.

Keywords: Fréchet distribution, FGM copula, AMH copula, Maximum Likelihood Estimation, Inference Function for Margins, Bootstrap

1 Introduction

The Fréchet distribution is a well-defined limiting distribution. It is a commonly used model for characterizing variables having extreme phenomena like floods, rains, cash flow and etc. A generalized Fréchet distribution is similar to the exponentiated exponential distribution which is considered as a general case of the exponential distribution. [1] and [2] discusses properties and parameter estimation for exponentiated generalized Fréchet distribution. [3] introduced an extended four-parameter Fréchet model called the exponentiated exponential Fréchet distribution. A random variable X has the two-parameter Fréchet distribution with scale and shape parameters α, λ respectively, if its CDF and PDF are given by

$$F(x;\alpha,\lambda) = e^{-(\frac{x}{\alpha})^{-\lambda}}, \tag{1}$$

and

$$f(x;\alpha,\lambda) = \frac{\lambda}{\alpha}(\frac{x}{\alpha})^{-\lambda-1}e^{-(\frac{x}{\alpha})^{-\lambda}}. \tag{2}$$

Recently, many researches considered the bivariate extension of the Fréchet distribution such as [5] which presented a class of multivariate copulas whose two-dimensional marginals belong to the family of bivariate Fréchet copulas. [4] discussed bivariate Fréchet distribution which is obtained by transforming a bivariate Rayleigh distribution. [6] discussed Patched bivariate Fréchet copula as a mixture of three simple structures: co-monotonicity, independence and counter-monotonicity.

A copula is a convenient approach to describe a multivariate distribution with dependence structure. [7] introduced copulas as following; a copula is a function that joins multivariate distribution functions with uniform [0, 1] margins. [8] introduced the pdf and cdf for the two dimension copula as follows: He considers the two random variables X_1 and X_2, with distribution functions $F_1(x_1)$ and $F_2(x_2)$ respectively, then the cdf and pdf for bivariate copula are respectively given as

$$F(x_1,x_2) = C(F_1(x_1),F_2(x_2)), \tag{3}$$

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and

$$f(x_1,x_2) = f_1(x_1)f_2(x_2)c(F_1(x_1),F_2(x_2)). \tag{4}$$

Many copulas had been defined based on Equations (3) and (4) such as Farlie-Gumbel-Morgenstern (FGM) and Ali-Mikhail-Haq (AMH). FGM copula is one of the most popular parametric families of copulas, the family was firstly introduced by [9]. [17] used FGM copula to introduce bivariate weibull distribution. [12] proposed AMH copula and [11] discussed correlation coefficient of AMH copula by Spearman and Kendall. To more information of the copula and application in copula see [21] and [20]. The joint cdf, pdf and correlation coefficient for FGM and AMH copulas are shown in Table (1).

Table 1: The Joint cdf and pdf for FGM and AMH Copulas

Copula	$C(u,v)$	c(u,v)	θ	Spearman	Kendall
FGM	$uv(\theta(1-u)(1-v))$	$1+\theta(1-2u)(1-2v)$	[-1,1]	[-0.333,0.333]	[-0.222,0.222]
AMH	$\frac{uv}{1-\theta(1-u)(1-v)}$	$\frac{1-\theta+2\theta(\frac{uv}{1-\theta(1-u)(1-v)})}{(1-\theta(1-u)(1-v))^2}$	[-1,1]	[-0.2711,0.4784]	[-0.1817,0.333]

In this paper, we study the bivariate extension of the Fréchet distribution based on FGM and AMH copula functions and discuss their statistical properties. Parameter estimation for FGM and AMH bivariate Fréchet distribution is introduced by two estimation methods which are: the maximum likelihood estimation (MLE) and inference functions for margins (IFM).

The rest of this paper is organized as follows: bivariate Fréchet distribution is obtained in Section 2. Some statistical properties of FGM bivariate copula distribution is given in section 3. Parameter estimation methods for the FGM and AMH bivariate Fréchet distributions is obtained in Section 4. In Section 5, asymptotic confidence and bootstrap confidence intervals are discussed. In Section 6, the potentiality of the new model is illustrated by simulation study. In Section 7, application of a two real data sets is discussed. Finally, conclusion of some remarks for bivariate Fréchet models are addressed in Section 8.

2 Bivariate Fréchet Distribution

According to Sklar theorem, using Equations (1, 2, 3, 4), we get the joint cdf and pdf of the bivariate Fréchet distribution for any copula as follows

$$F(x_1,x_2) = C(e^{-(\frac{x_1}{\alpha_1})^{-\lambda_1}}, e^{-(\frac{x_2}{\alpha_2})^{-\lambda_2}}), \tag{5}$$

and

$$f(x_1,x_2) = \frac{\lambda_1}{\alpha_1}(\frac{x_1}{\alpha_1})^{-\lambda_1-1}e^{-(\frac{x_1}{\alpha_1})^{-\lambda_1}}\frac{\lambda_2}{\alpha_2}(\frac{x_2}{\alpha_2})^{-\lambda_2-1}e^{-(\frac{x_2}{\alpha_2})^{-\lambda_2}}c(e^{-(\frac{x_1}{\alpha_1})^{-\lambda_1}}, e^{-(\frac{x_2}{\alpha_2})^{-\lambda_2}}). \tag{6}$$

In the following sub-sections, we use Equations (5, 6) to define a two bivariate Fréchet distributions based on FGM and AMH copulas.

2.1 FGM Bivariate Fréchet (FGMBF) Distribution

Appling the transformation of the FGM copula that is given in Table (1) and Equations (5, 6), we get the cdf, pdf, reliability function and hazard functions for FGMBF distribution as follows:

$$F_{FGMBF}(x_1,x_2) = e^{-(\frac{x_1}{\alpha_1})^{-\lambda_1}}e^{-(\frac{x_2}{\alpha_2})^{-\lambda_2}}(1+\theta(1-e^{-(\frac{x_1}{\alpha_1})^{-\lambda_1}})(1-e^{-(\frac{x_2}{\alpha_2})^{-\lambda_2}})), \tag{7}$$

$$f_{FGMBF}(x_1,x_2) = \frac{\lambda_1}{\alpha_1}(\frac{x_1}{\alpha_1})^{-\lambda_1-1}e^{-(\frac{x_1}{\alpha_1})^{-\lambda_1}}\frac{\lambda_2}{\alpha_2}(\frac{x_2}{\alpha_2})^{-\lambda_2-1}e^{-(\frac{x_2}{\alpha_2})^{-\lambda_2}}(1+\theta(1-2e^{-(\frac{x_1}{\alpha_1})^{-\lambda_1}})(1-2e^{-(\frac{x_2}{\alpha_2})^{-\lambda_2}})), \tag{8}$$

$$R_{FGMBF}(x_1,x_2) = (1-e^{-(\frac{x_1}{\alpha_1})^{-\lambda_1}})(1-e^{-(\frac{x_2}{\alpha_2})^{-\lambda_2}})(1+\theta(e^{-(\frac{x_1}{\alpha_1})^{-\lambda_1}}e^{-(\frac{x_2}{\alpha_2})^{-\lambda_2}}), \tag{9}$$

and

$$h_{FGMBF}(x_1,x_2)=\frac{\frac{\lambda_1}{\alpha_1}(\frac{x_1}{\alpha_1})^{-\lambda_1-1}e^{-(\frac{x_1}{\alpha_1})^{-\lambda_1}}\frac{\lambda_2}{\alpha_2}(\frac{x_2}{\alpha_2})^{-\lambda_2-1}e^{-(\frac{x_2}{\alpha_2})^{-\lambda_2}}(1+\theta(1-2e^{-(\frac{x_1}{\alpha_1})^{-\lambda_1}})(1-2e^{-(\frac{x_2}{\alpha_2})^{-\lambda_2}}))}{(1-e^{-(\frac{x_1}{\alpha_1})^{-\lambda_1}})(1-e^{-(\frac{x_2}{\alpha_2})^{-\lambda_2}})(1+\theta(e^{-(\frac{x_1}{\alpha_1})^{-\lambda_1}}e^{-(\frac{x_2}{\alpha_2})^{-\lambda_2}})}. \tag{10}$$

respectively, where α_1, λ_1, α_2, $\lambda_2>0$, $-1<\theta<1$ and $x_1,x_2>0$. Figures (1, 2) show the 3-dimension plots for the pdf, cdf, reliability and hazard rate function of FGMBF distribution with different values of parameters. As shown in Figure (1), the figures of pdf, cdf, reliability function and hazard function of FGMBF Distribution in case 1 when $\alpha_1=1.2$, $\lambda_1=2.3$, $\alpha_2=1.7$, $\lambda_2=1.9$ and $\theta=0.25$. In Figure (2), the figures of pdf, cdf, reliability function and hazard function of FGMBF Distribution in case 2 when $\alpha_1=0.8$, $\lambda_1=1.9$, $\alpha_2=1.4$, $\lambda_2=1.6$ and $\theta=0.5$. The hazard function behavior has more than one direction, where it takes an increasing and decreasing, which will have many applications in life testing.

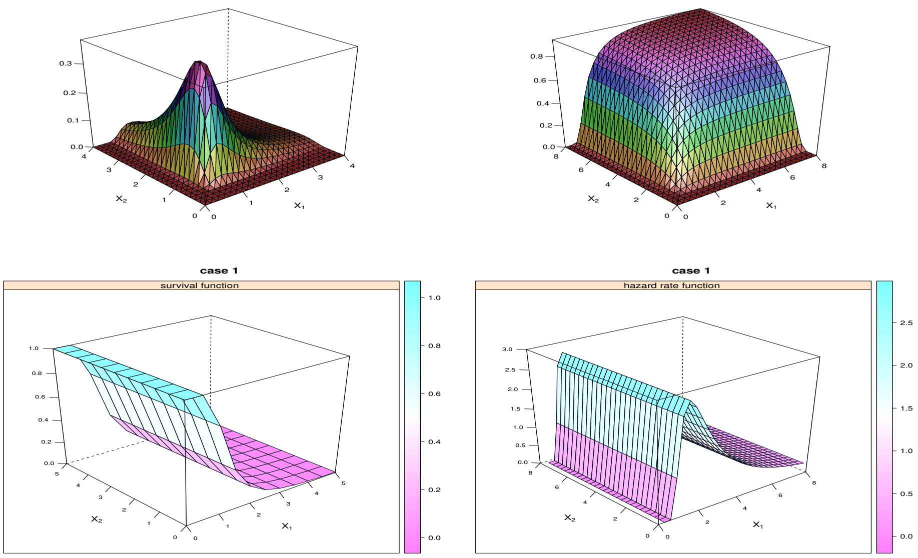

Fig. 1: pdf, cdf, reliability function and hazard function of FGMBF Distribution when $\alpha_1=1.2$, $\lambda_1=2.3$, $\alpha_2=1.7$, $\lambda_2=1.9$ and $\theta=0.25$.

2.2 AMH Bivariate Fréchet (AMHBF) Distribution

Applying the transformation of the AMH copula that is given in Table (1) and Equations (5, 6), we get the cdf, pdf, reliability function and hazard functions for AMHBF distribution respectively, as follows:

$$F_{AMHBF}(x_1,x_2)=\frac{e^{-(\frac{x_1}{\alpha_1})^{-\lambda_1}}e^{-(\frac{x_2}{\alpha_2})^{-\lambda_2}}}{1-\theta(1-e^{-(\frac{x_1}{\alpha_1})^{-\lambda_1}})(1-e^{-(\frac{x_2}{\alpha_2})^{-\lambda_2}})}, \tag{11}$$

$$f_{AMHBF}(x_1,x_2)=\frac{\lambda_1}{\alpha_1}(\frac{x_1}{\alpha_1})^{-\lambda_1-1}e^{-(\frac{x_1}{\alpha_1})^{-\lambda_1}}\frac{\lambda_2}{\alpha_2}(\frac{x_2}{\alpha_2})^{-\lambda_2-1}e^{-(\frac{x_2}{\alpha_2})^{-\lambda_2}}\frac{1-\theta+2\theta(\frac{e^{-(\frac{x_1}{\alpha_1})^{-\lambda_1}}e^{-(\frac{x_2}{\alpha_2})^{-\lambda_2}}}{1-\theta(1-e^{-(\frac{x_1}{\alpha_1})^{-\lambda_1}})(1-e^{-(\frac{x_2}{\alpha_2})^{-\lambda_2}})})}{(1-\theta(1-e^{-(\frac{x_1}{\alpha_1})^{-\lambda_1}})(1-e^{-(\frac{x_2}{\alpha_2})^{-\lambda_2}}))^2} \tag{12}$$

$$R_{AMHBF}(x_1,x_2)=\frac{(1-e^{-(\frac{x_1}{\alpha_1})^{-\lambda_1}})(1-e^{-(\frac{x_2}{\alpha_2})^{-\lambda_2}})}{1-\theta e^{-(\frac{x_1}{\alpha_1})^{-\lambda_1}}e^{-(\frac{x_2}{\alpha_2})^{-\lambda_2}}}, \tag{13}$$

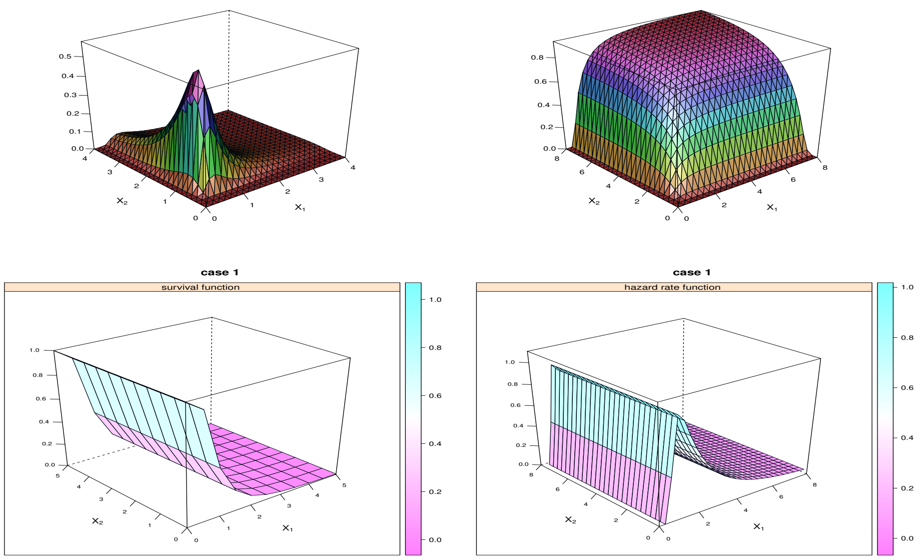

Fig. 2: pdf, cdf, reliability function and hazard function of FGMBF Distribution when $\alpha_1 = 0.8$, $\lambda_1 = 1.9$, $\alpha_2 = 1.4$, $\lambda_2 = 1.6$ and $\theta = 0.5$.

and

$$h_{AMHBF}(x_1,x_2) = \frac{f_{AMHBF}(x_1,x_2)}{R_{AMHBF}(x_1,x_2)}. \tag{14}$$

By using Equations (11, 12, 13, 14) then Figures (3, 4) show the 3-dimension plots for the pdf and cdf of AMHBF distribution with different values of Ω.

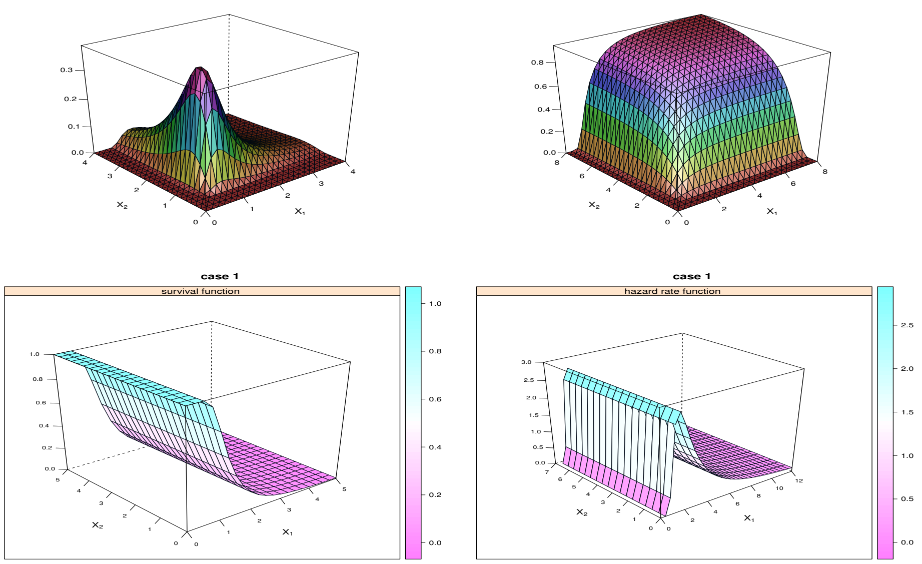

Fig. 3: pdf, cdf, reliability function and hazard function of AMHBF Distribution when $\alpha_1 = 1.2$, $\lambda_1 = 2.3$, $\alpha_2 = 1.7$, $\lambda_2 = 1.9$ and $\theta = 0.25$.

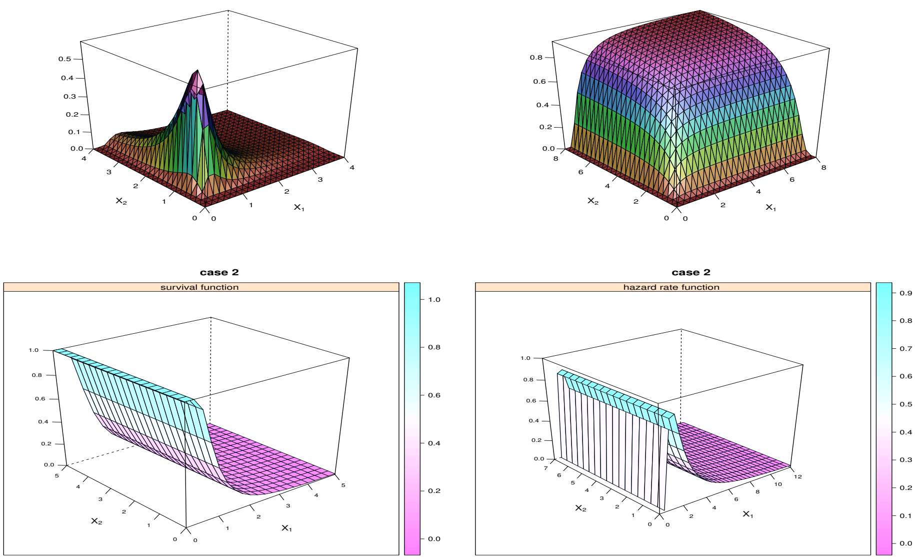

Fig. 4: pdf, cdf, reliability function and hazard function of AMHBF Distribution when $\alpha_1 = 0.8$, $\lambda_1 = 1.9$, $\alpha_2 = 1.4$, $\lambda_2 = 1.6$ and $\theta = 0.5$.

3 Properties of FGMBF Distribution

In this section, we get some important statistical properties of the FGMBF distribution such as marginal distributions, product moments and moment generating function.

3.1 The Marginal Distributions

The marginal density functions for X_1 and X_2 respectively are,

$$f(x_1;\alpha_1,\lambda_1) = \frac{\lambda_1}{\alpha_1}(\frac{x_1}{\alpha_1})^{-\lambda_1-1}e^{-(\frac{x_1}{\alpha_1})^{-\lambda_1}}, \tag{15}$$

and

$$f(x_2;\alpha_2,\lambda_2) = \frac{\lambda_2}{\alpha_2}(\frac{x_2}{\alpha_2})^{-\lambda_2-1}e^{-(\frac{x_2}{\alpha_2})^{-\lambda_2}}, \tag{16}$$

which are Fréchet distributed as shown in Equations (15, 16).

3.2 Moment Generating Function

Let (X_1,X_2) denote a random variable with the pdf in Equation (8). Then, the moment generating function of (X_1,X_2) is given by:

$$M_{x_1,x_2}(t_1,t_2) = \sum_{n=0}^{\infty}(\frac{t_1^n\alpha_1^n}{n!}\Gamma(1-\frac{n}{\lambda_1}))\sum_{m=0}^{\infty}(\frac{t_2^m\alpha_2^m}{m!}\Gamma(1+\frac{m}{\lambda_2}))(1+\theta-\theta 2^{\frac{n}{\lambda_1}}-\theta 2^{\frac{m}{\lambda_2}}+\theta 2^{\frac{n\lambda_2+m\lambda_1}{\lambda_1\lambda_2}}). \tag{17}$$

Such that $n < \lambda_1$.

3.3 Product Moments

If the random variable (X_1, X_2) is distributed as FGMBF, then its r^{th} and s^{th} joint moments around zero denoted by μ'_{rs} can be expressed as follows

$$\mu'_{rs} = \alpha_1^r \Gamma(1-\frac{r}{\lambda_1})\alpha_2^s \Gamma(1+\frac{s}{\lambda_2})(1+\theta-\theta 2^{\frac{s}{\lambda_2}}-\theta 2^{\frac{r}{\lambda_1}}+\theta 2^{\frac{s\alpha_1+r\alpha_2}{\alpha_1\alpha_2}}). \quad (18)$$

Such that $r<\lambda_1$.

4 Parameter Estimation Methods

In this section, we introduce two estimation methods that are used to estimate the unknown parameters of FGMBF and AMHBF distributions, such as: maximum likelihood estimation (MLE) and inference functions for margins (IFM). For more information about these methods see [13].

4.1 Maximum Likelihood Estimation (MLE)

The likelihood function of bivariate Fréchet distribution for any copula is as follows

$$L(\Omega) = (\frac{\lambda_1\lambda_2}{\alpha_1\alpha_2})^n \prod_{i=1}^{n}[(\frac{x_{1i}}{\alpha_1})^{-\lambda_1-1}(\frac{x_{2i}}{\alpha_2})^{-\lambda_2-1}]e^{-\sum_{i=0}^{n}(\frac{x_{1i}}{\alpha_1})^{-\lambda_1}}e^{-\sum_{i=0}^{n}(\frac{x_{2i}}{\alpha_2})^{-\lambda_2}}\prod_{i=0}^{n}c(e^{-(\frac{x_{1i}}{\alpha_1})^{-\lambda_1}},e^{-(\frac{x_{2i}}{\alpha_2})^{-\lambda_2}}). \quad (19)$$

and the log-likelihood function can be written as

$$l(\Omega) = n(\ln(\lambda_1)+\ln(\lambda_2)) - n(\ln(\alpha_1)+\ln(\alpha_2)) - (\lambda_1+1)\sum_{i=1}^{n}\ln(\frac{x_{1i}}{\alpha_1}) - (\lambda_2+1)\sum_{i=1}^{n}\ln(\frac{x_{2i}}{\alpha_2}) + \lambda_1\sum_{i=0}^{n}(\frac{x_{1i}}{\alpha_1}) + \lambda_2\sum_{i=0}^{n}(\frac{x_{2i}}{\alpha_2}) + \sum_{i=0}^{n}\ln(c(e^{-(\frac{x_{1i}}{\alpha_1})^{-\lambda_1}},e^{-(\frac{x_{2i}}{\alpha_2})^{-\lambda_2}})), \quad (20)$$

where Ω is a vector of parameters. The estimates of all parameters are obtained by differentiating the log-likelihood function in (20) with respect to each parameter separately, as following

$$\frac{\partial l(\Omega)}{\partial \alpha_j} = \frac{-n}{\alpha_j} + \frac{n(\lambda_j+1)}{\alpha_j} - \lambda_j\sum_{i=1}^{n}\frac{x_{ji}}{\alpha_j^2} + \frac{\partial}{\partial \alpha_j}\sum_{i=0}^{n}\ln(c(e^{-(\frac{x_{1i}}{\alpha_1})^{-\lambda_1}},e^{-(\frac{x_{2i}}{\alpha_2})^{-\lambda_2}})), \quad (21)$$

and

$$\frac{\partial l(\Omega)}{\partial \lambda_j} = \frac{n}{\lambda_j} + \sum_{i=1}^{n}\ln(\frac{x_{ji}}{\alpha_j}) - \lambda_j\sum_{i=1}^{n}\frac{x_{ji}}{\alpha_j} + \frac{\partial}{\partial \lambda_j}\sum_{i=0}^{n}\ln(c(e^{-(\frac{x_{1i}}{\alpha_1})^{-\lambda_1}},e^{-(\frac{x_{2i}}{\alpha_2})^{-\lambda_2}})), \quad (22)$$

where $j=l=1,2; j \neq l$ For FGMBF distribution

$\frac{\partial}{\partial \alpha_j}\sum_{i=0}^{n}\ln(c(e^{-(\frac{x_{1i}}{\alpha_1})^{-\lambda_1}},e^{-(\frac{x_{2i}}{\alpha_2})^{-\lambda_2}})) = \sum_{i=1}^{n} a(x_{ji},x_{li},\lambda_j,\lambda_l,\alpha_j,\alpha_l,\theta)$

$\frac{\partial}{\partial \lambda_j}\sum_{i=0}^{n}\ln(c(e^{-(\frac{x_{1i}}{\alpha_1})^{-\lambda_1}},e^{-(\frac{x_{2i}}{\alpha_2})^{-\lambda_2}})) = \sum_{i=1}^{n} b(x_{ji},x_{li},\lambda_j,\lambda_l,\alpha_j,\alpha_l,\theta)$

and $\frac{\partial l(\Omega)}{\partial \theta} = \sum_{i=1}^{n} d(x_{ji},x_{li},\lambda_j,\lambda_l,\alpha_j,\alpha_l,\theta)$

For AMHBF distribution

$\frac{\partial}{\partial \alpha_j}\sum_{i=1}^{n}\ln\frac{1-\theta+2\theta(F_{AMHBF}(x_{1i},x_{2i}))}{(1-\theta w(x_{1i},x_{2i}))^2} = \sum_{i=1}^{n}\zeta(x_{ji},x_{li},\lambda_j,\lambda_l,\alpha_j,\alpha_l,\theta) - \sum_{i=1}^{n}\eta(x_{ji},x_{li},\lambda_j,\lambda_l,\alpha_j,\alpha_l,\theta)$,

$\frac{\partial}{\partial \lambda_j}\sum_{i=1}^{n}\ln\frac{1-\theta+2\theta(F_{AMHBF}(x_{1i},x_{2i}))}{(1-\theta w(x_{1i},x_{2i}))^2} = \sum_{i=1}^{n}\varsigma(x_{ji},x_{li},\lambda_j,\lambda_l,\alpha_j,\alpha_l,\theta) - \sum_{i=1}^{n}\rho(x_{ji},x_{li},\lambda_j,\lambda_l,\alpha_j,\alpha_l,\theta)$,

and $\frac{\partial l(\Omega)}{\partial \theta} = \Upsilon(x_{ji},x_{li},\lambda_j,\lambda_l,\alpha_j,\alpha_l,\theta)$,

where

$w(x_{1i},x_{2i}) = (1-e^{-(\frac{x_1}{\alpha_1})^{-\lambda_1}})(1-e^{-(\frac{x_2}{\alpha_2})^{-\lambda_2}})$,

$$a(x_{ji},x_{li},\lambda_j,\lambda_l,\alpha_j,\alpha_l,\theta)=\frac{2\theta\lambda_j e^{-(\frac{x_{ji}}{\alpha_j})^{-\lambda_j}}(\frac{x_{ji}}{\alpha_j})^{-\lambda_j}(1-2e^{-(\frac{x_{ji}}{\alpha_j})^{-\lambda_j}})}{\alpha_j(1+\theta(1-2e^{-(\frac{x_{ji}}{\alpha_j})^{-\lambda_j}})(1-2e^{-(\frac{x_{li}}{\alpha_l})^{-\lambda_l}}))},$$

$$b(x_{ji},x_{li},\lambda_j,\lambda_l,\alpha_j,\alpha_l,\theta)=\frac{-2\theta e^{-(\frac{x_{ji}}{\alpha_j})^{-\lambda_j}}(\frac{x_{ji}}{\alpha_j})^{-\lambda_j}\ln(\frac{x_{ji}}{\alpha_j})(1-2e^{-(\frac{x_{ji}}{\alpha_j})^{-\lambda_j}})}{(1+\theta(1-2e^{-(\frac{x_{ji}}{\alpha_j})^{-\lambda_j}})(1-2e^{-(\frac{x_{li}}{\alpha_l})^{-\lambda_l}}))},$$

$$d(x_{ji},x_{li},\lambda_j,\lambda_l,\alpha_j,\alpha_l,\theta)=\frac{(1-2e^{-(\frac{x_{ji}}{\alpha_j})^{-\lambda_j}})(1-2e^{-(\frac{x_{li}}{\alpha_l})^{-\lambda_l}})}{(1+\theta(1-2e^{-(\frac{x_{ji}}{\alpha_j})^{-\lambda_j}})(1-2e^{-(\frac{x_{li}}{\alpha_l})^{-\lambda_l}}))},$$

$$\zeta(x_{ji},x_{li},\lambda_j,\lambda_l,\alpha_j,\alpha_l,\theta)=\frac{2\theta\lambda_j(1-\theta+2\theta\frac{e^{-(\frac{x_{ji}}{\alpha_j})^{-\lambda_j}}e^{-(\frac{x_{li}}{\alpha_l})^{-\lambda_l}}}{(1-\theta w(x_{1i},x_{2i}))})(\frac{x_{ji}}{\alpha_j})^{-\lambda_j}(1-e^{-(\frac{x_{li}}{\alpha_l})^{-\lambda_l}})}{\alpha_j(1-\theta w(x_{1i},x_{2i}))^3},$$

$$\eta(x_{ji},x_{li},\lambda_j,\lambda_l,\alpha_j,\alpha_l,\theta)=\frac{2\theta\lambda_j(-1+\theta w(x_{1i},x_{2i})+\theta e^{-(\frac{x_{ji}}{\alpha_j})^{-\lambda_j}}(1-e^{-(\frac{x_{li}}{\alpha_l})^{-\lambda_l}}))e^{-(\frac{x_{li}}{\alpha_l})^{-\lambda_l}}e^{-(\frac{x_{ji}}{\alpha_j})^{-\lambda_j}}(\frac{x_{ji}}{\alpha_j})^{-\lambda_j}}{\alpha_j(1-\theta w(x_{1i},x_{2i}))^4},$$

$$\varsigma(x_{ji},x_{li},\lambda_j,\lambda_l,\alpha_j,\alpha_l,\theta)=\frac{2\theta\lambda_j(1-\theta+2\theta\frac{e^{-(\frac{x_{ji}}{\alpha_j})^{-\lambda_j}}e^{-(\frac{x_{li}}{\alpha_l})^{-\lambda_l}}}{(1-\theta w(x_{1i},x_{2i}))})(1-e^{-(\frac{x_{li}}{\alpha_l})^{-\lambda_l}})e^{-(\frac{x_{ji}}{\alpha_j})^{-\lambda_j}}(\frac{x_{ji}}{\alpha_j})^{-\lambda_j}\ln(\frac{x_{ji}}{\alpha_j})}{(1-\theta w(x_{1i},x_{2i}))^3},$$

$$\rho(x_{ji},x_{li},\lambda_j,\lambda_l,\alpha_j,\alpha_l,\theta)=\frac{2\theta\lambda_j(-1+\theta w(x_{1i},x_{2i})+\theta e^{-(\frac{x_{ji}}{\alpha_j})^{-\lambda_j}}(1-e^{-(\frac{x_{li}}{\alpha_l})^{-\lambda_l}}))e^{-(\frac{x_{li}}{\alpha_l})^{-\lambda_l}}e^{-(\frac{x_{ji}}{\alpha_j})^{-\lambda_j}}(\frac{x_{ji}}{\alpha_j})^{-\lambda_j}\ln(\frac{x_{ji}}{\alpha_j})}{(1-\theta w(x_{1i},x_{2i}))^4},$$

$$\Upsilon(x_{ji},x_{li},\lambda_j,\lambda_l,\alpha_j,\alpha_l,\theta)=\frac{2w(x_{1i},x_{2i})(\theta+\frac{2\theta w(x_{1i},x_{2i})}{\theta w(x_{1i},x_{2i})-1}-1)}{(\theta w(x_{1i},x_{2i})-1)^3}-\frac{\frac{2w(x_{1i},x_{2i})}{(\theta w(x_{1i},x_{2i})-1)}-\frac{2\theta(w(x_{1i},x_{2i}))^2}{(\theta w(x_{1i},x_{2i})-1)^2}+1}{(\theta w(x_{1i},x_{2i})-1)^3},$$

4.2 Estimation by Inference Functions for Margins (IFM)

[14] introduced this parametric method with two-steps of estimation. Firstly, each marginal distribution is estimated separately. $\ln(L_1)=\sum_{i=1}^{n}\ln(f(x_{1i};\alpha_1,\lambda_1));\ln(L_2)=\sum_{i=1}^{n}\ln(f(x_{2i};\alpha_2,\lambda_2))$. Then, in the second step the copula parameter is estimated by maximizing the log-likelihood function of the copula density using the MLE estimates of the marginal $F(x_{1i};\alpha_1,\lambda_1)$ and $F(x_{2i};\alpha_2,\lambda_2)$. Considering the Equations (15, 16), the log-likelihood function of a Fréchet distribution is defined as:

$$l(\alpha_j,\lambda_j)=n\ln(\lambda_j-\alpha_j)-(\lambda_1+1)\sum_{i=1}^{n}\ln(\frac{x_{ji}}{\alpha_j})-\sum_{i=1}^{n}(\frac{x_{ji}}{\alpha_j})^{-\lambda_j}, \tag{23}$$

By Equation (23), the MLE of $(\lambda_j,\lambda_l,\alpha_j,\alpha_l)$ can be obtained by solving simultaneously the likelihood equations.

$\frac{\partial l(\alpha_j,\lambda_j)}{\partial\alpha_j}|_{\alpha_j=\hat{\alpha_j}}=0,\frac{\partial l(\alpha_j,\lambda_j)}{\partial\lambda_j}|_{\lambda_j=\hat{\lambda_j}}=0;j=1,2.$

then

$\hat{F}_j(x_{ji})=e^{-(\frac{x_{ji}}{\hat{\alpha_j}})^{-\hat{\lambda_j}}}$,

and considering the previous step, the IFM estimate of any copula function is defined as

$$l(\theta)=\sum_{i=1}^{n}\ln(c(\hat{F}_j(x_{ji}),\hat{F}_l(x_{li})) \tag{24}$$

The estimators of all parameters are obtained by differentiating the log-likelihood function in Equation 24 with respect to each parameter separately. Based on this, differentiating the log-likelihood function with respect to θ for FGMBF distribution is given as

$\frac{\partial l)IFM(\theta)}{\partial\theta}=\sum_{i=1}^{n}d(x_{ji},x_{li},\hat{\lambda}_j,\hat{\lambda}_l,\hat{\alpha}_j,\hat{\alpha}_l)$

While the differentiation of the log-likelihood function with respect to θ for AMHBF distribution is given as

$\frac{\partial l_{IFM}(\theta)}{\partial\theta}=\Upsilon(x_{ji},x_{li},\hat{\lambda}_j,\hat{\lambda}_l,\hat{\alpha}_j,\hat{\alpha}_l)$.

The estimates of the parameters are handled numerically by solving simultaneously the likelihood equations

$\frac{\partial l_{IFM}(\theta)}{\partial \theta}|_{\theta=\hat{\theta}} = 0$

There is no closed-form for the MLE $\hat{\theta}$ and its computation has to be performed numerically using a nonlinear optimization algorithm.

5 Confidence Interval

In this section, we propose two different methods to construct confidence intervals (CI) for the unknown parameters of bivariate Fréchet distribution, which are asymptotic confidence interval (ACI) and bootstrap confidence interval of α_k, λ_k where $k = 1,2$ and θ. Bootstrap approach is subdivided into percentile bootstrap and bootstrap-t.

5.1 Asymptotic Confidence Intervals

The most common method to set confidence bounds for the parameters is to use the asymptotic normal distribution of the MLE. In relation to the asymptotic variance-covariance matrix of the MLE of the parameters, Fisher information matrix $I(\Omega)$, where it is composed of the negative second derivatives of the natural logarithm of the likelihood function evaluated at $\hat{\Omega} = (\hat{\alpha_1}, \hat{\lambda_1}, \hat{\alpha_2}, \hat{\lambda_2}, \hat{\theta})$. Suppose the asymptotic variance-covariance matrix of the parameter vector Ω is

$$I(\hat{\Omega}) = \begin{bmatrix} I_{\hat{\alpha_1}\hat{\alpha_1}} & & & & \\ I_{\hat{\lambda_1}\hat{\alpha_1}} & I_{\hat{\lambda_1}\hat{\lambda_1}} & & & \\ I_{\hat{\alpha_2}\hat{\alpha_1}} & I_{\hat{\alpha_2}\hat{\lambda_1}} & I_{\hat{\alpha_2}\hat{\alpha_2}} & & \\ I_{\hat{\lambda_2}\hat{\alpha_1}} & I_{\hat{\lambda_2}\hat{\lambda_1}} & I_{\hat{\lambda_2}\hat{\alpha_2}} & I_{\hat{\lambda_2}\hat{\lambda_2}} & \\ I_{\hat{\theta}\hat{\alpha_1}} & I_{\hat{\theta}\hat{\lambda_1}} & I_{\hat{\theta}\hat{\alpha_2}} & I_{\hat{\theta}\hat{\lambda_2}} & I_{\hat{\theta}\hat{\theta}} \end{bmatrix}$$

where $V(\hat{\Omega}) = I^{-1}(\hat{\Omega})$
A $100(1-\gamma)\%$ confidence interval for parameter Ω can be constructed based on the asymptotic normality of the MLE.
$\hat{\alpha_j} \pm Z_{0.025}\sqrt{I_{\hat{\alpha_j}\hat{\alpha_j}}}, \hat{\lambda_j} \pm Z_{0.025}\sqrt{I_{\hat{\lambda_j}\hat{\lambda_j}}}$
and $\hat{\theta_j} \pm Z_{0.025}\sqrt{I_{\hat{\theta_j}\hat{\theta_j}}}$,
where $Z_{0.025}$ is the percentile of the standard normal distribution with right tail probability $\frac{\gamma}{2}$.

5.2 Bootstrap Confidence Interval

The bootstrap is a resampling method for statistical inference. It is commonly used to estimate confidence intervals. For more information see [15]. In this subsection, we use the parametric bootstrap method to construct confidence intervals for the unknown parameters α_k, λ_k where $k = 1,2$ and θ. We introduced two parametric bootstrap methods, percentile bootstrap (B-P) and bootstrap-t (B-t) CI.

5.2.1 Percentile Bootstrap Confidence Interval

1.Compute the MLE of Ω of distributions.
2.Generate a bootstrap samples using $\alpha_k, \lambda_k$and θ to obtain the bootstrap estimate of α_k say α_k^b , λ_k say λ_k^b and θ say θ^b using the bootstrap sample.
3.Repeat step (2) $\mathscr{B}$ times to have $(\alpha_k^{b(1)}, \alpha_k^{b(2)}, \ldots, \alpha_k^{b(\mathscr{B})}), (\lambda_k^{b(1)}, \lambda_k^{b(2)}, \ldots \lambda_k^{b(\mathscr{B})}) and (\theta_k^{b(1)}, \theta_k^{b(2)}, \ldots, \theta_k^{b(\mathscr{B})})$.
4.Arrange $(\alpha_k^{b(1)}, \alpha_k^{b(2)}, \ldots, \alpha_k^{b(\mathscr{B})}), (\lambda_k^{b(1)}, \lambda_k^{b(2)}, \ldots \lambda_k^{b(\mathscr{B})}) and (\theta_k^{b(1)}, \theta_k^{b(2)}, \ldots, \theta_k^{b(\mathscr{B})})$ in ascending order as $(\alpha_k^{b[1]}, \alpha_k^{b[2]}, \ldots, \alpha_k^{b([\mathscr{B}])}), (\lambda_k^{b[1]}, \lambda_k^{b[2]}, \ldots \lambda_k^{b[\mathscr{B}]}) and (\theta_k^{b[1]}, \theta_k^{b[2]}, \ldots, \theta_k^{b[\mathscr{B}]})$.
5.Two side $100(1-\gamma)\%$ percentile bootstrap confidence interval for the unknown parameters α_k, λ_k where $k = 1,2$ and θ are given by $[\alpha_k^{b([\mathscr{B}\frac{\gamma}{2}])}, \alpha_k^{b([\mathscr{B}(1-\frac{\gamma}{2})])}], [\lambda_k^{b([\mathscr{B}\frac{\gamma}{2}])}, \lambda_k^{b([\mathscr{B}(1-\frac{\gamma}{2})])}]$ and $[\theta^{b([\mathscr{B}\frac{\gamma}{2}])}, \theta^{b([\mathscr{B}(1-\frac{\gamma}{2})])}]$.

5.2.2 Bootstrap-t Confidence Interval

1. Same as the steps (1,2) in Boot-p.
2. Compute the t-statistic of Ω as $T = \frac{\hat{\Omega}^b - \hat{\Omega}}{\sqrt{V(\hat{\Omega}^b)}}$ where $V(\hat{\Omega}^b)$ is asymptotic variances of $\hat{\Omega}^b$ and it can be obtained using the Fisher information matrix.
3. Repeat steps II-III $\mathscr{B}$ times and obtain $(T^{(1)}, T^{(2)}, \ldots, T^{(\mathscr{B})})$.
4. Arrange $(T^{(1)}, T^{(2)}, \ldots, T^{(\mathscr{B})})$ in ascending order as $(T^{[1]}, T^{[2]}, \ldots, T^{[\mathscr{B}]})$.
5. A two side $100(1-\gamma)\%$ percentile bootstrap confidence interval for the unknown parameters α_k, λ_k where $k = 1,2$ and θ are given by $[\alpha_k + T_k^{b([\mathscr{B}\frac{\gamma}{2}])}\sqrt{V(\alpha_k^b)}, \alpha_k + \sqrt{V(\alpha_k^b)}T_k^{b([\mathscr{B}(1-\frac{\gamma}{2})])}]$, $[\lambda_k + T_k^{b([\mathscr{B}\frac{\gamma}{2}])}\sqrt{V(\lambda_k^b)}, \lambda_k + \sqrt{V(\lambda_k^b)}T_k^{b([\mathscr{B}(1-\frac{\gamma}{2})])}]$ and $[\theta + T_k^{b([\mathscr{B}\frac{\gamma}{2}])}\sqrt{V(\theta^b)}, \theta + \sqrt{V(\theta^b)}T_k^{b([\mathscr{B}(1-\frac{\gamma}{2})])}]$

6 Simulation Study

In this section; a Monte Carlo simulation is done for comparison between estimation methods based on copula such as: MLE and IFM. For estimating FGMBF and AMGBF distributions parameters R language was used.
To generate random variables: [7] discussed generating a sample from a specified joint distribution. By conditional distribution method, the joint distribution function is as follows $f(x_1,x_2) = f(x_1)f(x_2|x_1)$.
By using the following steps, we can generate a bivariate sample by using the conditional approach:

1. Generate U and V independently from a uniform(0,1) distribution.
2. Set $X_1 = \lambda_1[-\ln(U)]^{\frac{-1}{\lambda_1}}$.
3. Set $F(x_2|x_1) = V$ to find x_2 by a numerical simulation.
4. Repeat Steps (1, 2, 3) (n) times to obtain $(x_{1i}, x_{2i}), i = 1,2,\ldots,n$.

A simulation algorithm: Simulation experiments were carried out based on the following data generated form Fréchet Distributions, where X_1, X_2 are distributed as Fréchet with λ_j shape parameters and α_j scale parameter, $j = 1,2$ the values of the parameters $\alpha_1, \lambda_1, \alpha_2, \lambda_2$ and θ are chosen as the following cases for the random variables generating:
Case 1: $(\alpha_1 = 1.2, \lambda_1 = 2.3, \alpha_2 = 1.7, \lambda_2 = 1.9, \theta = 0.25)$,
Case 2: $(\alpha_1 = 0.8, \lambda_1 = 1.9, \alpha_2 = 1.4, \lambda_2 = 1.6, \theta = 0.5)$,
Case 3: $(\alpha_1 = 1.2, \lambda_1 = 0.9, \alpha_2 = 1.1, \lambda_2 = 0.75, \theta = -0.15)$.
For different sample size $n = 30, 50, 100$, and 200. The simulation methods are compared using the criteria of parameters estimation, the comparison is performed by calculating the Bias, the MSE, the length of asymptotic and bootstrap confidence intervals (L.CI) for each method of estimation as following

$Bias = (\hat{\Omega} - \Omega)$
Where $\hat{\Omega}$ is the estimated value of Ω.
$MSE = Mean(\hat{\Omega} - \Omega)^2$.
and L.CI=Upper.CI-Lower.CI
We restricted the number of repeated-samples to 1000.

On the basis of the results summarized in Tables (2, 3, 4) and figures (5, 6, 7, 8, 9, 10). It is observed that as sample size increases while fixed the vector value of Ω, the Bias, MSE and Length of confidence interval of the estimates decrease in all the considered methods. At large sample size, the estimators for different methods are nearly equivalent, where the difference is less and there are no significant differences in Bias and MSE values for IFM method and MLE method. It is noted that the IFM method gives better results that the MLE method. IFM method is the best method because it has two steps of estimation, first, the marginal distribution parameters are estimated and second, the copula parameter is estimated, taking into consideration previous parameter estimates of marginal distribution.

7 Application of Real Data

In this section, we analyze a real economic Egyptian data set for world bank national accounts data, and reanalyze a medical data set that was analyzed by [16] and [17]. We study the parameter estimation of the appropriate distribution of each data set, where the correlation between the two variables (bivariate data) is low. And through this, we access a fit model specialized in the study of weak relations and the extent of their impact and effectiveness.

7.1 The Economic Data

The economic data set introduced in [10], which is reproduced in Table (5) consists of 31 observations 1980 : 2010 on a response variable: Exports of goods and services (x_1) and GDP growth (x_2). The main reasons for selecting the economic data for the present study is due to the fact that, economy is an important sector for many developed and developing countries. Thus, the government is interested in increasing GDP growth and exports of goods and services. To show the usefulness of the proposed bivariate estimators obtained from Sections (2, 3) with real situations, we considered here the real economic data to estimate parameters of distribution for the GDP growth and exports of goods and services. where x_1 is exports of goods and services (EGS) (1980:2010) x_2 is GDP growth (1980:2010)
Source: The economics data are collected by World Bank National Accounts data and OECD National Accounts data.
The Goodness of Fit Test for Copulas: The simplest goodness-of-fit test for copulas lies in comparing the distance between a nonparametric estimate $\hat{C}_n$ of C and a parametric estimate C_θ derived from an estimator θ which is consistent when the null hypothesis H_0 holds. Let (F_{1n}, F_{2n}) be the empirical distribution functions of (F_1, F_2), respectively. A natural substitute for the unobservable $U_i = (F_{1i}, F_{2i})$ where $i = 1, \ldots, n$, is given by $(\hat{U_{1i}}, \hat{U_{1i}}) = (\frac{n}{n+1}F_{1n}(X_1 i), \frac{n}{n+1}F_{2n}(X_2 i))$.
The scaling factor $\frac{n}{n+1}$ used in defining $\hat{U_i}$ avoids numerical issues that sometimes occur, when a parametric copula density is evaluated at pseudo-observations. A natural estimator $\hat{C_n}$ of C, called the empirical copula, is then defined, for all $(u_1, u_2); u \in [0, 1]$, by
$\hat{C_n}(u_1, u_2) = \frac{1}{n}\sum_{i=1}^{n} I(\hat{U_{1i}} \leq u_{1i}, \hat{U_{2i}} \leq u_{2i})$
[18] introduced Multiplier bootstrap-based goodness-of-fit test, this consideration leads naturally to Anderson Darling-type statistics such as
$R_n = n\int_0^1 (\frac{\hat{C_n}(u_1,u_2) - \hat{C_{\theta n}}(u_1,u_2)}{(C_{\theta n}(u_1,u_2)(1-C_{\theta n}(u_1,u_2)) + \delta_m)^m})^2 d\hat{C_n}(u_1, u_2)$
Involving a consistent, rank-based estimator θ_n of θ, and tuning parameters $m \geq \theta$ and $\delta_m \geq 0$.
We use the conclusions of Genest to fit FGM and AMH copulas by R package.
This by using a parametric bootstrap N=10000 time and the empirical copula estimate see Table (6).
The comparison was done between FGMBF and AMHBF distributions for economic data by using different criteria as AIC, BIC, HQIC and CAiC see Table (7). The comparison of method estimation was done between MLE and IFM methods for economic data by using different standard deviation (std), see Table (8).

7.2 The Medical Data

The data set for 30 patients from [19], where Y_1: refers to first recurrence time, Y_2: to second recurrence time. [16] discussed the estimation of the parameters of FGM bivariate generalized exponential distribution for this data. By using a parametric bootstrap N=10000 time and the empirical copula estimate for medical data, see Table (9). The comparison was done between FGMBF and AMHBF distributions for medical data by using different criteria as AIC, BIC, HQIC and CAiC see Table (10). The comparison of method estimation was done between MLE and IFM methods for medical data by using different standard deviation (std), see Table (11).

The best method for of Estimation of the FGMBF and AMHBF distributions is IFM method since it has the least standard deviation of the copula parameter.

8 Conclusion

In this paper, we have proposed a class of bivariate Fréchet distributions based on FGM and AMH copula functions. Moreover, we obtained the reliability functions for FGMBF and AMHBF distributions; therefore, it can be used quite effectively in life testing data. Additionally, the new AMHBF model can be used as an alternative to any bivariate Fréchet distribution for different applications, but mathematically the new FGMBF model is works better because the marginal functions of FGMBF distribution has the same basic distribution and it has closed forms for moment generating function and product moments. A comparison between different estimation methods of the bivariate Fréchet distributions are concluded. The result shows that the best method of estimation is IFM method. Hence, we can argue that IFM estimator is the best performing estimator for FGMBF and AMHBF distributions.

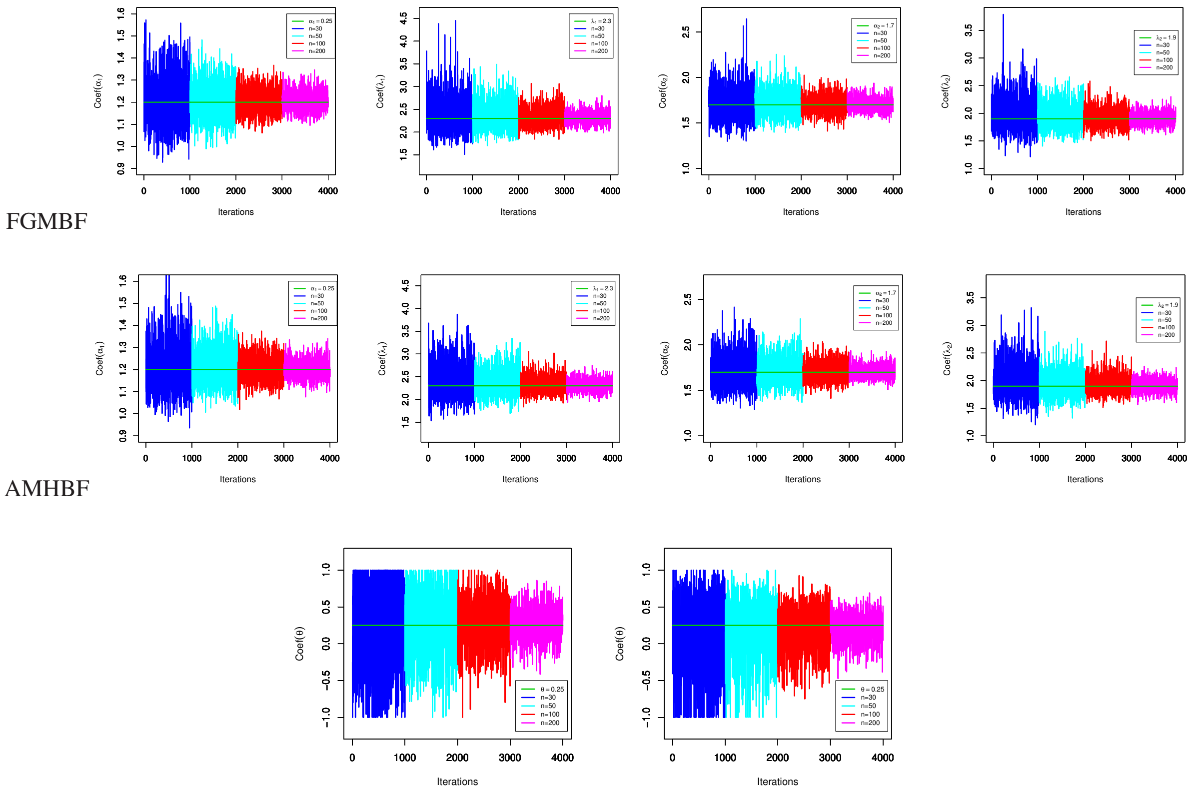

Fig. 5: Parameters Coefficient for FGMBF and AMHBF distributions by Using MLE in case 1.

Table 2: Estimation of the Parameters of FGMBF and AMGBF Distributions: Case 1.

			FGMBF					AMHBF				
n			Bias	MSE	ACI	Bt	BP	Bias	MSE	ACI	Bt	BP
30	MLE	$\hat{\alpha}_1$	0.0089	0.0119	0.4266	0.0547	0.0542	0.0087	0.0116	0.4209	0.0507	0.0502
		$\hat{\lambda}_1$	0.1177	0.1523	1.4593	0.2182	0.2124	0.1211	0.1478	1.4309	0.1895	0.1961
		$\hat{\alpha}_2$	0.0137	0.0289	0.6649	0.0856	0.0872	0.0151	0.0310	0.6882	0.0788	0.0801
		$\hat{\lambda}_2$	0.1004	0.0944	1.1387	0.1572	0.1613	0.1034	0.1027	1.1900	0.1619	0.1603
		$\hat{\theta}$	-0.0169	0.2757	2.0584	0.2397	0.2442	-0.0952	0.2589	1.9603	0.2221	0.2243
	IFM	$\hat{\alpha}_1$	0.0089	0.0118	0.4245	0.0536	0.0556	0.0080	0.0115	0.4202	0.0533	0.0549
		$\hat{\lambda}_1$	0.1154	0.1510	1.4554	0.1839	0.1837	0.1218	0.1480	1.4310	0.2010	0.2030
		$\hat{\alpha}_2$	0.0132	0.0287	0.6627	0.0707	0.0734	0.0144	0.0308	0.6862	0.0764	0.0778
		$\hat{\lambda}_2$	0.0988	0.0937	1.1361	0.1468	0.1501	0.1037	0.1020	1.1845	0.1382	0.1416
		$\hat{\theta}$	-0.0191	0.2690	2.0326	0.2692	0.2781	-0.1606	0.6410	3.0762	0.3005	0.2999
50	MLE	$\hat{\alpha}_1$	0.0029	0.0068	0.3232	0.0320	0.0324	0.0078	0.0062	0.3075	0.0288	0.0295
		$\hat{\lambda}_1$	0.0703	0.0791	1.0679	0.1062	0.1054	0.0708	0.0736	1.0270	0.0906	0.0897
		$\hat{\alpha}_2$	0.0104	0.0183	0.5288	0.0562	0.0571	0.0169	0.0194	0.5426	0.0553	0.0541
		$\hat{\lambda}_2$	0.0553	0.0502	0.8517	0.0872	0.0875	0.0472	0.0537	0.8901	0.0921	0.0945
		$\hat{\theta}$	-0.0183	0.1762	1.6446	0.1641	0.1687	-0.0678	0.1505	1.4981	0.1362	0.1353
	IFM	$\hat{\alpha}_1$	0.0027	0.0068	0.3226	0.0350	0.0343	0.0076	0.0062	0.3065	0.0306	0.0305
		$\hat{\lambda}_1$	0.0702	0.0788	1.0663	0.1067	0.1085	0.0715	0.0739	1.0285	0.0981	0.0965
		$\hat{\alpha}_2$	0.0103	0.0182	0.5272	0.0637	0.0646	0.0165	0.0194	0.5424	0.0488	0.0494
		$\hat{\lambda}_2$	0.0553	0.0502	0.8511	0.0795	0.0771	0.0473	0.0536	0.8886	0.0911	0.0922
		$\hat{\theta}$	-0.0203	0.1730	1.6291	0.1739	0.1732	-0.0782	0.1825	1.6472	0.1922	0.1941
100	MLE	$\hat{\alpha}_1$	0.0041	0.0030	0.2158	0.0145	0.0145	0.0025	0.0029	0.2122	0.0140	0.0140
		$\hat{\lambda}_1$	0.0344	0.0378	0.7507	0.0506	0.0500	0.0342	0.0326	0.6952	0.0476	0.0491
		$\hat{\alpha}_2$	0.0063	0.0087	0.3651	0.0273	0.0270	0.0077	0.0095	0.3820	0.0247	0.0245
		$\hat{\lambda}_2$	0.0273	0.0254	0.6153	0.0400	0.0411	0.0256	0.0251	0.6128	0.0418	0.0424
		$\hat{\theta}$	0.0179	0.0904	1.1770	0.0669	0.0679	-0.0329	0.0772	1.0824	0.0777	0.0813
	IFM	$\hat{\alpha}_1$	0.0042	0.0030	0.2147	0.0149	0.0148	0.0025	0.0029	0.2119	0.0164	0.0161
		$\hat{\lambda}_1$	0.0341	0.0377	0.7495	0.0564	0.0571	0.0338	0.0327	0.6969	0.0449	0.0452
		$\hat{\alpha}_2$	0.0060	0.0086	0.3637	0.0251	0.0255	0.0076	0.0096	0.3825	0.0277	0.0275
		$\hat{\lambda}_2$	0.0275	0.0253	0.6147	0.0437	0.0437	0.0257	0.0250	0.6124	0.0397	0.0410
		$\hat{\theta}$	0.0164	0.0893	1.1700	0.0766	0.0759	-0.0340	0.0763	1.0754	0.0677	0.0691
200	MLE	$\hat{\alpha}_1$	0.0024	0.0017	0.1602	0.0083	0.0080	0.0010	0.0015	0.1526	0.0076	0.0075
		$\hat{\lambda}_1$	0.0115	0.0161	0.4949	0.0261	0.0276	0.0132	0.0159	0.4917	0.0259	0.0264
		$\hat{\alpha}_2$	0.0064	0.0048	0.2718	0.0134	0.0134	0.0046	0.0044	0.2602	0.0130	0.0132
		$\hat{\lambda}_2$	0.0085	0.0123	0.4332	0.0219	0.0226	0.0041	0.0112	0.4156	0.0212	0.0220
		$\hat{\theta}$	0.0070	0.0414	0.7978	0.0407	0.0409	-0.0137	0.0344	0.7251	0.0348	0.0338
	IFM	$\hat{\alpha}_1$	0.0025	0.0017	0.1605	0.0081	0.0083	0.0010	0.0015	0.1528	0.0077	0.0076
		$\hat{\lambda}_1$	0.0115	0.0160	0.4942	0.0235	0.0237	0.0132	0.0160	0.4927	0.0251	0.0245
		$\hat{\alpha}_2$	0.0065	0.0048	0.2716	0.0144	0.0146	0.0044	0.0044	0.2603	0.0127	0.0124
		$\hat{\lambda}_2$	0.0084	0.0122	0.4324	0.0217	0.0208	0.0042	0.0112	0.4151	0.0199	0.0206
		$\hat{\theta}$	0.0064	0.0412	0.7957	0.0409	0.0399	-0.0142	0.0342	0.7237	0.0355	0.0356

Table 3: Estimation of the Parameters of FGMBF and AMGBF Distributions: Case 2.

			FGMBF					AMHBF				
n			Bias	MSE	ACI	Bt	BP	Bias	MSE	ACI	Bt	BP
30	MLE	$\hat{\alpha}_1$	0.0080	0.0078	0.3440	0.0444	0.0448	0.0083	0.0076	0.3412	0.0408	0.0407
		$\hat{\lambda}_1$	0.0995	0.1044	1.2057	0.1812	0.1755	0.0999	0.1003	1.1787	0.1575	0.1657
		$\hat{\alpha}_2$	0.0147	0.0280	0.6540	0.0812	0.0803	0.0177	0.0301	0.6766	0.0802	0.0801
		$\hat{\lambda}_2$	0.0861	0.0670	0.9574	0.1300	0.1331	0.0882	0.0745	1.0133	0.1373	0.1357
		$\hat{\theta}$	-0.0451	0.2300	1.8727	0.2089	0.2111	-0.0957	0.1923	1.6783	0.1865	0.1856
	IFM	$\hat{\alpha}_1$	0.0080	0.0078	0.3441	0.0435	0.0453	0.0072	0.0076	0.3413	0.0435	0.0447
		$\hat{\lambda}_1$	0.0953	0.1031	1.2023	0.1519	0.1518	0.1006	0.1010	1.1821	0.1661	0.1677
		$\hat{\alpha}_2$	0.0142	0.0279	0.6531	0.0680	0.0695	0.0152	0.0298	0.6743	0.0751	0.0769
		$\hat{\lambda}_2$	0.0831	0.0664	0.9567	0.1238	0.1257	0.0889	0.0742	1.0095	0.1209	0.1206
		$\hat{\theta}$	-0.0504	0.2249	1.8493	0.2344	0.2424	-0.1192	0.3241	2.1834	0.2263	0.2296
50	MLE	$\hat{\alpha}_1$	0.0029	0.0044	0.2604	0.0257	0.0261	0.0069	0.0041	0.2498	0.0237	0.0239
		$\hat{\lambda}_1$	0.0589	0.0540	0.8820	0.0874	0.0865	0.0585	0.0500	0.8462	0.0744	0.0745
		$\hat{\alpha}_2$	0.0109	0.0176	0.5181	0.0548	0.0561	0.0183	0.0188	0.5326	0.0550	0.0540
		$\hat{\lambda}_2$	0.0484	0.0364	0.7233	0.0735	0.0723	0.0404	0.0378	0.7458	0.0754	0.0763
		$\hat{\theta}$	-0.0309	0.1496	1.5123	0.1409	0.1453	-0.0602	0.1034	1.2387	0.1125	0.1120
	IFM	$\hat{\alpha}_1$	0.0027	0.0044	0.2608	0.0284	0.0278	0.0066	0.0041	0.2485	0.0249	0.0248
		$\hat{\lambda}_1$	0.0580	0.0538	0.8809	0.0881	0.0897	0.0591	0.0504	0.8496	0.0811	0.0797
		$\hat{\alpha}_2$	0.0107	0.0175	0.5173	0.0626	0.0638	0.0176	0.0188	0.5328	0.0481	0.0490
		$\hat{\lambda}_2$	0.0478	0.0363	0.7237	0.0675	0.0656	0.0402	0.0377	0.7448	0.0790	0.0796
		$\hat{\theta}$	-0.0349	0.1479	1.5021	0.1634	0.1680	-0.0646	0.1055	1.2487	0.1465	0.1536
100	MLE	$\hat{\alpha}_1$	0.0036	0.0020	0.1743	0.0117	0.0118	0.0023	0.0019	0.1715	0.0112	0.0113
		$\hat{\lambda}_1$	0.0286	0.0258	0.6195	0.0416	0.0413	0.0282	0.0222	0.5736	0.0392	0.0404
		$\hat{\alpha}_2$	0.0067	0.0085	0.3597	0.0268	0.0267	0.0082	0.0091	0.3724	0.0239	0.0239
		$\hat{\lambda}_2$	0.0223	0.0178	0.5165	0.0336	0.0345	0.0220	0.0181	0.5203	0.0349	0.0344
		$\hat{\theta}$	0.0135	0.0798	1.1068	0.0626	0.0629	-0.0315	0.0522	0.8873	0.0642	0.0673
	IFM	$\hat{\alpha}_1$	0.0036	0.0020	0.1735	0.0120	0.0120	0.0023	0.0019	0.1712	0.0133	0.0131
		$\hat{\lambda}_1$	0.0282	0.0257	0.6192	0.0466	0.0472	0.0280	0.0223	0.5757	0.0371	0.0373
		$\hat{\alpha}_2$	0.0064	0.0084	0.3585	0.0247	0.0249	0.0078	0.0092	0.3741	0.0273	0.0272
		$\hat{\lambda}_2$	0.0224	0.0179	0.5167	0.0369	0.0368	0.0222	0.0181	0.5200	0.0336	0.0345
		$\hat{\theta}$	0.0106	0.0788	1.0999	0.0717	0.0730	-0.0335	0.0518	0.8831	0.0560	0.0573
200	MLE	$\hat{\alpha}_1$	0.0020	0.0011	0.1291	0.0067	0.0065	0.0011	0.0010	0.1229	0.0061	0.0060
		$\hat{\lambda}_1$	0.0094	0.0110	0.4088	0.0216	0.0228	0.0109	0.0108	0.4054	0.0214	0.0215
		$\hat{\alpha}_2$	0.0065	0.0046	0.2658	0.0130	0.0130	0.0050	0.0042	0.2547	0.0127	0.0128
		$\hat{\lambda}_2$	0.0068	0.0086	0.3635	0.0184	0.0191	0.0038	0.0080	0.3497	0.0180	0.0184
		$\hat{\theta}$	0.0070	0.0396	0.7800	0.0399	0.0404	-0.0157	0.0223	0.5822	0.0281	0.0271
	IFM	$\hat{\alpha}_1$	0.0021	0.0011	0.1297	0.0066	0.0067	0.0009	0.0010	0.1234	0.0062	0.0062
		$\hat{\lambda}_1$	0.0095	0.0109	0.4083	0.0194	0.0196	0.0109	0.0109	0.4070	0.0207	0.0203
		$\hat{\alpha}_2$	0.0067	0.0046	0.2661	0.0141	0.0143	0.0044	0.0043	0.2556	0.0125	0.0120
		$\hat{\lambda}_2$	0.0067	0.0086	0.3628	0.0182	0.0176	0.0040	0.0080	0.3499	0.0170	0.0179
		$\hat{\theta}$	0.0057	0.0393	0.7773	0.0398	0.0400	-0.0165	0.0222	0.5811	0.0281	0.0288

Table 4: Estimation of the Parameters of FGMBF and AMGBF Distributions: Case 3.

			FGMBF					AMHBF				
n			Bias	MSE	ACI	Bt	BP	Bias	MSE	ACI	Bt	BP
30	MLE	$\hat{\alpha}_1$	0.0435	0.0867	1.1424	0.1419	0.1434	0.0400	0.0866	1.1431	0.1297	0.1284
		$\hat{\lambda}_1$	0.0456	0.0232	0.5706	0.0858	0.0838	0.0484	0.0228	0.5611	0.0741	0.0772
		$\hat{\alpha}_2$	0.0451	0.0899	1.1623	0.1479	0.1482	0.0477	0.0972	1.2083	0.1262	0.1277
		$\hat{\lambda}_2$	0.0393	0.0146	0.4480	0.0639	0.0632	0.0407	0.0157	0.4640	0.0627	0.0628
		$\hat{\theta}$	0.0143	0.2911	2.1154	0.2614	0.2631	-0.0603	0.2959	2.1203	0.2536	0.2527
	IFM	$\hat{\alpha}_1$	0.0425	0.0853	1.1331	0.1449	0.1525	0.0398	0.0862	1.1406	0.1486	0.1522
		$\hat{\lambda}_1$	0.0452	0.0231	0.5695	0.0719	0.0719	0.0477	0.0227	0.5599	0.0787	0.0794
		$\hat{\alpha}_2$	0.0443	0.0899	1.1628	0.1306	0.1358	0.0482	0.0965	1.2033	0.1309	0.1338
		$\hat{\lambda}_2$	0.0390	0.0145	0.4471	0.0578	0.0591	0.0399	0.0154	0.4610	0.0521	0.0531
		$\hat{\theta}$	0.0159	0.2827	2.0845	0.2626	0.2676	-0.2708	1.3477	4.4275	0.5900	0.5879
50	MLE	$\hat{\alpha}_1$	0.0187	0.0467	0.8448	0.0853	0.0853	0.0296	0.0441	0.8151	0.0761	0.0768
		$\hat{\lambda}_1$	0.0275	0.0121	0.4177	0.0417	0.0414	0.0282	0.0113	0.4027	0.0358	0.0355
		$\hat{\alpha}_2$	0.0315	0.0543	0.9053	0.0996	0.1003	0.0409	0.0572	0.9239	0.0922	0.0900
		$\hat{\lambda}_2$	0.0211	0.0076	0.3319	0.0346	0.0351	0.0186	0.0084	0.3530	0.0376	0.0383
		$\hat{\theta}$	-0.0075	0.1853	1.6879	0.1723	0.1753	-0.0620	0.1943	1.7118	0.1512	0.1505
	IFM	$\hat{\alpha}_1$	0.0182	0.0464	0.8418	0.0933	0.0926	0.0298	0.0440	0.8141	0.0830	0.0828
		$\hat{\lambda}_1$	0.0275	0.0121	0.4173	0.0417	0.0425	0.0280	0.0113	0.4025	0.0384	0.0378
		$\hat{\alpha}_2$	0.0318	0.0545	0.9069	0.1134	0.1176	0.0407	0.0572	0.9239	0.0833	0.0828
		$\hat{\lambda}_2$	0.0211	0.0076	0.3315	0.0315	0.0310	0.0184	0.0084	0.3521	0.0347	0.0351
		$\hat{\theta}$	-0.0059	0.1817	1.6717	0.1728	0.1722	-0.1273	0.4626	2.6205	0.2718	0.2711
100	MLE	$\hat{\alpha}_1$	0.0156	0.0204	0.5574	0.0379	0.0383	0.0110	0.0195	0.5466	0.0355	0.0363
		$\hat{\lambda}_1$	0.0134	0.0058	0.2938	0.0199	0.0199	0.0134	0.0050	0.2726	0.0187	0.0194
		$\hat{\alpha}_2$	0.0167	0.0237	0.5996	0.0457	0.0452	0.0197	0.0273	0.6429	0.0410	0.0416
		$\hat{\lambda}_2$	0.0114	0.0040	0.2432	0.0159	0.0163	0.0098	0.0038	0.2389	0.0167	0.0169
		$\hat{\theta}$	0.0202	0.0926	1.1910	0.0718	0.0725	-0.0339	0.1126	1.3096	0.0947	0.1008
	IFM	$\hat{\alpha}_1$	0.0158	0.0203	0.5556	0.0388	0.0390	0.0114	0.0196	0.5477	0.0426	0.0423
		$\hat{\lambda}_1$	0.0133	0.0058	0.2933	0.0221	0.0223	0.0132	0.0050	0.2727	0.0176	0.0177
		$\hat{\alpha}_2$	0.0159	0.0235	0.5980	0.0421	0.0416	0.0199	0.0272	0.6424	0.0458	0.0460
		$\hat{\lambda}_2$	0.0115	0.0040	0.2427	0.0172	0.0175	0.0098	0.0038	0.2388	0.0156	0.0156
		$\hat{\theta}$	0.0209	0.0916	1.1845	0.0797	0.0785	-0.0353	0.1165	1.3313	0.0852	0.0876
200	MLE	$\hat{\alpha}_1$	0.0091	0.0113	0.4149	0.0216	0.0209	0.0049	0.0101	0.3940	0.0197	0.0199
		$\hat{\lambda}_1$	0.0045	0.0025	0.1935	0.0102	0.0107	0.0052	0.0024	0.1929	0.0101	0.0103
		$\hat{\alpha}_2$	0.0139	0.0135	0.4525	0.0225	0.0227	0.0106	0.0123	0.4326	0.0214	0.0218
		$\hat{\lambda}_2$	0.0036	0.0019	0.1713	0.0086	0.0088	0.0015	0.0018	0.1640	0.0083	0.0083
		$\hat{\theta}$	0.0064	0.0412	0.7957	0.0409	0.0415	-0.0093	0.0515	0.8890	0.0431	0.0422
	IFM	$\hat{\alpha}_1$	0.0091	0.0113	0.4157	0.0212	0.0213	0.0050	0.0101	0.3940	0.0198	0.0193
		$\hat{\lambda}_1$	0.0045	0.0025	0.1934	0.0092	0.0093	0.0052	0.0024	0.1928	0.0098	0.0096
		$\hat{\alpha}_2$	0.0137	0.0135	0.4521	0.0240	0.0240	0.0107	0.0123	0.4321	0.0211	0.0211
		$\hat{\lambda}_2$	0.0036	0.0019	0.1711	0.0087	0.0087	0.0015	0.0018	0.1640	0.0078	0.0078
		$\hat{\theta}$	0.0068	0.0410	0.7939	0.0410	0.0396	-0.0089	0.0512	0.8868	0.0448	0.0437

Table 5: Economic Data.

years	x_1	x_2	years	x_1	x_2
1980	30.51	10.01	1995	22.55	4.64
1981	33.37	3.76	1996	20.75	4.99
1982	27.03	9.91	1997	18.84	5.49
1983	25.48	7.4	1998	16.21	4.04
1984	22.35	6.09	1999	15.05	6.11
1985	19.91	6.6	2000	16.2	5.37
1986	15.73	2.65	2001	17.48	3.54
1987	12.56	2.52	2002	18.32	2.37
1988	17.32	7.93	2003	21.8	3.19
1989	17.89	4.97	2004	28.23	4.09
1990	20.05	5.7	2005	30.34	4.48
1991	27.82	1.08	2006	29.95	6.85
1992	28.4	4.43	2007	30.25	7.09
1993	25.84	2.9	2008	33.04	7.16
1994	22.57	3.97	2009	24.96	4.67
			2010	21.35	5.15

Table 6: Goodness of Fit test of FGM and AMH Copulas for Economic Data.

	statistic	$\hat{\theta}$	p-value
FGM	0.52631	0.62707	0.1794
AMH	0.40758	0.59126	0.1833

Table 7: The Model Criterion Selection of FGMBF and AMHBF Distributions for Economic Data.

Criterion	FGMBF	AMHBF
LL	-174.1272	-173.9889
AIC	358.2544	357.9778
BIC	365.4244	365.1478
HQIC	360.5916	360.315
CAIC	360.6544	360.3778

Table 8: The Estimates and the Corresponding Stander Deviation of Parameters of FGMBF and AMHBF Distributions for Economic Data.

	FGMBF				AMHBF			
	MLE		IFM		MLE		IFM	
	Coef	std	coef	Std	coef	Std	coef	std
$\hat{\alpha}_1$	19.4831	0.9568	19.5465	0.9331	19.3297	0.9608	19.5465	0.9331
$\hat{\lambda}_1$	3.9925	0.5318	3.9921	0.5272	3.9772	0.535	3.9921	0.5272
$\hat{\alpha}_2$	3.6548	0.3883	3.6674	0.3932	3.6783	0.3845	3.6674	0.3932
$\hat{\lambda}_2$	1.7968	0.2089	1.7843	0.2092	1.7986	0.208	1.7843	0.2092
$\hat{\theta}$	0.7862	0.6332	0.7869	0.6327	0.6615	0.3243	0.6339	0.3182

Table 9: Goodness of Fit test of FGM Copula for Medical Data.

	statistic	$\hat{\theta}$	p-value
FGM	0.29031	0.46704	0.3944
AMH	0.21138	0.49748	0.5464

Table 10: The Model Criterion Selection of FGMBF and AMHBF Distributions for Medical Data.

	FGMBF	AMHBF
LL	-340.713	-340.641
AIC	691.425	691.282
BIC	698.431	698.288
HQIC	693.667	693.524
CAIC	693.925	693.782

Table 11: The Estimates and the Corresponding Stander Deviation of Parameters of FGMBF and AMHBF Distributions for Medical Data.

	FGMBF				**AMHBF**			
	MLE		IFM		MLE		IFM	
	Coef	std	coef	Std	coef	Std	coef	std
$\hat{\alpha}_1$	22.7492	6.1231	23.5331	6.2942	23.0461	6.1098	23.5331	6.2942
$\hat{\lambda}_1$	0.7286	0.0979	0.7234	0.0977	0.7267	0.0978	0.7234	0.0977
$\hat{\alpha}_2$	27.9023	6.2958	28.6622	6.4521	28.151	6.2881	28.6622	6.4521
$\hat{\lambda}_2$	0.8663	0.1175	0.8601	0.1166	0.8638	0.1171	0.8601	0.1166
$\hat{\theta}$	0.6098	0.5618	0.598	0.5514	0.5384	0.3549	0.5369	0.3528

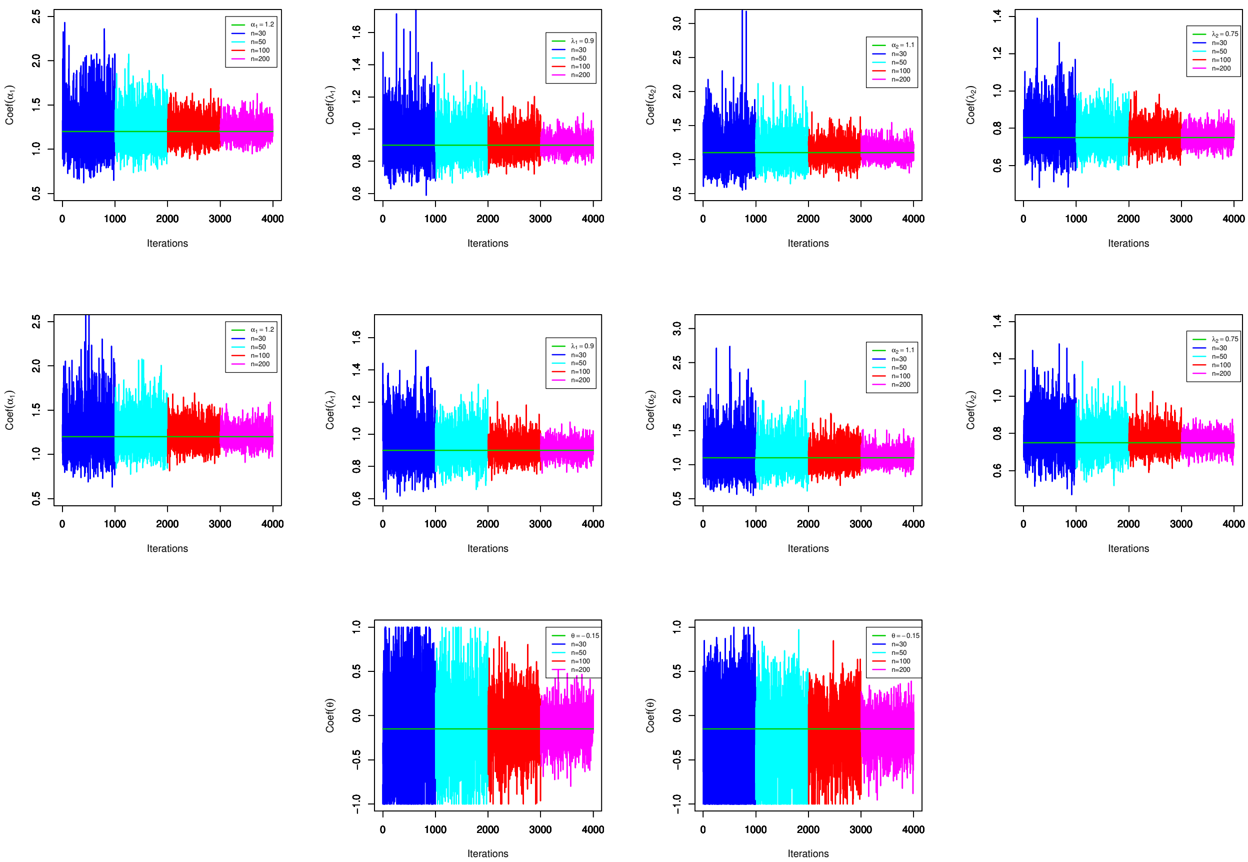

Fig. 6: Parameters Coefficient for FGMBF and AMHBF Distributions by Using MLE in Case 3.

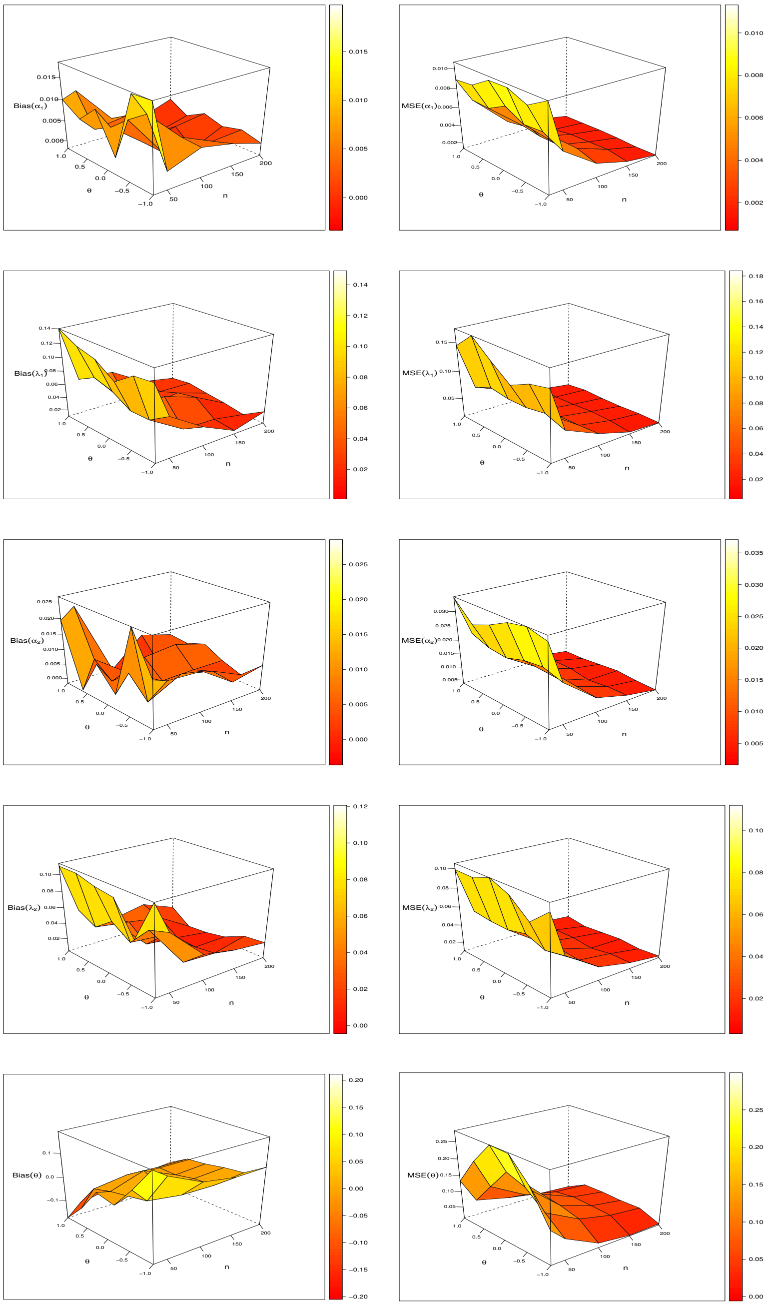

Fig. 7: The Bias and MSE of Parameters for FGMBF Distribution with Various Value of the Copula Parameter and Sample Size.

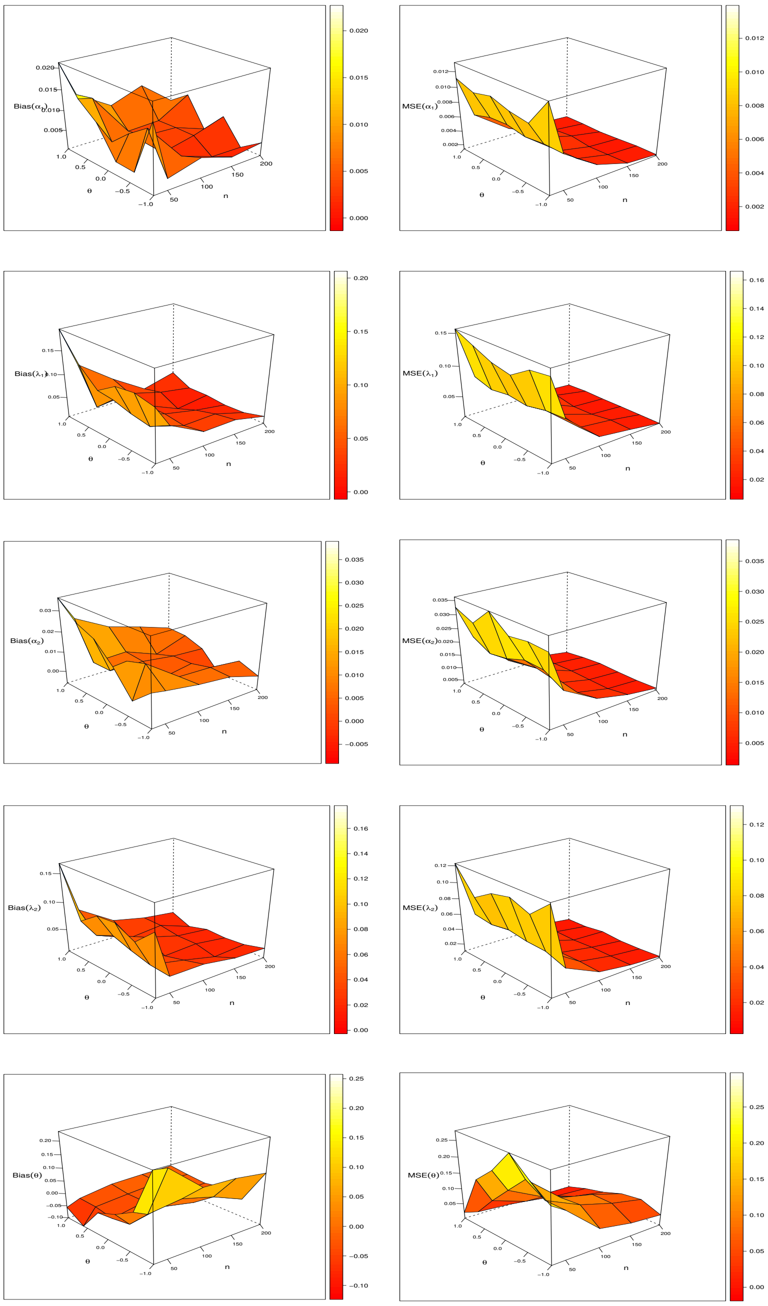

Fig. 8: The Bias and MSE of Parameters for AMHBF Distribution with Various Value of the Copula Parameter and Sample Size.

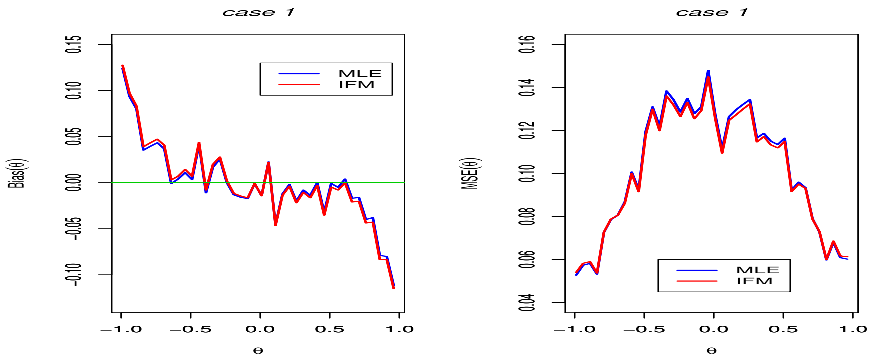

Fig. 9: The Bias and MSE of Copula Parameter for FGMBF Distribution with Various Value of Copula Parameters and Different Estimation Methods.

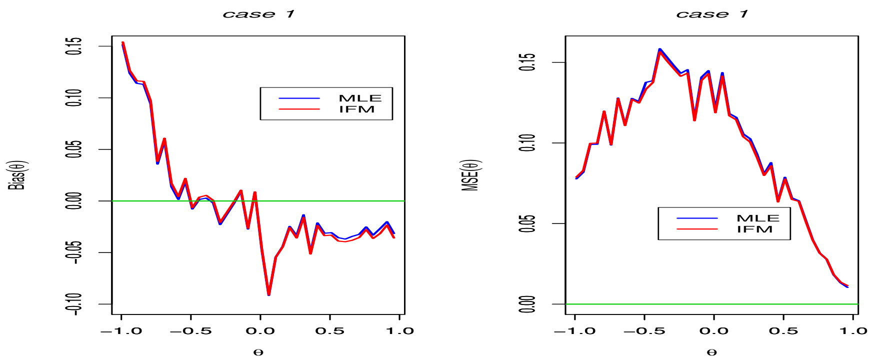

Fig. 10: The Bias and MSE of Copula Parameter for AMHBF Distribution with Various Value of Copula Parameters and Different Estimation Methods.

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