Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/090210

A Generalization of Transmuted Laplace Distribution

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Received: 28 Jun. 2019, Revised: 2 Oct. 2019, Accepted: 21 Oct. 2019
Published online: 1 Jul. 2020

Abstract: Two new parameters are added to the transmuted Laplace distribution. The proposed model is the generalized transmuted Laplace distribution. The statistical properties provide the moments, moment generating function and differential entropy. Estimation of parameters using the maximum likelihood method is investigated, as well. A real data set is used to show that the new distribution is a better model.

Keywords: Transmuted Laplace Distribution, Moment generating function, Maximum Likelihood Estimators, Differential entropy.

1 Introduction

The Laplace distribution, one of the important and earliest known probability distributions, is a continuous probability distribution named after the French mathematician Pierre-Simon Laplace. Sometimes, it is called the double exponential distribution, where it looks like two exponential distributions spliced together back-to-back. It is unimodal and symmetrical, but it has a sharper peak than the normal.

Recently, various statisticians have extensively addressed transmuted probability model, which arises from transforming a basic distribution into its generalized counterpart. Generalizing the two parameters Weibull distribution using the quadratic rank transmutation map, Aryaland Tsokos [1] investigated the transmuted Weibull distribution. Ashour and Eltehiwy [2] introduced the generalization of the Lomax distribution and various structural properties. Merovci [3] provided the transmuted generalized Rayleigh distribution using the quadratic rank transmution map. Afifyet et.al [4] provided a new generalization of the complementary Weibull geometric distribution. Hussian [5] introduced the transmuted exponentiated gamma distribution and some of its properties. The transmuted inverse exponential distribution has been addressed by Adejumo and Oguntunde [6]. Also Nassaret et al. [7,8] introduced the transmuted kumaraswamy logistic distribution and the transmuted Weibull logistic distribution. Abdel Hady and Shalaby [9] explored the transmuted Laplace distribution.

In this article, the transmuted Laplace distribution is generalized through adding two new parameters,The cumulative distribution function (cdf) and probability density function (pdf) of generalized transmuted Laplace distribution are introduced, along with the hazard rate and survival function in Section Two. Section Three covers the moments and the moment generating function. Estimation of the parameters is discussed in Section Four using the maximum likelihood method, so the information matrix. The differential entropy is obtained in Section Five. Section Six compares between the introduced distribution and other distributions using real data, which shows that the introduced one is a better model.

2 General transmuted Laplace distribution

A random variable X has a transmuted distribution if its cumulative distribution function (cdf) is given by:

$$F(x) = (1+\lambda)G(x) - \lambda[G(x)]^2, \qquad |\lambda| \leq 1 \tag{1}$$

where $G(x)$ is the cumulative distribution function of the base distribution, for $\lambda = 0$. Equation(1) reduces to the base distribution. This article handles a generalization using a transmuted distribution by adding two parameters, so the random

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variable X has a general transmuted distribution if it satisfies the relation

$$F(x) = (1+\lambda)[G(x)]^{\delta} - \lambda[G(x)]^{\alpha} \qquad \delta,\alpha > 0\,,\ |\lambda| \le 1 \tag{2}$$

Observe that for $\delta = 1$ and $\alpha = 2$, this reduces to Equation (1).
Let us define the Laplace distribution, as follows

$$G(x) = \begin{cases} \frac{1}{2} e^{\frac{x}{b}} & x < 0 \\ 1 - \frac{1}{2} e^{-\frac{x}{b}} & x \ge 0. \end{cases} \tag{3}$$

Applying the general transmuted distribution given in (3), we obtain the cdf of the generalized transmuted Laplace distribution (GTL)

$$F(x) = \begin{cases} (1+\lambda)\left[\frac{1}{2} e^{\frac{x}{b}}\right]^{\delta} - \lambda\left[\frac{1}{2} e^{\frac{x}{b}}\right]^{\alpha} & x < 0 \\ (1+\lambda)\left[1 - \frac{1}{2} e^{\frac{-x}{b}}\right]^{\delta} - \lambda\left[1 - \frac{1}{2} e^{\frac{-x}{b}}\right]^{\alpha} & x \ge 0 \end{cases} \qquad \alpha, b, \delta > 0, |\lambda| \le 1 \tag{4}$$

Hence, the pdf of GTL distribution given in (4) is:

$$f(x) = \begin{cases} \frac{\delta(1+\lambda)}{2^{\delta} b}\left[e^{\frac{x}{b}}\right]^{\delta} - \frac{\alpha\lambda}{2^{\alpha} b}\left[e^{\frac{x}{b}}\right]^{\alpha} & x < 0 \\ \frac{\delta(1+\lambda)}{2b} e^{\frac{-x}{b}}\left[1 - \frac{1}{2} e^{\frac{-x}{b}}\right]^{\delta-1} - \frac{\alpha\lambda}{2b} e^{\frac{-x}{b}}\left[1 - \frac{1}{2} e^{\frac{-x}{b}}\right]^{\alpha-1} & x \ge 0 \end{cases} \tag{5}$$

The hazard rate function of a given distribution $F(x)$ is defined by:

$$h(x;\alpha,b,\delta,\lambda) = \frac{f(x)}{1 - F(x)},$$

From (4) & (5), the hazard rate of GTL distribution is written as

$$h(x;\alpha,b,\delta,\lambda) = \begin{cases} \dfrac{\frac{\delta(1+\lambda)}{2^{\delta} b}\left[e^{\frac{x}{b}}\right]^{\delta} - \frac{\alpha\lambda}{2^{\alpha} b}\left[e^{\frac{x}{b}}\right]^{\alpha}}{1 - \left[\left[\frac{1}{2} e^{\frac{x}{b}}\right]^{\delta} + \lambda\left\{\left[\frac{1}{2} e^{\frac{x}{b}}\right]^{\alpha} - \left[\frac{1}{2} e^{\frac{x}{b}}\right]^{\delta}\right\}\right]} & x < 0, \\ \dfrac{\frac{\delta(1+\lambda)}{2b} e^{\frac{-x}{b}}\left[1 - \frac{1}{2} e^{\frac{-x}{b}}\right]^{\delta-1} - \frac{\alpha\lambda}{2b} e^{\frac{-x}{b}}\left[1 - \frac{1}{2} e^{\frac{-x}{b}}\right]^{\alpha-1}}{1 - \left[1 - \frac{1}{2} e^{\frac{-x}{b}}\right]^{\delta} + \lambda\left\{\left[1 - \frac{1}{2} e^{\frac{-x}{b}}\right]^{\alpha} - \left[1 - \frac{1}{2} e^{\frac{-x}{b}}\right]^{\delta}\right\}} & x \ge 0 \end{cases} \tag{6}$$

It is important to know the cumulative hazard rate function defined as: $H(x) = -Ln(1-F(x))$ The cumulative hazard rate function of GTL distribution is given by

$$H(x;\alpha,b,\delta,\lambda) = \begin{cases} -Ln\left[1-\left\{\left[\frac{1}{2}e^{\frac{x}{b}}\right]^{\delta}+\lambda\left\{\left[\frac{1}{2}e^{\frac{x}{b}}\right]^{\alpha}-\left[\frac{1}{2}e^{\frac{x}{b}}\right]^{\delta}\right\}\right\}\right] & x<0, \\ -Ln\left[1-\left\{\left[1-\frac{1}{2}e^{\frac{-x}{b}}\right]^{\delta}+\lambda\left\{\left[1-\frac{1}{2}e^{\frac{-x}{b}}\right]^{\alpha}-\left[1-\frac{1}{2}e^{\frac{-x}{b}}\right]^{\delta}\right\}\right\}\right] & x\geq 0 \end{cases} \quad (7)$$

Also, the survival function is:

$$S(x;\alpha,b,\delta,\lambda) = \begin{cases} 1-\left[\frac{1}{2}e^{\frac{x}{b}}\right]^{\delta}+\lambda\left\{\left[\frac{1}{2}e^{\frac{x}{b}}\right]^{\alpha}-\left[\frac{1}{2}e^{\frac{x}{b}}\right]^{\delta}\right\} & x<0, \\ 1-\left[1-\frac{1}{2}e^{\frac{-x}{b}}\right]^{\delta}+\lambda\left\{\left[1-\frac{1}{2}e^{\frac{-x}{b}}\right]^{\alpha}-\left[1-\frac{1}{2}e^{\frac{-x}{b}}\right]^{\delta}\right\} & x\geq 0 \end{cases} \quad (8)$$

Figure 1, 2, 3 and 4 show the different shapes of cdf, pdf, hazard rate and the survival function respectively at different values of the parameters $\alpha, b, \delta, \lambda$.

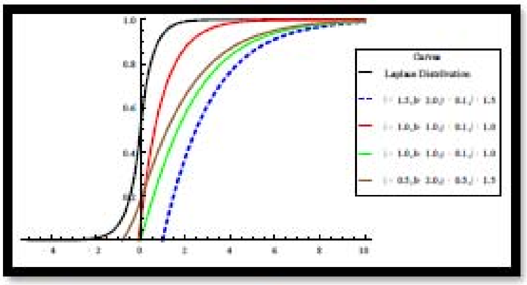


Fig (1): plots of cdf at different values of $\alpha, b, \delta, \lambda$

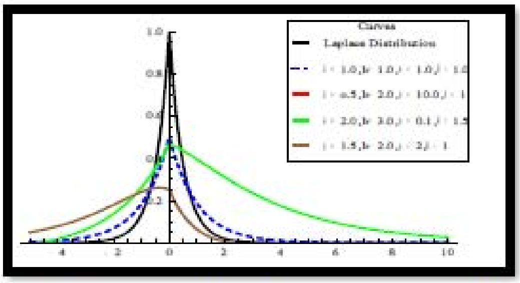
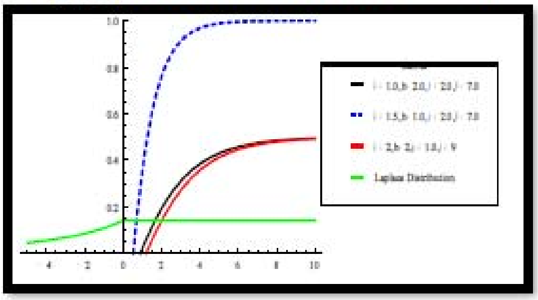
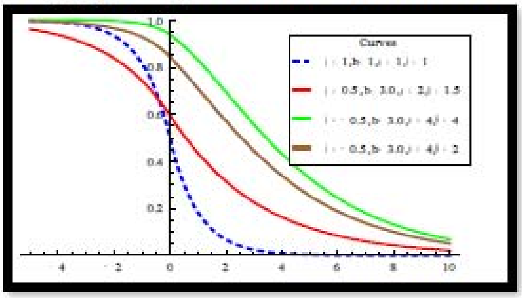


Fig (2): plots of pdf at different values of $\alpha, b, \delta, \lambda$



Fig(3): plots of hazard rate function at different values of $\alpha, b, \delta, \lambda$



Fig(4): plots of survival function at different values of $\alpha, b, \delta, \lambda$

The plots of the hazard function, reveal that the hazard rate increases,then it assumes constant hazard rate

3 Moments

Suppose X is a random variable has a GTL distribution given by Equations (4) and (5). Then the r-th moment can be written as

$$E\left(x^{r}\right)=\int_{-\infty}^{\infty} x^{r} f(x)\, dx$$

$$E\left(x^{r}\right)=\int_{-\infty}^{0} x^{r}\left[\frac{\delta(1+\lambda)}{2^{\delta} b}[e^{\frac{x}{b}}]^{\delta}-\frac{\alpha\lambda}{2^{\alpha} b}[e^{\frac{x}{b}}]^{\alpha}\right] dx+\int_{0}^{\infty} x^{r}\left[\frac{\delta(1+\lambda)}{2b} e^{\frac{-x}{b}}\left[1-\frac{1}{2} e^{\frac{-x}{b}}\right]^{\delta-1}-\frac{\alpha\lambda}{2b} e^{\frac{-x}{b}}\left[1-\frac{1}{2} e^{\frac{-x}{b}}\right]^{\alpha-1}\right] dx$$

$$\begin{aligned} E\left(x^{r}\right)= & (-1)^{r}\Gamma(r+1)\left\{\frac{1+\lambda}{2^{\delta}}\left(\frac{\delta}{b}\right)^{-r}-\frac{\lambda}{2^{\alpha}}\left(\frac{\alpha}{b}\right)^{-r}\right\} \\ & +\Gamma(r+1)\frac{1}{2b}\left\{\delta(1+\lambda)\sum_{l=0}^{\infty}\left(\frac{-1}{2}\right)^{l}\binom{\delta-1}{l}\left(\frac{b}{l+1}\right)^{r+1}-\alpha\lambda\sum_{t=0}^{\infty}\left(\frac{-1}{2}\right)^{t}\binom{\alpha-1}{l}\left(\frac{b}{t+1}\right)^{r+1}\right\} \end{aligned} \quad (9)$$

At $r=1$ we can find the mean of GTL random variable, which is given by:

$$E(x)=b\left\{\frac{\lambda}{\alpha\, 2^{\alpha}}-\frac{1+\lambda}{\delta\, 2^{\delta}}\right\}+\frac{b}{2}\left\{\delta(1+\lambda)\sum_{l=0}^{\infty}\left(\frac{-1}{2}\right)^{l}\binom{\delta-1}{l}\frac{1}{(l+1)^{2}}-\alpha\lambda\sum_{k=0}^{\infty}\left(\frac{-1}{2}\right)^{k}\binom{\alpha-1}{k}\frac{1}{(k+1)^{2}}\right\} \quad (10)$$

Also, at $r=2$ we can obtain the second moment which is given by,

$$E\left(x^{2}\right)=2b^{2}\left\{\frac{1+\lambda}{\delta^{2}2^{\delta}}-\frac{\lambda}{\alpha^{2}2^{\alpha}}\right\}+\left\{\delta(1+\lambda)\sum_{l=0}^{\infty}\left(\frac{-1}{2}\right)^{l}\binom{\delta-1}{l}\frac{b^{2}}{(l+1)^{3}}-\alpha\lambda\sum_{k=0}^{\infty}\left(\frac{-1}{2}\right)^{k}\binom{\alpha-1}{k}\frac{b^{2}}{(k+1)^{3}}\right\} \quad (11)$$

From (10)&(11) we obtain the variance of (GTLD) as

$$V(x)=E\left(x^{2}\right)-(E(x))^{2}.$$

The moment generating function of (GTLD) can be obtained by:

$$\begin{aligned} M_{x}(t)=E\left(e^{tx}\right)= & \int_{-\infty}^{\infty} e^{tx}\ f(x)\, dx \\ = & \frac{\delta(1+\lambda)}{2^{\delta}\, b}\int_{-\infty}^{0}(e^{\frac{x}{b}})^{\delta}\ e^{xt}\ dx-\frac{\alpha\,\lambda}{2^{\alpha}\, b}\int_{-\infty}^{0}(e^{\frac{x}{b}})^{\alpha}\ e^{xt}\ dx+\frac{\delta(1+\lambda)}{2\, b}\int_{0}^{\infty} e^{x\left(t-\frac{1}{b}\right)}(1-\frac{1}{2}e^{-\frac{x}{b}})^{\delta-1}\ dx \\ & -\frac{\alpha\,\lambda}{2\, b}\int_{0}^{\infty} e^{x\left(t-\frac{1}{b}\right)}(1-\frac{1}{2}e^{-\frac{x}{b}})^{\alpha-1}\ dx \\ = & \frac{\delta(1+\lambda)}{2\, b}\left[\frac{1}{2^{\delta-1}}\left\{\frac{1}{t+\frac{\delta}{b}}+\sum_{l=0}^{\infty}\left(\frac{-1}{2}\right)^{l}\binom{\delta-1}{l}\left(\frac{b}{l+1-bt}\right)\right\}\right] \\ & -\frac{\alpha\,\lambda}{2\, b}\left[\frac{1}{2^{\alpha-1}}\left\{\frac{1}{t+\frac{\alpha}{b}}+\sum_{k=0}^{\infty}\left(\frac{-1}{2}\right)^{k}\binom{\alpha-1}{k}\left(\frac{k}{k+1-bt}\right)\right\}\right] \end{aligned} \quad (12)$$

4 Maximum Likelihood Estimators

Let $X_1, X_2, \cdots, X_n$ be a random sample drawn from GTL distribution, given by Equation (5), then the parameters of GTL distribution are estimated by the maximum likelihood method. The Log -Likelihood function is given by:

$$Log\left(L\left(x_1,x_2,\ldots,x_n;\alpha,b,\delta,\lambda\right)\right)=\begin{cases} n\left[Log\left(\frac{2^{\alpha}\delta(1+\lambda)}{2^{\delta}\alpha\lambda}\right)\right]+\sum_{i=1}^{n}\left(\frac{x_i}{b}\right)^{\delta}-\sum_{i=1}^{n}\left(\frac{x_i}{b}\right)^{\alpha} & x_i<0 \\ n\left[Log\left(\frac{2^{\alpha}\delta(1+\lambda)}{2^{\delta}\alpha\lambda}\right)\right]+\sum_{i=1}^{n}Log\left(1-\frac{1}{2}e^{\frac{-x_i}{b}}\right)^{\delta-1}-\sum_{i=1}^{n}Log\left(1-\frac{1}{2}e^{\frac{-x_i}{b}}\right)^{\alpha-1} & x_i\geq 0 \end{cases} \tag{13}$$

Define $\Delta_i=\{1\ If\ \ x_i<0,\ \ 0\ \ If\ x_i\geq 0\}$, which an indicator function. Thus,the log Likelihood function can be written in the following form

$$\begin{aligned} Log\left(L\left(x_1,x_2,\cdots,x_n;\alpha,b,\delta,\lambda\right)\right)=&\sum_{i=1}^{n}\Delta_i\{\ n(Log\left(\frac{2^{\alpha}\delta(1+\lambda)}{2^{\delta}\alpha\lambda}\right))\}+\sum_{i=1}^{n}\Delta_i\left\{\left(\frac{x_i}{b}\right)^{\delta}-\left(\frac{x_i}{b}\right)^{\alpha}\right\} \\ &+\sum_{i=1}^{n}(1-\Delta_i)\left\{n\left(\frac{2^{\alpha}\delta(1+\lambda)}{2^{\delta}\alpha\lambda}\right)\right\} \\ &+\sum_{i=1}^{n}(1-\Delta_i)Log\left(1-\frac{1}{2}e^{\frac{-x_i}{b}}\right)^{\delta-1}-Log\left(1-\frac{1}{2}e^{\frac{-x_i}{b}}\right)^{\alpha-1} \end{aligned} \tag{14}$$

Differentiating Equation (14), with respect to the parameters $\alpha, b, \delta and \lambda$, we have:

$$\frac{\partial LogL}{\partial\alpha}=-\sum_{i=1}^{n}\Delta_i\left[\left(\frac{x_i}{b}\right)^{\alpha}Log\left(\frac{x_i}{b}\right)+\frac{n}{\alpha}\{1-\alpha log\,(2)\ \}\right]+\sum_{i=1}^{n}(1-\Delta_i)\left[-Log(\ 1-\frac{1}{2}e^{-\frac{x_i}{b}})-\frac{n}{\alpha}\{1-\alpha Log\,(2)\}\right] \tag{15}$$

$$\frac{\partial LogL}{\partial b}=-\sum_{i=1}^{n}\Delta_i\frac{x_i}{b^2}\left[-\delta(\frac{x_i}{b})^{\delta-1}-\alpha(\frac{x_i}{b})^{\alpha-1}\right]-\sum_{i=1}^{n}(1-\Delta_i)\left[\frac{x_i e^{-x_i}}{2b^2\left(1-\frac{1}{2}e^{-\frac{x_i}{b}}\right)}(2-\alpha-\delta)\right] \tag{16}$$

$$\frac{\partial LogL}{\partial\delta}=\sum_{i=1}^{n}\Delta_i[\left(\frac{x_i}{b}\right)^{\delta}Log\left(\frac{x_i}{b}\right)+\frac{n}{\delta}\{1-\delta\ Log\,(2)\}+\sum_{i=1}^{n}(1-\Delta_i)\left[Log\left(1-\frac{1}{2}e^{-\frac{x_i}{b}}\right)+\frac{n}{\delta}\{1-\delta Log\,(2)\}\right] \tag{17}$$

$$\frac{\partial LogL}{\partial\lambda}=\left(\frac{-n}{\lambda\,(1+\lambda)}\right)\{\sum_{i=1}^{n}\Delta_i+\sum_{i=1}^{n}(1-\Delta_i)\} \tag{18}$$

The maximum likelihood estimators (MLE) of the parameters $\alpha, b, \delta and \lambda$ are obtained through setting the previous equations to zero, and solving them using Mathematica program. Normal approximation of the MLE can be used for constructing approximate confidence intervals and for testing hypotheses on the parameter $\alpha, b, \delta and \lambda$. Now we derive the Fisher information matrix for interval estimation and hypotheses testing for the model parameters. The 4×4 Fisher information matrix is given by:

$$\begin{pmatrix} \frac{\partial^2 LogL}{\partial\lambda^2} & \frac{\partial^2 LogL}{\partial\lambda\partial b} & \frac{\partial^2 LogL}{\partial\lambda\partial\alpha} & \frac{\partial^2 LogL}{\partial\delta\partial\lambda} \\ \frac{\partial^2 LogL}{\partial b\partial\lambda} & \frac{\partial^2 LogL}{\partial b^2} & \frac{\partial^2 LogL}{\partial b\partial\alpha} & \frac{\partial^2 LogL}{\partial\delta\partial b} \\ \frac{\partial^2 LogL}{\partial\alpha\partial\lambda} & \frac{\partial^2 LogL}{\partial\alpha\partial b} & \frac{\partial^2 LogL}{\partial\alpha^2} & \frac{\partial^2 LogL}{\partial\delta\partial\alpha} \\ \frac{\partial^2 LogL}{\partial\delta\partial\lambda} & \frac{\partial^2 LogL}{\partial\delta\partial b} & \frac{\partial^2 LogL}{\partial\delta\partial\alpha} & \frac{\partial^2 LogL}{\partial\delta^2} \end{pmatrix}$$

where its elements are given by:

$$\frac{\partial^2 LogL}{\partial\alpha^2} = \sum_{i=1}^{n} \Delta_i(-(\frac{x_i}{b})^{\alpha} \; Log\left(\frac{x_i}{b}\right)^2 + \frac{n}{\alpha^2}) + \sum_{i=1}^{n}(1-\Delta_i)\frac{n}{\alpha^2} \tag{19}$$

$$\frac{\partial^2 LogL}{\partial\alpha\partial\delta} = 0$$

$$\frac{\partial^2 LogL}{\partial b^2} = -\sum_{i=1}^{n}\Delta_i\,(\frac{x_i^2}{b^4}\left\{\left(\frac{x_i}{b}\right)^{\alpha-2}\alpha(\alpha-1)+\left(\frac{x_i}{b}\right)^{\delta-2}\delta(\delta-1)\right\}+\frac{2x_i}{b^3}\left\{\left(\frac{x_i}{b}\right)^{\alpha-1}\alpha+\left(\frac{x_i}{b}\right)^{\delta-1}\delta\right\})$$
$$+\sum_{i=1}^{n}(1-\Delta_i)\,(\frac{x_i\, e^{\frac{-x_i}{b}}}{b^3\left(1-\frac{1}{2}e^{-\frac{x_i}{b}}\right)}(\delta-\alpha)+\frac{x_i{}^2 e^{\frac{-x_i}{b}}}{2b^4\left(1-\frac{1}{2}e^{-\frac{x_i}{b}}\right)}(\alpha-\delta)\left\{1+\frac{e^{\frac{-x_i}{b}}}{2\left(1-\frac{1}{2}e^{-\frac{x_i}{b}}\right)}\right\}) \tag{20}$$

$$\frac{\partial^2 LogL}{\partial b\partial\lambda} = 0$$

$$\frac{\partial^2 LogL}{\partial b\partial\alpha} = \sum_{i=1}^{n}\Delta_i\,(\frac{x_i\left(\frac{x_i}{b}\right)^{\alpha-1}}{b^2}\{1+\alpha\; Log\left(\frac{x_i}{b}\right)\})+\sum_{i=1}^{n}(1-\Delta_i)\,(\frac{x_i\, e^{\frac{-x_i}{b}}}{2b^2\left(1-\frac{1}{2}e^{-\frac{x_i}{b}}\right)} \tag{21}$$

$$\frac{\partial^2 LogL}{\partial b\partial\delta} = \sum_{i=1}^{n}\Delta_i\,(-\frac{x_i\left(\frac{x_i}{b}\right)^{\delta-1}}{b^2}\{1+\delta\; Log\left(\frac{x_i}{b}\right)\})-\sum_{i=1}^{n}(1-\Delta_i)\,(\frac{x_i\, e^{\frac{-x_i}{b}}}{2b^2\left(1-\frac{1}{2}e^{-\frac{x_i}{b}}\right)} \tag{22}$$

$$\frac{\partial^2 LogL}{\partial\delta^2} = \sum_{i=1}^{n}\Delta_i((\frac{x_i}{b})^{\delta} \; Log\left(\frac{x_i}{b}\right)^2 - \frac{n}{\delta^2}) + \sum_{i=1}^{n}(1-\Delta_i)\frac{-n}{\delta^2} \tag{23}$$

$$\frac{\partial^2 LogL}{\partial\lambda^2} = \sum_{i=1}^{n}\Delta_i\; n(\frac{1}{\lambda^2}-\frac{1}{(1+\lambda)^2}\;) + \sum_{i=1}^{n} n\,(1-\Delta_i)(\frac{1}{\lambda^2}-\frac{1}{(1+\lambda)^2}) \tag{24}$$

$\frac{\partial^2 LogL}{\partial\lambda\partial b}$, $\frac{\partial^2 LogL}{\partial\lambda\partial\alpha}$ and $\frac{\partial^2 LogL}{\partial\lambda\partial\delta} = 0$

Now, we derive the Expected information matrix, whose elements are given by the relation $E\left[-\frac{\partial^2 L}{\partial\delta_i\partial\delta_j}\right]$; $i,j = 1,2,3,4$.

$$E\left[-\frac{\partial^2 LogL}{\partial\alpha^2}\right] = -\left[\sum_{i=1}^{n}\Delta_i\{(\frac{n}{\alpha^2}\right)\left(\frac{1+\lambda}{2^\delta}-\frac{\lambda}{2^\alpha}\right)-\frac{(-1)^\alpha\delta(1+\lambda)\Gamma(\delta+1)}{2^\delta\delta^{\alpha+1}}\left(log\left(\frac{-1}{\delta}\right)+\psi(1+\alpha)+\psi^{(1)}(1+\alpha)\right)$$
$$-\frac{(-1)^\alpha\lambda\alpha\Gamma(\alpha)}{2^\alpha\alpha^\alpha}\left(log\left(\frac{-1}{\alpha}\right)+\psi^{(0)}(1+\alpha)+\psi^{(1)}(1+\alpha)\right)]$$
$$+\sum_{i=1}^{n}(1-\Delta_i)[\frac{n}{\alpha^2}((1+\lambda)\left(1-2^{-\delta}\right)-\lambda\left(1-2^{-\alpha}\right))]] \tag{25}$$

$$E\left[-\frac{\partial^2 LogL}{\partial b^2}\right]=-\left[\sum_{i=1}^{n}\Delta_i(\frac{(-1)^{\alpha}(\alpha+1)\Gamma(\alpha)}{b^2}(\frac{(1+\lambda)\alpha^2}{2^{\delta}\delta^{\alpha}}-\frac{\lambda}{2^{\alpha}\alpha^{\alpha-2}})\right)$$
$$+\sum_{i=1}^{n}(1-\Delta_i)((\frac{\delta(1+\lambda)}{2b^2}\sum_{l=0}^{\infty}\left(\frac{-1}{2}\right)^{l}\binom{\delta-2}{l}\frac{1}{(l+2)^2}-\frac{\alpha\lambda}{2b^2}\sum_{k=0}^{\infty}\left(\frac{-1}{2}\right)^{k}\binom{\alpha-2}{k}\frac{1}{(k+2)^2})(\delta-\alpha)$$
$$+\left(\frac{\delta(1+\lambda)}{4b^2}\sum_{l=0}^{\infty}\left(\frac{-1}{2}\right)^{l}\binom{\delta-3}{l}\frac{1}{(l+2)^2}-\frac{\alpha\lambda}{4b^2}\sum_{k=0}^{\infty}\left(\frac{-1}{2}\right)^{k}\binom{\alpha-3}{k}\frac{1}{(k+2)^2}\right)(\alpha-\delta)$$
$$+\left(\frac{\delta(1+\lambda)}{2b^2}\sum_{l=0}^{\infty}\left(\frac{-1}{2}\right)^{l}\binom{\delta-2}{l}\frac{1}{(l+2)^3}-\frac{\alpha\lambda}{2b^2}\sum_{k=0}^{\infty}\left(\frac{-1}{2}\right)^{k}\binom{\alpha-2}{k}\frac{1}{(k+2)^3}\right)(\alpha-\delta))] \tag{26}$$

$$E\left[-\frac{\partial^2 LogL}{\partial b\partial\alpha}\right]=-[-\sum_{i=1}^{n}\Delta_i[\frac{(-1)^{\alpha}\Gamma(\alpha)}{b}\left(\frac{(1+\lambda)\alpha}{2^{\delta}\delta^{\alpha}}+\frac{\lambda}{2^{\alpha}\alpha^{\alpha-1}}\right)$$
$$-\frac{(-1)^{\alpha}(1+\lambda)\Gamma(\alpha+1)}{b2^{\delta}\delta^{\alpha}}\left(log\left(\frac{-1}{b}\right)-log\left(\frac{\delta}{b}\right)+\psi(1+\alpha)\right)$$
$$+\frac{(-1)^{\alpha}\lambda\Gamma(\alpha)}{b^2 2^{\alpha}\alpha^{\alpha-2}}\left(log\left(\frac{-1}{b}\right)-log\left(\frac{\alpha}{b}\right)+\psi(1+\alpha)\right)] \tag{27}$$
$$+\sum_{i=1}^{n}(1-\Delta_i)[\frac{-\delta(1+\lambda)}{4b}\sum_{l=0}^{\infty}(\frac{-1}{2})^{l}\binom{\delta-2}{l}\frac{1}{(l+2)^2}$$
$$+\frac{\alpha\lambda}{4b}\sum_{k=0}^{\infty}\left(\frac{-1}{2}\right)^{k}\binom{\alpha-2}{k}\frac{1}{(k+2)^2}]]$$

$$E\left[-\frac{\partial^2 LogL}{\partial b\partial\delta}\right]=-[\sum_{i=1}^{n}\Delta_i[\frac{(-1)^{\delta}\Gamma(\delta)}{b}\left(\frac{(1+\lambda)}{2^{\delta}\delta^{\delta-1}}+\frac{\alpha\delta\lambda}{2^{\alpha}\alpha^{\delta}}\right)$$
$$-\frac{(-1)^{\delta}(1+\lambda)\Gamma(\delta)}{b2^{\delta}\delta^{\delta+2}}\left(log\left(\frac{-1}{b}\right)-log\left(\frac{\delta}{b}\right)+\psi(1+\delta)\right)$$
$$-\frac{(-1)^{\delta}\lambda\delta\Gamma(\delta+1)}{b2^{\alpha}\alpha^{\delta}}\left(log\left(\frac{-1}{b}\right)-log\left(\frac{\alpha}{b}\right)+\psi(1+\delta)\right) \tag{28}$$
$$+\sum_{i=1}^{n}(1-\Delta_i)[\frac{-\delta(1+\lambda)}{4b}\sum_{l=0}^{\infty}(\frac{-1}{2})^{l}\binom{\delta-2}{l}\frac{1}{(l+2)^2}$$
$$+\frac{\alpha\lambda}{4b}\sum_{k=0}^{\infty}\left(\frac{-1}{2}\right)^{k}\binom{\alpha-2}{k}\frac{1}{(k+2)^2}]]$$

$$E\left[-\frac{\partial^2 LogL}{\partial\delta^2}\right]=-[\sum_{i=1}^{n}\Delta_i(\frac{-n}{\delta^2}\left(\frac{1+\lambda}{2^{\delta}}-\frac{\lambda}{2^{\alpha}}\right)-\frac{(-1)^{\delta}\delta(1+\lambda)\Gamma(\delta)}{2^{\delta}\delta^{\delta}}\left(log\left(\frac{-1}{\delta}\right)+\psi(1+\delta)+\psi^{(1)}(1+\delta)\right)$$
$$-\frac{(-1)^{\delta}\lambda\alpha\Gamma(\delta+1)}{2^{\delta}\alpha^{\delta+1}}\left(log\left(\frac{-1}{\alpha}\right)+\psi^{(0)}(1+\delta)+\psi^{(1)}(1+\delta)\right)] \tag{29}$$
$$+\sum_{i=1}^{n}(1-\Delta_i)[\frac{-n}{\delta^2}((1+\lambda)\left(1-2^{-\delta}\right)-\lambda\left(1-2^{-\alpha}\right))]]$$

$$E\left[-\frac{\partial^2 LogL}{\partial\lambda^2}\right]=-[\sum_{i=1}^{n}\Delta_i\ n\left(\frac{1}{\lambda^2}-\frac{1}{(1+\lambda^2}\right)\left(\frac{(1+\lambda)}{2^{\delta}}-\frac{\lambda}{2^{\alpha}}\right)$$
$$+\sum_{i=1}^{n}(1-\Delta_i)n\left(\frac{1}{\lambda^2}-\frac{1}{(1+\lambda^2}\right)\left((1+\lambda)\left(1-2^{-\delta}\right)-\lambda\left(1-2^{-\alpha}\right)\right) \tag{30}$$

where $\psi^{(n)}(z)$ is the n^{th} derivative of the digamma function with respect to z.

5 The Differential Entropy

The differential entropy is a concept in information theory that began as an attempt by Shannon to extend the idea of entropy to the continuous probability distributions. Let X be a random variable having pdf $f(x)$ then the differential entropy $h(x)$ is defined as:

$$h(x) = -\int_{-\infty}^{\infty} f(x) Log\, f(x)\, dx \tag{31}$$

$$\begin{aligned} h(x) = &- \Bigg\{ \int_{-\infty}^{0} \left\{ \frac{\delta(1+\lambda)}{2^{\delta} b} \left[e^{\frac{x}{b}} \right]^{\delta} - \frac{\alpha\lambda}{2^{\alpha} b} \left[e^{\frac{x}{b}} \right]^{\alpha} \right\} \left\{ Log \left[\frac{\delta(1+\lambda)}{2^{\delta} b} \left[e^{\frac{x}{b}} \right]^{\delta} - \frac{\alpha\lambda}{2^{\alpha} b} \left[e^{\frac{x}{b}} \right]^{\alpha} \right] \right\} \\ &+ \int_{0}^{\infty} \left\{ \frac{\delta(1+\lambda)}{2b} e^{\frac{-x}{b}} \left[1 - \frac{1}{2} e^{\frac{-x}{b}} \right]^{\delta-1} - \frac{\alpha\lambda}{2b} e^{\frac{-x}{b}} \left[1 - \frac{1}{2} e^{\frac{-x}{b}} \right]^{\alpha-1} \right\} \times \\ &\left\{ Log \left[\frac{\delta(1+\lambda)}{2b} e^{\frac{-x}{b}} \left[1 - \frac{1}{2} e^{\frac{-x}{b}} \right]^{\delta-1} - \frac{\alpha\lambda}{2b} e^{\frac{-x}{b}} \left[1 - \frac{1}{2} e^{\frac{-x}{b}} \right]^{\alpha-1} \right] \right\} \Bigg\} \end{aligned} \tag{32}$$

i.e. we have

$$\begin{aligned} h(x) = &\frac{\delta(1+\lambda)}{b 2^{\delta}} \left\{ \frac{b}{\delta} \left[Log \left(\frac{2^{\alpha}\delta(1+\lambda)}{2^{\delta}\alpha\lambda} \right) \right] + b \left(\frac{b}{\delta} \right)^{\delta} \Gamma(\delta) - \left(\frac{b}{\delta} \right)^{\alpha+1} \Gamma(\alpha+1) \right\} \\ &- \frac{\alpha\lambda}{2^{\alpha} b} \left\{ \frac{b}{\alpha} Log \left(\frac{\delta(1+\lambda)}{b 2^{\delta}} \right) + \left(\frac{b}{\alpha} \right)^{\delta+1} \Gamma(\delta+1) - \frac{b}{\alpha} Log \frac{\alpha\lambda}{2^{\alpha} b} - b \left(\frac{b}{\alpha} \right)^{\alpha+1} \Gamma(\alpha) \right\} \\ &+ \frac{\delta(1+\lambda)}{2b} \left\{ \left[Log \left(\frac{2^{\alpha}\delta(1+\lambda)}{2^{\delta}\alpha\lambda} \right) \right] \sum_{l=0}^{\infty} \left(\frac{-1}{2} \right)^{l} \binom{\delta-1}{l} \frac{b}{1+l} \right. \\ &\left. + (\delta-\alpha) \left[\sum_{l=0}^{\infty} \left(\frac{-1}{2} \right)^{l} \binom{\delta-1}{l} b\Gamma(l+1)\, {}_2F_1[1,1,3+l,-1] - \frac{bLog(2)}{1+l} \right] \right\} \\ &- \frac{\alpha\lambda}{2b} \left\{ \left[Log \left(\frac{\delta(1+\lambda)}{\alpha\lambda} \right) \right] \sum_{l=0}^{\infty} \left(\frac{-1}{2} \right)^{l} \binom{\alpha-1}{l} \frac{b}{1+l} \right. \\ &\left. + (\delta-\alpha) \left[\sum_{l=0}^{\infty} \left(\frac{-1}{2} \right)^{l} \binom{\alpha-1}{l} b\Gamma(l+1)\, {}_2F_1[1,1,3+l,-1] - \frac{bLog(2)}{1+l} \right] \right\} \end{aligned} \tag{33}$$

where ${}_2F_1(a;b;c;z)$ is the Hypergeometric series defined by

$$ {}_2F_1[a;b;c;z] = \sum_{n=0}^{\infty} \frac{(a)_n \ (b)_n}{(c)_n} \frac{z^n}{n!}$$

where

$$(q)_n = \begin{cases} 1 & n = 0 \\ q(q+1)\cdots(q+n-1) & n > 0 \end{cases}$$

6 Applications

In this section, we fit the introduced generalized transmuted Laplace distribution (GTL), the Laplace distribution given in (3) and the transmuted Laplace distribution, which is given by the following

$$F(x) = \frac{1}{2} \left\{ sgn(x) \left[1 - Exp \left(-\frac{1}{b}|x| \right) \right] + 1 \right\} \left\{ (1+\lambda) - \frac{\lambda}{2} \left\{ sgn(x) \left[1 - Exp \left(-\frac{1}{b}|x| \right) \right] + 1 \right\} \right\}$$

Thus , we compare them using the real data set in Table 1, given by Lawless [10]. First, we estimate unknown parameters of the models by the maximum likelihood method. Then, we obtain the values of Akaike Information criterion (AIC), Bayesian information criterion (BIC) and corrected Akaike information criterion (AICC), where

$$AIC = 2k - 2lnL, \quad \text{where } k \text{ is a number of parameter.}$$
$$BIC = -2lnL + kln(n), \quad \text{where } n \text{ is a number of observations.}$$
$$AICC = AIC + 2k(k+1)/n - k - 1.$$

A summary of computations is presented in Table 2.

Table 1: The number of million revolutions before failure for each of the 23 ball

17.88	**28.92**	**33.00**	**41.52**	**42.12**	**45.60**
48.80	51.84	51.96	54.12	55.56	67.80
68.44	68.64	68.88	84.12	93.12	98.64
105.12	105.84	127.92	128.04	173.40	

Table 2: Criteria comparison for the data set.

Distribution	Max. Likelihood Estimates	Log-Likelihood	AIC	BIC	AICC
Laplace	$\hat{b}$=28.306	-115.79	233.58	234.71	233.77
Transmuted Laplace	$\hat{b}$= 28.099,$\widehat{\lambda}$=0.025	-183.05	370.10	372.37	370.7
Generalization of transmuted Laplace	$\widehat{\lambda}$=0.259552,$\hat{b}$=3.35567, $\widehat{\alpha}$=4.937,$\widehat{\delta}$=1.442	-802.885	161.77	161.31	162.99

Table (2) exhibits that , the generalized transmuted Laplace distribution fits better than the other distributions, where the GTL distribution gives the minimum value for Log likelihood and minimum values for the AIC, BIC and AICC criteria.

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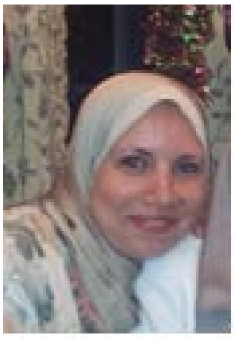

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