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On the diameter of the Kronecker product graph

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Abstract: Let G_1 and G_2 be two undirected nontrivial graphs. The Kronecker product of G_1 and G_2 denoted by $G_1 \otimes G_2$ with vertex set $V(G_1) \times V(G_2)$, two vertices x_1x_2 and y_1y_2 are adjacent if and only if $(x_1, y_1) \in E(G_1)$ and $(x_2, y_2) \in E(G_2)$. This paper presents a formula for computing the diameter of $G_1 \otimes G_2$ by means of the diameters and primitive exponents of factor graphs.

Keywords: graph theory, diameter, Kronecker product, primitive exponent.

1 Introduction

For notation and graph-theoretical terminology not defined here we follow [18]. Specifically, let $G = (V, E)$ be a nontrivial graph with no parallel edges, but loops allowed, where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set.

For two graphs G and H, Kronecker product $G \otimes H$ is a graph with vertex set $V(G) \times V(H)$ and two vertices x_1x_2 and y_1y_2 are adjacent when $(x_1, y_1) \in E(G)$ and $(x_2, y_2) \in E(G)$.

As an operation of graphs, Kronecker product $G \otimes H$ was introduced first by Weichesel [15] in 1962. It has been shown that the Kronecker product is a good method to construct lager networks that can generate many good properties of the factor graphs (see [9]), and has received much research attention recently. Some properties and graphic parameters have been investigated [1,2,5,8,11]. The connectivity and diameter are two important parameters to measure reliability and efficiency of a network. Very recently, the connectivity of Kronecker product graph has been deeply studied (see, [3,6,7,11,12,14,16,17]). However, the diameter of Kronecker product graph has been not investigated yet.

In this paper, we determine the diameter of Kronecker product graph by means of primitive exponents and diameters of factor graphs. In particular, we obtain that

$$d(G_1 \otimes G_2) = \begin{cases} \gamma_1 & \text{if } \gamma_1 = \gamma_2; \\ \max\{\gamma_2 + 1, d_1\} & \text{if } \gamma_1 > \gamma_2; \\ \max\{\gamma_1 + 1, d_2\} & \text{if } \gamma_1 < \gamma_2, \end{cases}$$

where γ_i and d_i are the primitive exponent and diameter of G_i for $i = 1, 2$, respectively.

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2 Some Lemmas

Let G be a graph. Denote $\gamma(G;x,y)$ to be the minimum integer such that there exists an (x,y)-walk of lengh k for any $k\geq\gamma(G;x,y)$ and $\gamma(G)$ be the minimum integer γ for which, for any two vertices x and y in G, there exists an (x,y)-walk of length k for any integer $k\geq\gamma$. Let $\gamma(G)=\max\{\gamma(G;x,y):x,y\in V(G)\}$.

If $\gamma(G)$ is well-defined, then G is said to be *primitive*, and $\gamma(G)$ is called the *primitive exponent*, *exponent* for short, of G. If $\gamma(G)$ does not exist, then denote $\gamma(G)=\infty$.

Let K_n^+ be a graph obtained from a complete graph K_n by appending a loop on each vertex. It is clear that for a graph G without parallel edges of order n, $\gamma(G)=1$ if and only if $G\cong K_n^+$.

Let A be the adjacency matrix of G. Equivalently, the exponent of G is the minimum integer γ for which $A^\gamma>0$ and $A^k\not>0$ for any positive integer $k<\gamma$. Let A_i be the adjacent matrix of G_i for $i=1,2$. Since for any positive integer k, $(A_1\otimes A_2)^k=A_1^k\otimes A_2^k$, by definition, we have the following result immediately.

Proposition 2.1 *Let G_i be a primitive graph with exponent γ_i for $i=1,2$, and $G=G_1\otimes G_2$. Then $\gamma(G)=\max\{\gamma_1,\gamma_2\}$.*

The following lemmas will be used in proofs of our main results.

Lemma 2.1 (Liu et al. [10]) *A graph G is primitive if and only if G is connected and contains odd cycles.*

Lemma 2.2 (Liu et al. [10]) *Let G be a primitive graph, and let x and y be any pair of vertices in $V(G)$. If there are two (x,y)-walks P_1 and P_2 with lengths k_1 and k_2, respectively, where k_1 and k_2 have different parity, then $\gamma(G;x,y)\leq\max\{k_1,k_2\}-1$.*

Lemma 2.3 (Delorme and Solé [4]) *If G is a primitive graph with diameter d, then $\gamma(G)\leq 2d$.*

Lemma 2.4 (Weichesel [15]) *Let G_1 and G_2 be two connected graphs and $G=G_1\otimes G_2$. Then G is connected if and only if either G_1 or G_2 contains an odd cycle.*

Lemma 2.5 *Let $G=G_1\otimes G_2$, x_i and y_i be any two vertices in G_i, P_i be an (x_i,y_i)-walk of length ℓ_i in G_i for $i=1,2$. If ℓ_1 and ℓ_2 have same parity, then there is an (x_1x_2,y_1y_2)-walk of length $\max\{\ell_1,\ell_2\}$ in G.*

Proof. Without loss of generality, suppose $\ell_1\geq\ell_2$. Let $k=\ell_1-\ell_2$. Then k is even. Let $P_1=(x_1,z_1\ldots,z_{\ell_1-1},y_1)$ and $P_2'=(x_2,u_1\ldots,u_{\ell_1-1},y_2)$ be an (x_2,y_2)-walk of length ℓ_1 in G_2 obtained from P_2 by repeating k times of some edge in P_2. Then $(x_1x_2,z_1u_1,\ldots,z_{\ell_1-1}u_{\ell_1-1},y_1y_2)$ is an (x_1x_2,y_1y_2)-walk of length ℓ_1 in G. $\square$

Lemma 2.6 *Let G be a primitive graph with exponent γ and order $n\geq 2$. We have (i) if γ is odd, then there exist two vertices x and y, and two different vertices u and v, such that the shortest odd (x,y)-walk and the shortest even (u,v)-walk are of length γ and $\gamma+1$, respectively; (ii) if γ is even, then there exist two different vertices p and q, and two vertices w and s, such that the shortest even (p,q)-walk and the shortest odd (w,s)-walk are of length γ and $\gamma+1$, respectively.*

Proof. (i) Assume that γ is odd. If $\gamma=1$, then $G\cong K_n^+$. Let u and v be two different vertices in G. Then the shortest odd (u,v)-walk and the shortest even (u,v)-walk are of length 1 and 2, respectively. Suppose now $\gamma\geq 3$.

Let A be the adjacency matrix of G. By definition of γ, $A^{\gamma-1}\not>0$ and $A^{\gamma-2}\not>0$. These imply that there exist four vertices x, y, u and v such that there are no odd (x,y)-walk and even (u,v)-walk with length $\gamma-2$ and $\gamma-1$, respectively. Hence there are no odd (x,y)-walk and even (u,v)-walk with length no more than $\gamma - 2$ and $\gamma - 1$, respectively. Therefore, the shortest odd (x, y)-walk and the shortest even (u, v)-walk are of length γ and $\gamma + 1$, respectively.

We now show $u \neq v$. If $u = v$, then (u, w, u) is an even (u, v)-walk of length 2 for any vertex w adjacent to u in G, a contradiction with $\gamma + 1 \geq 3$.

(ii) Assume γ is even. If $\gamma = 2$, then $d = d(G) = 1$ or 2 since $d \leq \gamma$. If $d = 1$, then iG is isomorphic to a complete graph K_n with m vertices having loops and $m < n$. Let p be a vertex with no loop and $q \neq p$ be another vertex in G. Then the shortest even (p, q)-walk and odd (p, p)-walk are of length 2 and 3, respectively. If $d = 2$, then there exist two different vertices p and q such that $d_G(p, q) = 2$, and hence the shortest even (p, q)-walk and odd (p, q)-walk are of length 2 and 3, respectively.

The case when $\gamma > 2$ can be proved by applying the similar discussion as in (i). □

Lemma 2.7 *Let G_i be a primitive graph with exponent γ_i for $i = 1, 2$, $G = G_1 \otimes G_2$, and $x = x_1x_2$ and $y = y_1y_2$ be two different vertices in G. If the shortest odd (resp. even) (x_1, y_1)-walk in G_1 and the shortest even (resp. odd) (x_2, y_2)-walk in G_2 are of length m and n, respectively, then $d_G(x, y) \geq \min\{m, n\}$.*

Proof. Without loss of generality, assume that m is odd and n is even. Let
$$P = (x_1x_2, \ldots, u_1u_2, \ldots, y_1y_2)$$
be a minimum (x, y)-path with length s in G. Then
$$(x_1, \ldots, u_1, \ldots, y_1) \text{ and } (x_2, \ldots, u_2, \ldots, y_2)$$
be an (x_1, y_1)-walk in G_1 and an (x_2, y_2)-walk in G_2, respectively, and both of them are of length s.

If s is odd, then $s \geq m$ since the shortest odd (x_1, y_1)-walk in G_1 is of length m; If s is even, then $s \geq n$ since the shortest even (x_2, y_2)-walk in G_2 is of length n. Therefore $d_G(x, y) = s \geq \min\{m, n\}$. □

3 Main results

Let G be a connected graph with odd cycles and $C^o(G)$ be the set of all odd cycles in G. For $C \in C^o(G)$ and $x \in V(G)$, let
$$d_G(x, C) = \min\{d_G(x, y) : y \in V(C)\},$$
and let
$$d^o_G(C) = \max\{d_G(x, C) : x \in V(G - C)\} \quad \text{for } C \in C^o(G),$$
$$l^o(G) = \min\{2d^o_G(C) + |V(C)| - 1 : C \in C^o(G)\}.$$
We define $l^o(G) = \infty$ if G is bipartite.

Theorem 3.1 *$\gamma(G) \leq l^o(G)$ for any connected graph G.*

Proof. If G contains no odd cycles, then $l^o(G) = \infty$, and so the conclusion holds. Suppose that G contains odd cycles. By Lemma 2.1, G is primitive. We only need to prove that for any two vertices x and y in G, $\gamma(G; x, y) \leq l^o(G)$.

By definition, there exists an odd cycle C such that $l^o(G) = 2d^o_G(C) + |V(C)| - 1$. Let $d_1 = d_G(x, C)$ and $d_2 = d_G(y, C)$. Then $d_1 \leq d^o_G(C)$ and $d_2 \leq d^o_G(C)$. Let $P_x = (x, x_1, \ldots, x_{d_1})$ and $P_y = (y, y_1, \ldots, y_{d_2})$ be two shortest paths from x and y to C, respectively, where $x_{d_1}, y_{d_2} \in V(C)$ (maybe $x_{d_1} = y_{d_2}$). Two vertices x_{d_1} and y_{d_2} partition C into two paths P_1 and P_2 with lengths p_1 and p_2, respectively. Then p_1 and p_2 have different parity, say $p_1 \geq p_2$. Thus, $P_x \cup P_1 \cup P_y$ and $P_x \cup P_2 \cup P_y$ are two (x, y)-walks with length of different parity and at most
$$d_1 + d_2 + p_1 \leq 2d^o_G(C) + |V(C)| = l^o(G) + 1.$$
By Lemma 2.2, $\gamma(G; x, y) \leq l^o(G)$. □

Corollary 3.1 *If G is a connected graph with loops and diameter d, then $\gamma(G) \leq 2d$.*

Let $H_{n,p}$ and $F_{n,p}$ $(p \geq 1)$ be two graphs, which are obtained by joining a complete graph K_p and a cycle C_p to the end-vertex x_{n-p} of a path $P_{n-p} = (x_1, x_2, \ldots, x_{n-p})$ with an edge, respectively.

The following result can be deduced by Theorem 3.1.

Corollary 3.2 (Wang and Wang [13]) *Let G be a primitive graph with order n and odd girth $p \geq 3$. Then $\gamma(G) \leq 2n-p-1$ with equality if and only if G is isomorphic to $F_{n,p}$.*

Proof. By Lemma 2.1, G is connected and contains an odd cycle C with $l^o(G) = 2d^o_G(C) + |V(C)| - 1$. Since $d^o_G(C) \leq n - |V(C)|$ and $|V(C)| \geq p$, by Theorem 3.1, we have that

$$\begin{aligned}\gamma(G) &\leq l^o(G) = 2d^o_G(C) + |V(C)| - 1 \\ &\leq 2(n - |V(C)|) + |V(C)| - 1 \\ &\leq 2n - p - 1.\end{aligned} \tag{1}$$

The equality implies that all equalities in (1) hold, in particular, $d^o_G(C) = n - |V(C)|$ and $|V(C)| = p$. Thus, there is a vertex x_1 such that $d_G(x_1, C) = n - p$ in G. Suppose $P = (x_1, x_2, \ldots, x_{n-p}, x_{n-p+1})$ is a shortest path from x_1 to C, where x_{n-p+1} is in C. By the minimality of P and primitivity of G, it is easy to see that G is isomorphic to $F_{n,p}$. Also, if G is isomorphic to $F_{n,p}$, then the shortest odd closed (x_1, x_1)-walk is of length $2(n-p) + p = 2n - p$. This implies there is no closed (x_1, x_1)-walk of length $2n - p - 2$. Hence, $\gamma(F_{n,p}) \geq 2n - p - 1$. □

Corollary 3.3 *If $p \geq 3$, then $\gamma(H_{n,p}) = 2n - 2p + 2$.*

Proof. Let $G = H_{n,p}$. Since G contains K_p and $p \geq 3$, G is primitive by Lemma 2.1, and so $d^o_G(C) = d_G(x_1, C) = n - p$ for any $C \in C^o(G)$. Let C be a cycle of length 3 in G. By Theorem 3.1,

$$\gamma(G) \leq l^o(G) \leq 2d^o_G(C) + |V(C)| - 1 = 2(n-p) + |V(C)| - 1 \leq 2n - 2p + 2.$$

It is clear that the shortest odd closed (x_1, x_1) - walk is of length $2(n-p)+3$. This implies there is no closed (x_1, x_1)-walk of length $2(n-p)+1$. Hence, $\gamma(G) \geq 2n - 2p + 2$. The conclusion follows. □

Theorem 3.2 *Let G_i be a connected graph with diameter $d_i \geq 1$ and exponent $\gamma_i = \gamma(G_i)$ for $i = 1, 2$, G_1 contains odd cycles, and $G = G_1 \otimes G_2$. Then the diameter $d(G)$ of G satisfies the following properties.*

(1) $d(G) \geq \max\{d_1, d_2\}$.

(2) If G_2 contains odd cycles, then

$$d(G) \geq \begin{cases} \gamma_1 & \text{if } \gamma_1 = \gamma_2; \\ \min\{\gamma_1, \gamma_2\} + 1 & \text{if } \gamma_1 \neq \gamma_2. \end{cases}$$

(3) $d(G) \leq \max\{\gamma_1, \gamma_2\}$.

(4) $d(G) \leq \min\{\max\{\gamma_1 + 1, d_2\}, \max\{\gamma_2 + 1, d_1\}\}$ with equality if G_2 is bipartite.

Proof. Since both G_1 and G_2 are connected and G_1 contains odd cycles, by Lemma 2.1 and Lemma 2.5, γ_1 is well-defined and G is connected. Since $d_1 \geq 1$ and $d_2 \geq 1$, the order of G_1 and G_2 are no less than 2.

(1) For $i = 1, 2$, let x_i and y_i be two vertices in G_i with $d_{G_i}(x_i, y_i) = d_i$ and let $P = (x_1x_2, \ldots, u_1u_2, \ldots, y_1y_2)$ be a shortest (x_1x_2, y_1y_2)-path in G. Then $(x_1, \ldots, u_1, \ldots, y_1)$ and $(x_2, \ldots, u_2, \ldots, y_2)$ are two walks in G_1 and G_2, respectively. Thus $d(G) \geq d(P) \geq \max\{d_1, d_2\}$.

(2) Since G_2 contains odd cycles, γ_2 is well-defined by Lemma 2.1. Without loss of generality, assume $\gamma_2 \geq \gamma_1$ and γ_1 is odd. By Lemma 2.6, there exist two different vertices x_1 and y_1 such that the shortest even (x_1, y_1)-walk is of length $\gamma_1 + 1$ in G_1; also there exist two vertices x_2 and y_2 such that the shortest odd (x_2, y_2)-walk is of length γ_2 or $\gamma_2 + 1$ in G_2. By Lemma 2.7, $d_G(x_1x_2, y_1y_2) \geq \min\{\gamma_1 + 1, \gamma_2\}$, and so

$$d(G) \geq \begin{cases} \gamma_1 & \text{if } \gamma_1 = \gamma_2; \\ \min\{\gamma_1, \gamma_2\} + 1 & \text{if } \gamma_1 \neq \gamma_2. \end{cases}$$

(3) Without loss of generality, suppose that γ_2 is well-defined and $\gamma_2 \leq \gamma_1$. Let $x = x_1x_2$ and $y = y_1y_2$ be any two different vertices in G. By definition of γ, there exist an (x_1, y_1)-walk and an (x_2, y_2)-walk of length γ_1 in G_1 and G_2, respectively. By Lemma 2.5, there exists an (x, y)-walk of length γ_1, and hence $d(G; x, y) \leq \gamma_1$. By the arbitrariness of x and y, we have $d(G) \leq \gamma_1$.

(4) Without loss of generality, suppose that γ_2 is well-defined, and only need to prove $d(G) \leq \max\{\gamma_1 + 1, d_2\}$. Let $x = x_1x_2$ and $y = y_1y_2$ be any two different vertices in G and

$d'_2 = d_{G_2}(x_2, y_2)$ (maybe $x_2 = y_2$). If $d'_2 \geq \gamma_1$, then there exists an (x_1, y_1)-walk of length d'_2 in G_1 by definition of γ. By Lemma 2.5, there exists an (x, y)-walk of length d'_2 in G. If $d'_2 < \gamma_1$, then one of $d'_2 + \gamma_1$ and $d'_2 + \gamma_1 + 1$ is even. By definition of γ, there exist two (x_1, y_1)-walks of lengths γ_1 and $\gamma_1 + 1$ in G_1, respectively. By Lemma 2.5, there exists an (x, y)-walk of length no more than $\gamma_1 + 1$ in G. Thus $d_G(x, y) \leq \max\{\gamma_1 + 1, d_2\}$, and hence $d(G) \leq \max\{\gamma_1 + 1, d_2\}$ by arbitrariness of x and y.

Now assume that G_2 is bipartite. Let x_2 and y_2 be two vertices in different parts in G_2. Then any (x_2, y_2)-walk and closed (x_2, x_2)-walk are of odd and even length in G_2, respectively. If $\gamma_1 = 1$, then $d(G) \geq d_G(x_1x_2, y_1x_2) \geq 2 = \gamma_1 + 1$ for any two different vertices x_1 and y_1 in G_1 since $|V(G_1)| \geq 2$. Next, assume $\gamma_1 \geq 2$.

By using the Lemma 2.6, we have the following conclusions. If γ_1 is odd, then there exist two different vertices x_1 and y_1 such that the shortest even (x_1, y_1)-walk is of length $\gamma_1 + 1$ in G_1, and hence $d(G) \geq d_G(x_1x_2, y_1x_2) \geq \gamma_1 + 1$. If γ_1 is even, then there exist two vertices x_1 and y_1 such that the shortest odd (x_1, y_1)-walk is of length $\gamma_1 + 1$ in G, and hence $d(G) \geq d_G(x_1x_2, y_1y_2) \geq \gamma_1 + 1$. By the conclusion (1), $d(G) \geq d_2$, and hence $d(G) = \max\{\gamma_1 + 1, d_2\}$.

The theorem follows. □

Corollary 3.4 *Let G_i be a connected graph with diameter $d_i \geq 1$ and $l_i = l^o(G_i)$ for $i = 1, 2$, $G = G_1 \otimes G_2$. Then*

$$d(G) \leq \min\{\max\{l_1 + 1, d_2\}, \max\{l_2 + 1, d_1\}\}.$$

Proof. Without loss of generality, we can suppose that both G_1 and G_2 contain odd cycles. By Theorem 3.1, $\gamma(G_1) \leq l_1$ and $\gamma(G_2) \leq l_2$. The conclusion follows by the conclusion (4) in Theorem 3.2. □

Corollary 3.5 *Let G_i be a connected graph with diameter $d_i \geq 1$ for $i = 1, 2$ and $G = G_1 \otimes G_2$. If G_1 contains odd cycles, then $d(G) \leq \max\{2d_1 + 1, d_2\}$.*

Proof. By Lemma 2.3, $\gamma(G_1) \leq 2d_1$. The Theorem follows by the conclusion (4) in Theorem 3.2. □

The following result, obtained by Leskovec et al. [9], can be deduced by Theorem 3.2 immediately.

Corollary 3.6 (Leskovec et al. [9]) *Let G_i be a connected graph with diameter $d_i \geq 1$ and there is a loop on every vertex of G_i for $i = 1, 2$. Then $d(G_1 \otimes G_2) = \max\{d_1, d_2\}$.*

Proof. It is clear that $\gamma(G_1) = d_1$ and $\gamma(G_2) = d_2$ since each of G_1 and G_2 has a loop on every vertex. The conclusion follows by the conclusions (1) and (3) in Theorem 3.2. □

Corollary 3.7 *Let G be a primitive graph with order $n \geq 2$. Then $\gamma(G) = d(G \otimes K_2) - 1$.*

By Theorem 3.2, we immediately obtain our main results in this paper.

Theorem 3.3 *Let G_i be a connected graph with diameter $d_i \geq 1$ and exponent $\gamma_i = \gamma(G_i)$ for $i = 1, 2$. If G_1 contains odd cycles, then*

$$d(G_1 \otimes G_2) = \begin{cases} \gamma_1 & \text{if } \gamma_1 = \gamma_2; \\ \max\{\gamma_2 + 1, d_1\} & \text{if } \gamma_1 > \gamma_2; \\ \max\{\gamma_1 + 1, d_2\} & \text{if } \gamma_1 < \gamma_2. \end{cases}$$

In Theorem 3.3, we consider the diameter of the Kronecker product of two graphs G_1 and G_2 with order no less than 2. Next, we consider the case that at least one of G_1 and G_2 with order 1. Let G be a connected graph with order n and no parallel edges. We have noted in Section 2, $\gamma(G) = 1$ if and only if $G \cong K_n^+$. For a graph H with order 1, if $G \otimes H$ is connected, then $H \cong K_1^+$ since $G \otimes K_1$ is empty. It is easy to see that $K_1^+ \otimes G \cong G$ and then $d(K_1^+ \otimes G) = d(G)$.

In the following, we show the diameters for some special Kronecker product of two graphs only by using the diameters of factor graphs.

Theorem 3.4 *Let G_i be a connected graph with order $n_i \geq 2$ for $i = 1, 2$. Then $d(G_1 \otimes G_2) = 1$ if and only if $G_1 \cong K_{n_1}^+$ and $G_2 \cong K_{n_2}^+$.*

Proof. The sufficiency is obviously.

Now we show the necessity. By contradiction. Without loss of generality, assume $G_1 \ncong K_{n_1}^+$. Then either there exists a vertex x such that it does not contain a loop or $d(G_1) \geq 2$. Then $d(G) \geq d_G(xy, xz) \geq 2$ for any two different vertices $y, z \in V(G_2)$ or $d(G) \geq d(G_1) \geq 2$ by the conclusion (1) in Theorem 3.2. $\square$

Theorem 3.5 *Let $G \ncong K_n^+$ be a connected graph with order $n \geq 2$ and $m \geq 2$. Then*

$$d(K_m^+ \otimes G) = \begin{cases} 2, & d(G) = 1; \\ d(G), & d(G) \geq 2. \end{cases}$$

Proof. The Theorem follows by Theorem 3.3 since $\gamma(K_m^+) = 1$ and $\gamma(G) \geq 2$. $\square$

Theorem 3.6 *Let G be a connected graph with diameter $d \geq 1$ and H be a complete t partite graph with $t \geq 3$. Then*

$$d(G \otimes H) = \begin{cases} d, & d \geq 3; \\ 2, & d \leq 2 \text{ and } \gamma(G) \leq 2; \\ 3, & d \leq 2 \text{ and } \gamma(G) > 2. \end{cases}$$

Proof. It is clear that $d(H) \geq 2$, H is primitive and $\gamma(H) = 2$. The Theorem follows by Theorem 3.3. $\square$

Corollary 3.8 *Let G be $H_{n,p}$ or $F_{n,p}$ with odd cycles and diameter $d_1 \geq 1$, and H be any connected graph with diameter $d_2 \geq 1$.*

(1) If H is bipartite, then $d(G \otimes H) = \max\{2d_1 + 1, d_2\}$.

(2) If $H = G_{n_2,p_2}$ is non-bipartite, then

$$d(G \otimes H) = \begin{cases} 2d_1 & \text{if } d_1 = d_2; \\ \max\{d_1, 2d_2 + 1\} & \text{if } d_1 > d_2; \\ \max\{d_2, 2d_1 + 1\} & \text{if } d_1 < d_2. \end{cases}$$

Proof. By Lemma 2.1, G is primitive since G contains odd cycles. By Lemma 2.5, $G \otimes H$ is connected. By Corollary 3.1 and 3.2, $\gamma(G) = 2d_1$. If H is bipartite, then H is not primitive by Lemma 2.1. Thus $\infty = \gamma(H) > \gamma(G)$, and hence, $d(G \otimes H) = \max\{2d_1 + 1, d_2\}$ by Theorem 3.3. If $H = G_{n_2,p_2}$ is non-bipartite, then $\gamma(H) = 2d_2$. The conclusion follows by Theorem 3.3 immediately. $\square$

Corollary 3.9 *Let C_m be an odd cycle and H be a connected graph with order n and diameter $d \geq 1$.*

(1) If H is bipartite, then $d(C_m \otimes H) = \max\{m, d\}$. Hence $d(C_m \otimes P_n) = \max\{m, n-1\}$, and $d(C_m \otimes C_n) = \max\{m, \frac{n}{2}\}$ if n is even.

(2) If $H = C_n$ and n is odd, then

$$d(C_m \otimes C_n) = \begin{cases} m - 1 & \text{if } m = n, \\ \max\{n, \frac{m-1}{2}\} & \text{if } m > n \\ \max\{m, \frac{n-1}{2}\} & \text{if } m < n. \end{cases}$$

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