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Multidimensional Fractional Iyengar Type Inequalities for Radial Functions

George A. Anastassiou

Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, U.S.A.

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Abstract: Here we derive a variety of multivariate fractional Iyengar type inequalities for radial functions defined on the shell and ball. Our approach is based on the polar coordinates in $\mathbb{R}^N$, $N \geq 2$, and the related multivariate polar integration formula. Via this method we transfer author's univariate fractional Iyengar type inequalities into multivariate fractional Iyengar inequalities.

Keywords: Iyengar inequality, Polar coordinates, radial function, shell and ball, fractional derivative.

1 Background

We are motivated by the following famous Iyengar inequality (1938), [1].

Theorem 1. *Let f be a differentiable function on $[a,b]$ and $|f'(x)| \leq M$. Then*

$$\left| \int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) \right| \leq \frac{M(b-a)^2}{4} - \frac{(f(b)-f(a))^2}{4M}. \tag{1}$$

We need

Definition 1. *([2], p. 394) Let $\nu > 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ the ceiling of the number), $f \in AC^n([a,b])$ (i.e. $f^{(n-1)}$ is absolutely continuous on $[a,b]$). The left Caputo fractional derivative of order ν is defined as*

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\,dt, \tag{2}$$

$\forall\, x \in [a,b]$, and it exists almost everywhere over $[a,b]$.

We need

Definition 2. *([3], p. 336-337) Let $\nu > 0$, $n = \lceil \nu \rceil$, $f \in AC^n([a,b])$. The right Caputo fractional derivative of order ν is defined as*

$$D_{b-}^{\nu} f(x) = \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (z-x)^{n-\nu-1} f^{(n)}(z)\,dz, \tag{3}$$

$\forall\, x \in [a,b]$, and exists almost everywhere over $[a,b]$.

In [4] we proved the following Caputo fractional Iyengar type inequalities:

* Corresponding author e-mail: ganastss@memphis.edu

Theorem 2.([4]) *Let $\nu > 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), and $f \in AC^n([a,b])$ (i.e. $f^{(n-1)}$ is absolutely continuous on $[a,b]$). We assume that $D_{*a}^{\nu} f, D_{b-}^{\nu} f \in L_\infty([a,b])$. Then*

i)

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leq$$

$$\frac{\max\left\{ \|D_{*a}^{\nu} f\|_{L_\infty([a,b])}, \|D_{b-}^{\nu} f\|_{L_\infty([a,b])} \right\}}{\Gamma(\nu+2)} \left[(t-a)^{\nu+1} + (b-t)^{\nu+1} \right], \tag{4}$$

$\forall\, t \in [a,b]$,

ii) at $t = \frac{a+b}{2}$, the right hand side of (4) is minimized, and we get:

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq$$

$$\frac{\max\left\{ \|D_{*a}^{\nu} f\|_{L_\infty([a,b])}, \|D_{b-}^{\nu} f\|_{L_\infty([a,b])} \right\}}{\Gamma(\nu+2)} \frac{(b-a)^{\nu+1}}{2^{\nu}}, \tag{5}$$

iii) if $f^{(k)}(a) = f^{(k)}(b) = 0$, for all $k = 0,1,\ldots,n-1$, we obtain

$$\left| \int_a^b f(x)\,dx \right| \leq \frac{\max\left\{ \|D_{*a}^{\nu} f\|_{L_\infty([a,b])}, \|D_{b-}^{\nu} f\|_{L_\infty([a,b])} \right\}}{\Gamma(\nu+2)} \frac{(b-a)^{\nu+1}}{2^{\nu}}, \tag{6}$$

which is a sharp inequality,

iv) more generally, for $j = 0,1,2,\ldots,N \in \mathbb{N}$, it holds

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right|$$

$$\leq \frac{\max\left\{ \|D_{*a}^{\nu} f\|_{L_\infty([a,b])}, \|D_{b-}^{\nu} f\|_{L_\infty([a,b])} \right\}}{\Gamma(\nu+2)} \left(\frac{b-a}{N} \right)^{\nu+1} \left[j^{\nu+1} + (N-j)^{\nu+1} \right], \tag{7}$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1,\ldots,n-1$, from (7) we get:

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{N} \right) [j f(a) + (N-j) f(b)] \right| \leq$$

$$\frac{\max\left\{ \|D_{*a}^{\nu} f\|_{L_\infty([a,b])}, \|D_{b-}^{\nu} f\|_{L_\infty([a,b])} \right\}}{\Gamma(\nu+2)} \left(\frac{b-a}{N} \right)^{\nu+1} \left[j^{\nu+1} + (N-j)^{\nu+1} \right], \tag{8}$$

$j = 0,1,2,\ldots,N$,

vi) when $N = 2$ and $j = 1$, (8) turns to

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2} \right) (f(a) + f(b)) \right| \leq$$

$$\frac{\max\left\{ \|D_{*a}^{\nu} f\|_{L_\infty([a,b])}, \|D_{b-}^{\nu} f\|_{L_\infty([a,b])} \right\}}{\Gamma(\nu+2)} \frac{(b-a)^{\nu+1}}{2^{\nu}}, \tag{9}$$

vii) when $0 < \nu \leq 1$, inequality (9) is again valid without any boundary conditions.

We mention

Theorem 3.([4]) *Let $\nu \geq 1$, $n = \lceil \nu \rceil$, and $f \in AC^n([a,b])$. We assume that $D_{*a}^{\nu} f, D_{b-}^{\nu} f \in L_1([a,b])$. Then*
i)

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leq$$

$$\frac{\max\left\{ \|D_{*a}^{\nu} f\|_{L_1([a,b])}, \|D_{b-}^{\nu} f\|_{L_1([a,b])} \right\}}{\Gamma(\nu+1)} \left[(t-a)^{\nu} + (b-t)^{\nu} \right], \tag{10}$$

$\forall t \in [a,b]$,
ii) when $\nu = 1$, from (10), we have

$$\left| \int_a^b f(x)\,dx - [f(a)(t-a) + f(b)(b-t)] \right| \leq$$

$$\|f'\|_{L_1([a,b])}(b-a), \quad \forall t \in [a,b], \tag{11}$$

iii) from (11), we obtain ($\nu = 1$ case)

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2} \right)(f(a)+f(b)) \right| \leq \|f'\|_{L_1([a,b])}(b-a), \tag{12}$$

iv) at $t = \frac{a+b}{2}$, $\nu > 1$, the right hand side of (10) is minimized, and we get:

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq$$

$$\frac{\max\left\{ \|D_{*a}^{\nu} f\|_{L_1([a,b])}, \|D_{b-}^{\nu} f\|_{L_1([a,b])} \right\}}{\Gamma(\nu+1)} \frac{(b-a)^{\nu}}{2^{\nu-1}}, \tag{13}$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, for all $k = 0, 1, \ldots, n-1$; $\nu > 1$, from (13), we obtain

$$\left| \int_a^b f(x)\,dx \right| \leq \frac{\max\left\{ \|D_{*a}^{\nu} f\|_{L_1([a,b])}, \|D_{b-}^{\nu} f\|_{L_1([a,b])} \right\}}{\Gamma(\nu+1)} \frac{(b-a)^{\nu}}{2^{\nu-1}}, \tag{14}$$

which is a sharp inequality,
vi) more generally, for $j = 0, 1, 2, \ldots, N \in \mathbb{N}$, it holds

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right|$$

$$\leq \frac{\max\left\{ \|D_{*a}^{\nu} f\|_{L_1([a,b])}, \|D_{b-}^{\nu} f\|_{L_1([a,b])} \right\}}{\Gamma(\nu+1)} \left(\frac{b-a}{N} \right)^{\nu} \left[j^{\nu} + (N-j)^{\nu} \right], \tag{15}$$

vii) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, \ldots, n-1$, from (15) we get:

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{N} \right) [j f(a) + (N-j) f(b)] \right| \leq$$

$$\frac{\max\left\{ \|D_{*a}^{\nu} f\|_{L_1([a,b])}, \|D_{b-}^{\nu} f\|_{L_1([a,b])} \right\}}{\Gamma(\nu+1)} \left(\frac{b-a}{N} \right)^{\nu} \left[j^{\nu} + (N-j)^{\nu} \right], \tag{16}$$

$j = 0, 1, 2, \ldots, N$,
viii) when $N = 2$ and $j = 1$, (16) turns to

$$\left| \int_a^b f(x)\,dx - \frac{(b-a)}{2}(f(a)+f(b)) \right| \leq$$

$$\frac{\max\left\{ \|D_{*a}^{\nu} f\|_{L_1([a,b])}, \|D_{b-}^{\nu} f\|_{L_1([a,b])} \right\}}{\Gamma(\nu+1)} \frac{(b-a)^{\nu}}{2^{\nu-1}}. \tag{17}$$

We mention

Theorem 4.([4]) *Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\nu>\frac{1}{q}$, $n=\lceil\nu\rceil$; $f\in AC^n([a,b])$, *with* $D_{*a}^{\nu}f, D_{b-}^{\nu}f\in L_q([a,b])$. *Then*

i)

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[f^{(k)}(a)(t-a)^{k+1}+(-1)^k f^{(k)}(b)(b-t)^{k+1}\right]\right|\leq$$

$$\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_q([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_q([a,b])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\left[(t-a)^{\nu+\frac{1}{p}}+(b-t)^{\nu+\frac{1}{p}}\right], \tag{18}$$

$\forall\, t\in[a,b]$,

ii) at $t=\frac{a+b}{2}$, *the right hand side of (18) is minimized, and we get:*

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right|\leq$$

$$\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_q([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_q([a,b])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\frac{(b-a)^{\nu+\frac{1}{p}}}{2^{\nu-\frac{1}{q}}}, \tag{19}$$

iii) if $f^{(k)}(a)=f^{(k)}(b)=0$, *for all* $k=0,1,\dots,n-1$, *we obtain*

$$\left|\int_a^b f(x)\,dx\right|\leq\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_q([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_q([a,b])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\frac{(b-a)^{\nu+\frac{1}{p}}}{2^{\nu-\frac{1}{q}}}, \tag{20}$$

which is a sharp inequality,

iv) more generally, for $j=0,1,2,\dots,N\in\mathbb{N}$, *it holds*

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{N}\right)^{k+1}\left[j^{k+1}f^{(k)}(a)+(-1)^k(N-j)^{k+1}f^{(k)}(b)\right]\right|$$

$$\leq\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_q([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_q([a,b])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\left(\frac{b-a}{N}\right)^{\nu+\frac{1}{p}}\left[j^{\nu+\frac{1}{p}}+(N-j)^{\nu+\frac{1}{p}}\right], \tag{21}$$

v) if $f^{(k)}(a)=f^{(k)}(b)=0$, $k=1,\dots,n-1$, *from (21) we get:*

$$\left|\int_a^b f(x)\,dx-\left(\frac{b-a}{N}\right)[jf(a)+(N-j)f(b)]\right|\leq$$

$$\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_q([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_q([a,b])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\left(\frac{b-a}{N}\right)^{\nu+\frac{1}{p}}\left[j^{\nu+\frac{1}{p}}+(N-j)^{\nu+\frac{1}{p}}\right], \tag{22}$$

for $j=0,1,2,\dots,N$,

vi) when $N=2$ *and* $j=1$, *(22) turns to*

$$\left|\int_a^b f(x)\,dx-\left(\frac{b-a}{2}\right)(f(a)+f(b))\right|\leq$$

$$\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_q([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_q([a,b])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\frac{(b-a)^{\nu+\frac{1}{p}}}{2^{\nu-\frac{1}{q}}}, \tag{23}$$

vii) when $1/q<\nu\leq 1$, *inequality (23) is again valid but without any boundary conditions.*

We need the following different fractional calculus background:

Let $\alpha > 0$, $m = [\alpha]$ ($[\cdot]$ is the integral part), $\beta = \alpha - m$, $0 < \beta < 1$, $f \in C([a,b])$, $[a,b] \subset \mathbb{R}$, $x \in [a,b]$. The gamma function Γ is given by $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$. We define the left Riemann-Liouville integral ([2], p. 24)

$$\left(J_\alpha^{a+} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \tag{24}$$

$a \le x \le b$. We define the subspace $C_{a+}^\alpha([a,b])$ of $C^m([a,b])$:

$$C_{a+}^\alpha([a,b]) = \left\{ f \in C^m([a,b]) : J_{1-\beta}^{a+} f^{(m)} \in C^1([a,b]) \right\}. \tag{25}$$

For $f \in C_{a+}^\alpha([a,b])$, we define the left generalized α-fractional derivative of f over $[a,b]$ as

$$D_{a+}^\alpha f := \left(J_{1-\beta}^{a+} f^{(m)} \right)', \tag{26}$$

see [2], p. 24. Canavati first in [5] introduced the above over $[0,1]$.

We have that $D_{a+}^n f = f^{(n)}$; $n \in \mathbb{N}$.

Notice that $D_{a+}^\alpha f \in C([a,b])$.

Furthermore we need:

Let again $\alpha > 0$, $m = [\alpha]$, $\beta = \alpha - m$, $f \in C([a,b])$, call the right Riemann-Liouville fractional integral operator by

$$\left(J_{b-}^\alpha f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt, \tag{27}$$

$x \in [a,b]$, see [6]. Define the subspace of functions

$$C_{b-}^\alpha([a,b]) := \left\{ f \in C^m([a,b]) : J_{b-}^{1-\beta} f^{(m)} \in C^1([a,b]) \right\}. \tag{28}$$

Define the right generalized α-fractional derivative of f over $[a,b]$ as

$$\overline{D}_{b-}^\alpha f = (-1)^{m-1} \left(J_{b-}^{1-\beta} f^{(m)} \right)', \tag{29}$$

see [6]. We set $\overline{D}_{b-}^0 f = f$. We have $\overline{D}_{b-}^n f = (-1)^n f^{(n)}$; $n \in \mathbb{N}$. Notice that $\overline{D}_{b-}^\alpha f \in C([a,b])$.

We mention the following Canavati fractional Iyengar type inequalities:

Theorem 5.([7]) *Let $\nu \ge 1$, $n = [\nu]$ and $f \in C_{a+}^\nu([a,b]) \cap C_{b-}^\nu([a,b])$. Then*

i)

$$\left| \int_a^b f(x)\, dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \le$$

$$\frac{\max\left\{ \left\| D_{a+}^\nu f \right\|_{\infty,([a,b])}, \left\| \overline{D}_{b-}^\nu f \right\|_{\infty,([a,b])} \right\}}{\Gamma(\nu+2)} \left[(t-a)^{\nu+1} + (b-t)^{\nu+1} \right], \tag{30}$$

$\forall\, t \in [a,b]$,

ii) at $t = \frac{a+b}{2}$, the right hand side of (30) is minimized, and we get:

$$\left| \int_a^b f(x)\, dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \le$$

$$\frac{\max\left\{ \left\| D_{a+}^\nu f \right\|_{\infty,([a,b])}, \left\| \overline{D}_{b-}^\nu f \right\|_{\infty,([a,b])} \right\}}{\Gamma(\nu+2)} \frac{(b-a)^{\nu+1}}{2^\nu}, \tag{31}$$

iii) if $f^{(k)}(a) = f^{(k)}(b) = 0$, for all $k = 0, 1, ..., n-1$, we obtain

$$\left| \int_a^b f(x)\,dx \right| \le \frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{\infty,([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{\infty,([a,b])} \right\}}{\Gamma(\nu+2)} \frac{(b-a)^{\nu+1}}{2^{\nu}}, \tag{32}$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, it holds

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right|$$

$$\le \frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{\infty,([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{\infty,([a,b])} \right\}}{\Gamma(\nu+2)} \left(\frac{b-a}{N} \right)^{\nu+1} \left[j^{\nu+1} + (N-j)^{\nu+1} \right], \tag{33}$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, ..., n-1$, from (33) we get:

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{N} \right) [j f(a) + (N-j) f(b)] \right| \le$$

$$\frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{\infty,([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{\infty,([a,b])} \right\}}{\Gamma(\nu+2)} \left(\frac{b-a}{N} \right)^{\nu+1} \left[j^{\nu+1} + (N-j)^{\nu+1} \right], \tag{34}$$

$j = 0, 1, 2, ..., N$,

vi) when $N = 2$ and $j = 1$, (34) turns to

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2} \right) (f(a) + f(b)) \right| \le$$

$$\frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{\infty,([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{\infty,([a,b])} \right\}}{\Gamma(\nu+2)} \frac{(b-a)^{\nu+1}}{2^{\nu}}. \tag{35}$$

We mention

Theorem 6.([7]) *Let $\nu \ge 1$, $n = [\nu]$, and $f \in C_{a+}^{\nu}([a,b]) \cap C_{b-}^{\nu}([a,b])$. Then*

i)

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \le$$

$$\frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{L_1([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{L_1([a,b])} \right\}}{\Gamma(\nu+1)} \left[(t-a)^{\nu} + (b-t)^{\nu} \right], \tag{36}$$

$\forall\, t \in [a,b]$,

ii) when $\nu = 1$, from (36), we have

$$\left| \int_a^b f(x)\,dx - [f(a)(t-a) + f(b)(b-t)] \right| \le$$

$$\left\| f' \right\|_{L_1([a,b])} (b-a), \ \ \forall\, t \in [a,b], \tag{37}$$

iii) from (37), we obtain ($\nu = 1$ case)

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2} \right) (f(a) + f(b)) \right| \le \left\| f' \right\|_{L_1([a,b])} (b-a), \tag{38}$$

iv) at $t=\frac{a+b}{2}$, $\nu>1$, the right hand side of (36) is minimized, and we get:

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right|\leq$$

$$\frac{\max\left\{\left\|D_{a+}^{\nu}f\right\|_{L_1([a,b])},\left\|\overline{D}_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\frac{(b-a)^{\nu}}{2^{\nu-1}}, \tag{39}$$

v) if $f^{(k)}(a)=f^{(k)}(b)=0$, for all $k=0,1,...,n-1$; $\nu>1$, from (39), we obtain

$$\left|\int_a^b f(x)\,dx\right|\leq\frac{\max\left\{\left\|D_{a+}^{\nu}f\right\|_{L_1([a,b])},\left\|\overline{D}_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\frac{(b-a)^{\nu}}{2^{\nu-1}}, \tag{40}$$

which is a sharp inequality,
vi) more generally, for $j=0,1,2,...,N\in\mathbb{N}$, it holds

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{N}\right)^{k+1}\left[j^{k+1}f^{(k)}(a)+(-1)^k(N-j)^{k+1}f^{(k)}(b)\right]\right|$$

$$\leq\frac{\max\left\{\left\|D_{a+}^{\nu}f\right\|_{L_1([a,b])},\left\|\overline{D}_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\left(\frac{b-a}{N}\right)^{\nu}\left[j^{\nu}+(N-j)^{\nu}\right], \tag{41}$$

vii) if $f^{(k)}(a)=f^{(k)}(b)=0$, $k=1,...,n-1$, from (41) we get:

$$\left|\int_a^b f(x)\,dx-\left(\frac{b-a}{N}\right)\left[jf(a)+(N-j)f(b)\right]\right|\leq$$

$$\frac{\max\left\{\left\|D_{a+}^{\nu}f\right\|_{L_1([a,b])},\left\|\overline{D}_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\left(\frac{b-a}{N}\right)^{\nu}\left[j^{\nu}+(N-j)^{\nu}\right], \tag{42}$$

$j=0,1,2,...,N,$
viii) when $N=2$ and $j=1$, (42) turns to

$$\left|\int_a^b f(x)\,dx-\frac{(b-a)}{2}\left(f(a)+f(b)\right)\right|\leq$$

$$\frac{\max\left\{\left\|D_{a+}^{\nu}f\right\|_{L_1([a,b])},\left\|\overline{D}_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\frac{(b-a)^{\nu}}{2^{\nu-1}}, \tag{43}$$

We mention

Theorem 7.(*[7]) Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\nu\geq1$, $n=[\nu]$; $f\in C_{a+}^{\nu}([a,b])\cap C_{b-}^{\nu}([a,b])$. Then*
i)

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[f^{(k)}(a)(t-a)^{k+1}+(-1)^k f^{(k)}(b)(b-t)^{k+1}\right]\right|\leq$$

$$\frac{\max\left\{\left\|D_{a+}^{\nu}f\right\|_{L_q([a,b])},\left\|\overline{D}_{b-}^{\nu}f\right\|_{L_q([a,b])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\left[(t-a)^{\nu+\frac{1}{p}}+(b-t)^{\nu+\frac{1}{p}}\right], \tag{44}$$

$\forall\, t\in[a,b],$

ii) at $t = \frac{a+b}{2}$*, the right hand side of (44) is minimized, and we get:*

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{L_q([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{L_q([a,b])} \right\}}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{\nu+\frac{1}{p}}}{2^{\nu-\frac{1}{q}}}, \tag{45}$$

iii) if $f^{(k)}(a) = f^{(k)}(b) = 0$*, for all* $k = 0,1,...,n-1$*, we obtain*

$$\left| \int_a^b f(x)\,dx \right| \leq \frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{L_q([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{L_q([a,b])} \right\}}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{\nu+\frac{1}{p}}}{2^{\nu-\frac{1}{q}}}, \tag{46}$$

which is a sharp inequality,

iv) more generally, for $j = 0,1,2,...,N \in \mathbb{N}$*, it holds*

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N}\right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right|$$

$$\leq \frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{L_q([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{L_q([a,b])} \right\}}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \left(\frac{b-a}{N}\right)^{\nu+\frac{1}{p}} \left[j^{\nu+\frac{1}{p}} + (N-j)^{\nu+\frac{1}{p}} \right], \tag{47}$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$*,* $k = 1,...,n-1$*, from (47) we get:*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{N}\right) \left[jf(a) + (N-j)f(b) \right] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{L_q([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{L_q([a,b])} \right\}}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \left(\frac{b-a}{N}\right)^{\nu+\frac{1}{p}} \left[j^{\nu+\frac{1}{p}} + (N-j)^{\nu+\frac{1}{p}} \right], \tag{48}$$

for $j = 0,1,2,...,N$*,*

vi) when $N = 2$ *and* $j = 1$*, (48) turns to*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2}\right) (f(a) + f(b)) \right| \leq$$

$$\frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{L_q([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{L_q([a,b])} \right\}}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{\nu+\frac{1}{p}}}{2^{\nu-\frac{1}{q}}}. \tag{49}$$

We need

Definition 3.*([8]) Let* $a,b \in \mathbb{R}$*. The left conformable fractional derivative starting from* a *of a function* $f : [a,\infty) \to \mathbb{R}$ *of order* $0 < \alpha \leq 1$ *is defined by*

$$\left(T_{\alpha}^{a} f\right)(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon (t-a)^{1-\alpha}\right) - f(t)}{\varepsilon}. \tag{50}$$

If $\left(T_{\alpha}^{a} f\right)(t)$ *exists on* (a,b)*, then*

$$\left(T_{\alpha}^{a} f\right)(a) = \lim_{t \to a+} \left(T_{\alpha}^{a} f\right)(t). \tag{51}$$

The right conformable fractional derivative of order $0<\alpha\leq 1$ *terminating at b of* $f:(-\infty,b]\to\mathbb{R}$ *is defined by*

$$\left({}_{\alpha}^{b}Tf\right)(t)=-\lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon(b-t)^{1-\alpha}\right)-f(t)}{\varepsilon}. \tag{52}$$

If $\left({}_{\alpha}^{b}Tf\right)(t)$ *exists on* (a,b)*, then*

$$\left({}_{\alpha}^{b}Tf\right)(b)=\lim_{t\to b-}\left({}_{\alpha}^{b}Tf\right)(t). \tag{53}$$

Note that if f is differentiable then

$$\left(T_{\alpha}^{a}f\right)(t)=(t-a)^{1-\alpha}f'(t), \tag{54}$$

and

$$\left({}_{\alpha}^{b}Tf\right)(t)=-(b-t)^{1-\alpha}f'(t). \tag{55}$$

In the higher order case we can generalize things as follows:

Definition 4.*([8]) Let* $\alpha\in(n,n+1]$*, and set* $\beta=\alpha-n$*. Then, the left conformable fractional derivative starting from a of a function* $f:[a,\infty)\to\mathbb{R}$ *of order* α*, where* $f^{(n)}(t)$ *exists, is defined by*

$$\left(\mathbf{T}_{\alpha}^{a}f\right)(t)=\left(T_{\beta}^{a}f^{(n)}\right)(t), \tag{56}$$

The right conformable fractional derivative of order α *terminating at b of* $f:(-\infty,b]\to\mathbb{R}$*, where* $f^{(n)}(t)$ *exists, is defined by*

$$\left({}_{\alpha}^{b}\mathbf{T}f\right)(t)=(-1)^{n+1}\left({}_{\beta}^{b}Tf^{(n)}\right)(t). \tag{57}$$

If $\alpha=n+1$ *then* $\beta=1$ *and* $\mathbf{T}_{n+1}^{a}f=f^{(n+1)}$*.*

If n is odd, then ${}_{n+1}^{b}\mathbf{T}f=-f^{(n+1)}$*, and if n is even, then* ${}_{n+1}^{b}\mathbf{T}f=f^{(n+1)}$*.*

When $n=0$ *(or* $\alpha\in(0,1]$*), then* $\beta=\alpha$*, and (56), (57) collapse to (50), (52), respectively.*

We need

Remark.([9]) We notice the following: let $\alpha\in(n,n+1]$ and $f\in C^{n+1}([a,b])$, $n\in\mathbb{N}$. Then ($\beta:=\alpha-n$, $0<\beta\leq 1$)

$$\left(\mathbf{T}_{\alpha}^{a}(f)\right)(x)=\left(T_{\beta}^{\alpha}f^{(n)}\right)(x)=(x-a)^{1-\beta}f^{(n+1)}(x), \tag{58}$$

and

$$\left({}_{\alpha}^{b}\mathbf{T}(f)\right)(x)=(-1)^{n+1}\left({}_{\beta}^{b}Tf^{(n)}\right)(x)=$$

$$(-1)^{n+1}(-1)(b-x)^{1-\beta}f^{(n+1)}(x)=(-1)^{n}(b-x)^{1-\beta}f^{(n+1)}(x). \tag{59}$$

Consequently we get that

$$\left(\mathbf{T}_{\alpha}^{a}(f)\right)(x),\ \left({}_{\alpha}^{b}\mathbf{T}(f)\right)(x)\in C([a,b]).$$

Furthermore it is obvious that

$$\left(\mathbf{T}_{\alpha}^{a}(f)\right)(a)=\left({}_{\alpha}^{b}\mathbf{T}(f)\right)(b)=0, \tag{60}$$

when $0<\beta<1$, i.e. when $\alpha\in(n,n+1)$.

We mention the following Conformable fractional Iyengar type inequalities:

Theorem 8.*([10]) Let* $\alpha\in(n,n+1]$ *and* $f\in C^{n+1}([a,b])$*,* $n\in\mathbb{N}$*;* $\beta=\alpha-n$*. Then*

i)

$$\left|\int_{a}^{b}f(t)\,dt-\sum_{k=0}^{n}\frac{1}{(k+1)!}\left[f^{(k)}(a)(z-a)^{k+1}+(-1)^{k}f^{(k)}(b)(b-z)^{k+1}\right]\right|\leq$$

$$\frac{\Gamma(\beta)\max\left\{\left\|\mathbf{T}_{\alpha}^{a}(f)\right\|_{\infty,[a,b]},\left\|{}_{\alpha}^{b}\mathbf{T}(f)\right\|_{\infty,[a,b]}\right\}}{\Gamma(\alpha+2)}\left[(z-a)^{\alpha+1}+(b-z)^{\alpha+1}\right], \tag{61}$$

$\forall z \in [a,b]$,

ii) at $z = \frac{a+b}{2}$, *the right hand side of (61) is minimized, and we get:*

$$\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq$$

$$\frac{\Gamma(\beta) \max\left\{ \left\| \mathbf{T}_\alpha^a(f) \right\|_{\infty,[a,b]}, \left\| {}_\alpha^b\mathbf{T}(f) \right\|_{\infty,[a,b]} \right\}}{\Gamma(\alpha+2)} \frac{(b-a)^{\alpha+1}}{2^\alpha}, \tag{62}$$

iii) assuming $f^{(k)}(a) = f^{(k)}(b) = 0$, *for* $k = 0,1,\ldots,n$, *we obtain*

$$\left| \int_a^b f(t)\,dt \right| \leq \frac{\Gamma(\beta) \max\left\{ \left\| \mathbf{T}_\alpha^a(f) \right\|_{\infty,[a,b]}, \left\| {}_\alpha^b\mathbf{T}(f) \right\|_{\infty,[a,b]} \right\}}{\Gamma(\alpha+2)} \frac{(b-a)^{\alpha+1}}{2^\alpha}, \tag{63}$$

which is a sharp inequality,

iv) more generally, for $j = 0,1,2,\ldots,N \in \mathbb{N}$, *it holds*

$$\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \left[f^{(k)}(a)\, j^{k+1} + (-1)^k f^{(k)}(b)\,(N-j)^{k+1} \right] \right|$$

$$\leq \frac{\Gamma(\beta) \max\left\{ \left\| \mathbf{T}_\alpha^a(f) \right\|_{\infty,[a,b]}, \left\| {}_\alpha^b\mathbf{T}(f) \right\|_{\infty,[a,b]} \right\}}{\Gamma(\alpha+2)} \left(\frac{b-a}{N} \right)^{\alpha+1} \left[j^{\alpha+1} + (N-j)^{\alpha+1} \right], \tag{64}$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1,\ldots,n$, *from (64) we get:*

$$\left| \int_a^b f(t)\,dt - \left(\frac{b-a}{N} \right) [j f(a) + (N-j) f(b)] \right| \leq$$

$$\frac{\Gamma(\beta) \max\left\{ \left\| \mathbf{T}_\alpha^a(f) \right\|_{\infty,[a,b]}, \left\| {}_\alpha^b\mathbf{T}(f) \right\|_{\infty,[a,b]} \right\}}{\Gamma(\alpha+2)} \left(\frac{b-a}{N} \right)^{\alpha+1} \left[j^{\alpha+1} + (N-j)^{\alpha+1} \right], \tag{65}$$

$j = 0,1,2,\ldots,N$,

vi) when $N = 2$ *and* $j = 1$, *(65) turns to*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2} \right) (f(a) + f(b)) \right| \leq$$

$$\frac{\Gamma(\beta) \max\left\{ \left\| \mathbf{T}_\alpha^a(f) \right\|_{\infty,[a,b]}, \left\| {}_\alpha^b\mathbf{T}(f) \right\|_{\infty,[a,b]} \right\}}{\Gamma(\alpha+2)} \frac{(b-a)^{\alpha+1}}{2^\alpha}. \tag{66}$$

We mention L_p conformable fractional Iyengar inequalities:

Theorem 9.*([10]) Let* $\alpha \in (n, n+1]$ *and* $f \in C^{n+1}([a,b])$, $n \in \mathbb{N}$; $\beta = \alpha - n$. *Let also* $p_1, p_2, p_3 > 1 : \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, *with* $\beta > \frac{1}{p_1} + \frac{1}{p_3}$. *Then*

i)

$$\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n} \frac{1}{(k+1)!} \left[f^{(k)}(a)(z-a)^{k+1} + (-1)^k f^{(k)}(b)(b-z)^{k+1} \right] \right| \leq$$

$$\frac{\max\left\{ \left\| \mathbf{T}_\alpha^a(f) \right\|_{p_3,[a,b]}, \left\| {}_\alpha^b\mathbf{T}(f) \right\|_{p_3,[a,b]} \right\}}{n!\,(p_1 n+1)^{\frac{1}{p_1}} (p_2(\beta-1)+1)^{\frac{1}{p_2}} \left(\alpha + \frac{1}{p_1} + \frac{1}{p_2} \right)} \tag{67}$$

$$\left[(z-a)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}} + (b-z)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}} \right],$$

$\forall z \in [a,b]$,

ii) at $z = \frac{a+b}{2}$, *the right hand side of (67) is minimized, and we get:*

$$\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq$$

$$\frac{\max\left\{ \left\| \mathbf{T}_{\alpha}^{a}(f) \right\|_{p_3,[a,b]}, \left\| {}_{\alpha}^{b}\mathbf{T}(f) \right\|_{p_3,[a,b]} \right\}}{n!\,(p_1 n+1)^{\frac{1}{p_1}} \left(p_2(\beta-1)+1\right)^{\frac{1}{p_2}} \left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)} \frac{(b-a)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}}{2^{\alpha-\frac{1}{p_3}}}, \tag{68}$$

iii) assuming $f^{(k)}(a) = f^{(k)}(b) = 0$, *for* $k = 0,1,\ldots,n$, *we obtain*

$$\left| \int_a^b f(t)\,dt \right| \leq \frac{\max\left\{ \left\| \mathbf{T}_{\alpha}^{a}(f) \right\|_{p_3,[a,b]}, \left\| {}_{\alpha}^{b}\mathbf{T}(f) \right\|_{p_3,[a,b]} \right\}}{n!\,(p_1 n+1)^{\frac{1}{p_1}} \left(p_2(\beta-1)+1\right)^{\frac{1}{p_2}} \left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)} \frac{(b-a)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}}{2^{\alpha-\frac{1}{p_3}}}, \tag{69}$$

which is a sharp inequality,

iv) more generally, for $j = 0,1,2,\ldots,N \in \mathbb{N}$, *it holds*

$$\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n} \frac{1}{(k+1)!} \left(\frac{b-a}{N}\right)^{k+1} \left[f^{(k)}(a)\, j^{k+1} + (-1)^k f^{(k)}(b)\,(N-j)^{k+1} \right] \right|$$

$$\leq \frac{\max\left\{ \left\| \mathbf{T}_{\alpha}^{a}(f) \right\|_{p_3,[a,b]}, \left\| {}_{\alpha}^{b}\mathbf{T}(f) \right\|_{p_3,[a,b]} \right\}}{n!\,(p_1 n+1)^{\frac{1}{p_1}} \left(p_2(\beta-1)+1\right)^{\frac{1}{p_2}} \left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)}$$

$$\left(\frac{b-a}{N}\right)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}} \left[j^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}} + (N-j)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}} \right], \tag{70}$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1,\ldots,n$, *from (70) we get:*

$$\left| \int_a^b f(t)\,dt - \left(\frac{b-a}{N}\right) \left[j f(a) + (N-j) f(b) \right] \right| \leq \tag{71}$$

$$\frac{\max\left\{ \left\| \mathbf{T}_{\alpha}^{a}(f) \right\|_{p_3,[a,b]}, \left\| {}_{\alpha}^{b}\mathbf{T}(f) \right\|_{p_3,[a,b]} \right\}}{n!\,(p_1 n+1)^{\frac{1}{p_1}} \left(p_2(\beta-1)+1\right)^{\frac{1}{p_2}} \left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)}$$

$$\left(\frac{b-a}{N}\right)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}} \left[j^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}} + (N-j)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}} \right],$$

for $j = 0,1,2,\ldots,N$,

vi) when $N = 2$ *and* $j = 1$, *(71) turns to*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2}\right) (f(a) + f(b)) \right| \leq$$

$$\frac{\max\left\{ \left\| \mathbf{T}_{\alpha}^{a}(f) \right\|_{p_3,[a,b]}, \left\| {}_{\alpha}^{b}\mathbf{T}(f) \right\|_{p_3,[a,b]} \right\}}{n!\,(p_1 n+1)^{\frac{1}{p_1}} \left(p_2(\beta-1)+1\right)^{\frac{1}{p_2}} \left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)} \frac{(b-a)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}}{2^{\alpha-\frac{1}{p_3}}}. \tag{72}$$

We need

Remark. We define the ball $B(0,R) = \{x \in \mathbb{R}^N : |x| < R\} \subseteq \mathbb{R}^N$, $N \geq 2$, $R > 0$, and the sphere

$$S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\},$$

where $|\cdot|$ is the Euclidean norm. Let $d\omega$ be the element of surface measure on S^{N-1} and

$$\omega_N = \int_{S^{N-1}} d\omega = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}$$

is the area of S^{N-1}.

For $x \in \mathbb{R}^N - \{0\}$ we can write uniquely $x = r\omega$, where $r = |x| > 0$ and $\omega = \frac{x}{r} \in S^{N-1}$, $|\omega| = 1$. Note that $\int_{B(0,R)} dy = \frac{\omega_N R^N}{N}$ is the Lebesgue measure on the ball, that is the volume of $B(0,R)$, which exactly is $Vol(B(0,R)) = \frac{\pi^{\frac{N}{2}} R^N}{\Gamma\left(\frac{N}{2}+1\right)}$.

Following [11, pp. 149-150, exercise 6], and [12, pp. 87-88, Theorem 5.2.2] we can write for $F : \overline{B(0,R)} \to \mathbb{R}$ a Lebesgue integrable function that

$$\int_{B(0,R)} F(x)\,dx = \int_{S^{N-1}} \left(\int_0^R F(r\omega)\, r^{N-1} dr\right) d\omega, \tag{73}$$

and we use this formula a lot.

Typically here the function $f : \overline{B(0,R)} \to \mathbb{R}$ is radial; that is, there exists a function g such that $f(x) = g(r)$, where $r = |x|$, $r \in [0,R]$, $\forall\, x \in \overline{B(0,R)}$.

We need

Remark. Let the spherical shell $A := B(0,R_2) - \overline{B(0,R_1)}$, $0 < R_1 < R_2$, $A \subseteq \mathbb{R}^N$, $N \geq 2$, $x \in \overline{A}$. Consider that $f : \overline{A} \to \mathbb{R}$ is radial; that is, there exists g such that $f(x) = g(r)$, $r = |x|$, $r \in [R_1,R_2]$, $\forall\, x \in \overline{A}$. Here x can be written uniquely as $x = r\omega$, where $r = |x| > 0$ and $\omega = \frac{x}{r} \in S^{N-1}$, $|\omega| = 1$, see ([11], p. 149-150 and [2], p. 421), furthermore for $F : \overline{A} \to \mathbb{R}$ a Lebesgue integrable function we have that

$$\int_A F(x)\,dx = \int_{S^{N-1}} \left(\int_{R_1}^{R_2} F(r\omega)\, r^{N-1} dr\right) d\omega. \tag{74}$$

Here

$$Vol(A) = \frac{\omega_N \left(R_2^N - R_1^N\right)}{N} = \frac{\pi^{\frac{N}{2}} \left(R_2^N - R_1^N\right)}{\Gamma\left(\frac{N}{2}+1\right)}. \tag{75}$$

In this article we derive multivariate fractional Iyengar type inequalities on the shell and ball of $\mathbb{R}^N$, $N \geq 2$, for radial function. Our following results are based on the presented background results.

2 Main Results

In the rest of this article we consider the functions:

i) $f : \overline{A} \to \mathbb{R}$ which is radial, i.e. there exists g such that $f(x) = g(r)$, $r = |x|$, $r \in [R_1,R_2]$, $\forall\, x \in \overline{A}$; where A is the spherical shell $A := B(0,R_2) - \overline{B(0,R_1)}$, $0 < R_1 < R_2$, $A \subseteq \mathbb{R}^N$, $N \geq 2$, also

ii) $f : \overline{B(0,R)} \to \mathbb{R}$ which is radial, i.e. there exists g such that $f(x) = g(r)$, where $r = |x|$, $r \in [0,R]$, $\forall\, x \in \overline{B(0,R)}$; where $B(0,R)$ is the ball, $B(0,R) \subseteq \mathbb{R}^N$, $N \geq 2$, $R > 0$.

We will employ the related function $h(s) := g(s)\, s^{N-1}$, where $s \in [R_1,R_2]$ or $s \in [0,R]$.

We present the following multivariate Caputo fractional Iyengar type inequalities:

Theorem 10. *Let the radial $f : \overline{A} \to \mathbb{R}$. Let $\nu > 0$, $n = \lceil \nu \rceil$, and $h \in AC^n([R_1,R_2])$ (i.e. $h^{(n-1)}$ is absolutely continuous on $[R_1,R_2]$). We assume that $D_{*R_1}^{\nu} h$, $D_{R_2-}^{\nu} h \in L_\infty([R_1,R_2])$. Then*

i)

$$\left| \int_A f(y)\,dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[h^{(k)}(R_1)(t-R_1)^{k+1} + \right.\right.\right.$$

$$(-1)^k h^{(k)}(R_2)(R_2-t)^{k+1}\Big]\Big\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\Bigg| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D^{\nu}_{*R_1}h\right\|_{L_\infty([R_1,R_2])},\left\|D^{\nu}_{R_2-}h\right\|_{L_\infty([R_1,R_2])}\right\}}{\Gamma(\nu+2)}\cdot$$

$$\left[(t-R_1)^{\nu+1}+(R_2-t)^{\nu+1}\right], \tag{76}$$

$\forall\, t\in[R_1,R_2]$,

ii) at $t=\frac{R_1+R_2}{2}$, *the right hand side of (76) is minimized, and we get:*

$$\Bigg|\int_A f(y)\,dy-\Bigg\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{(R_2-R_1)^{k+1}}{2^k}$$

$$\left[h^{(k)}(R_1)+(-1)^k h^{(k)}(R_2)\right]\Bigg\}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\Bigg| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D^{\nu}_{*R_1}h\right\|_{L_\infty([R_1,R_2])},\left\|D^{\nu}_{R_2-}h\right\|_{L_\infty([R_1,R_2])}\right\}}{\Gamma(\nu+2)}\frac{(R_2-R_1)^{\nu+1}}{2^{\nu-1}}, \tag{77}$$

iii) if $h^{(k)}(R_1)=h^{(k)}(R_2)=0$, *for all* $k=0,1,\ldots,n-1$, *we obtain*

$$\left|\int_A f(y)\,dy\right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\cdot$$

$$\max\left\{\left\|D^{\nu}_{*R_1}h\right\|_{L_\infty([R_1,R_2])},\left\|D^{\nu}_{R_2-}h\right\|_{L_\infty([R_1,R_2])}\right\}\frac{(R_2-R_1)^{\nu+1}}{\Gamma(\nu+2)\,2^{\nu-1}}, \tag{78}$$

which is a sharp inequality,

iv) more generally, for $j=0,1,2,\ldots,N\in\mathbb{N}$, *it holds*

$$\Bigg|\int_A f(y)\,dy-\Bigg\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{R_2-R_1}{N}\right)^{k+1}$$

$$\left[j^{k+1}h^{(k)}(R_1)+(-1)^k(N-j)^{k+1}h^{(k)}(R_2)\right]\Bigg\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\Bigg| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D^{\nu}_{*R_1}h\right\|_{L_\infty([R_1,R_2])},\left\|D^{\nu}_{R_2-}h\right\|_{L_\infty([R_1,R_2])}\right\}}{\Gamma(\nu+2)}$$

$$\left(\frac{R_2-R_1}{N}\right)^{\nu+1}\left[j^{\nu+1}+(N-j)^{\nu+1}\right], \tag{79}$$

v) if $h^{(k)}(R_1)=h^{(k)}(R_2)=0$, $k=1,\ldots,n-1$, *from (79) we get:*

$$\left|\int_A f(y)\,dy-\left(\frac{R_2-R_1}{N}\right)\left[jh(R_1)+(N-j)h(R_2)\right]\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|$$

$$\leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D^{\nu}_{*R_1}h\right\|_{L_\infty([R_1,R_2])},\left\|D^{\nu}_{R_2-}h\right\|_{L_\infty([R_1,R_2])}\right\}}{\Gamma(\nu+2)}$$

$$\left(\frac{R_2-R_1}{N}\right)^{\nu+1}\left[j^{\nu+1}+(N-j)^{\nu+1}\right], \tag{80}$$

$j=0,1,2,...,N,$

vi) when $N=2$ and $j=1$, (80) turns to

$$\left|\int_A f(y)\,dy-(R_2-R_1)\left(h(R_1)+h(R_2)\right)\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{*R_1}^{\nu}h\right\|_{L_\infty([R_1,R_2])},\left\|D_{R_2-}^{\nu}h\right\|_{L_\infty([R_1,R_2])}\right\}}{\Gamma(\nu+2)}\frac{(R_2-R_1)^{\nu+1}}{2^{\nu-1}}, \tag{81}$$

vii) when $0<\nu\leq 1$, inequality (81) is again valid without any boundary conditions.

*Proof.*By Theorem 2 and (74). See in the 3. Appendix the general proving method in this article.

We give

Corollary 1.*(to Theorem 10) Let the radial $f:\overline{B(0,R)}\to\mathbb{R}$. Let $\nu>0$, $n=\lceil\nu\rceil$, and $h\in AC^n([0,R])$. We assume that $D_{*0}^{\nu}h$, $D_{R-}^{\nu}h\in L_\infty([0,R])$. Then*

i)

$$\left|\int_{B(0,R)} f(y)\,dy-\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[h^{(k)}(0)\,t^{k+1}+\right.\right.\right.$$

$$\left.\left.\left.(-1)^k h^{(k)}(R)\,(R-t)^{k+1}\right]\right\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{*0}^{\nu}h\right\|_{L_\infty([0,R])},\left\|D_{R-}^{\nu}h\right\|_{L_\infty([0,R])}\right\}}{\Gamma(\nu+2)}\cdot$$

$$\left[t^{\nu+1}+(R-t)^{\nu+1}\right], \tag{82}$$

$\forall\, t\in[0,R],$

ii) at $t=\frac{R}{2}$, the right hand side of (82) is minimized, and we get:

$$\left|\int_{B(0,R)} f(y)\,dy-\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{R^{k+1}}{2^k}\right.\right.$$

$$\left.\left.\left[h^{(k)}(0)+(-1)^k h^{(k)}(R)\right]\right\}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{*0}^{\nu}h\right\|_{L_\infty([0,R])},\left\|D_{R-}^{\nu}h\right\|_{L_\infty([0,R])}\right\}}{\Gamma(\nu+2)}\frac{R^{\nu+1}}{2^{\nu-1}}, \tag{83}$$

iii) if $h^{(k)}(0)=h^{(k)}(R)=0$, for all $k=0,1,...,n-1$, we obtain

$$\left|\int_{B(0,R)} f(y)\,dy\right|\leq\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\cdot$$

$$\max\left\{\left\|D_{*0}^{\nu}h\right\|_{L_\infty([0,R])},\left\|D_{R-}^{\nu}h\right\|_{L_\infty([0,R])}\right\}\frac{R^{\nu+1}}{\Gamma(\nu+2)\,2^{\nu-1}}, \tag{84}$$

which is a sharp inequality,

iv) more generally, for $j = 0,1,2,...,N \in \mathbb{N}$*, it holds*

$$\left| \int_{B(0,R)} f(y)\,dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{R}{N}\right)^{k+1} \right.\right.$$

$$\left.\left.\left[j^{k+1} h^{(k)}(0) + (-1)^k (N-j)^{k+1} h^{(k)}(R) \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\|D_{*0}^{\nu} h\right\|_{L_\infty([0,R])}, \left\|D_{R-}^{\nu} h\right\|_{L_\infty([0,R])} \right\}}{\Gamma(\nu+2)}$$

$$\left(\frac{R}{N}\right)^{\nu+1} \left[j^{\nu+1} + (N-j)^{\nu+1} \right], \tag{85}$$

v) if $h^{(k)}(0) = h^{(k)}(R) = 0$, $k = 1,...,n-1$*, from (85) we get:*

$$\left| \int_{B(0,R)} f(y)\,dy - \left(\frac{R}{N}\right)(N-j)\,h(R) \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right|$$

$$\leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\|D_{*0}^{\nu} h\right\|_{L_\infty([0,R])}, \left\|D_{R-}^{\nu} h\right\|_{L_\infty([0,R])} \right\}}{\Gamma(\nu+2)}$$

$$\left(\frac{R}{N}\right)^{\nu+1} \left[j^{\nu+1} + (N-j)^{\nu+1} \right], \tag{86}$$

$j = 0,1,2,...,N$,
vi) when $N = 2$ *and* $j = 1$*, (86) turns to*

$$\left| \int_{B(0,R)} f(y)\,dy - Rh(R) \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\|D_{*0}^{\nu} h\right\|_{L_\infty([0,R])}, \left\|D_{R-}^{\nu} h\right\|_{L_\infty([0,R])} \right\}}{\Gamma(\nu+2)} \frac{R^{\nu+1}}{2^{\nu-1}}, \tag{87}$$

vii) when $0 < \nu \leq 1$*, inequality (87) is again valid without any boundary conditions.*

*Proof.*Based on Theorem 10, just set there $R_1 = 0$, $R_2 = R$, the assumptions now are on $B(0,R)$, and use (73).

We continue with

Theorem 11.*Let the radial* $f : \overline{A} \to \mathbb{R}$*. Let* $\nu \geq 1$, $n = \lceil \nu \rceil$*, and* $h \in AC^n([R_1,R_2])$ *(i.e.* $h^{(n-1)}$ *is absolutely continuous on* $[R_1,R_2]$*). We assume that* $D_{*R_1}^{\nu} h$, $D_{R_2-}^{\nu} h \in L_1([R_1,R_2])$*. Then*
i)

$$\left| \int_A f(y)\,dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[h^{(k)}(R_1)(t-R_1)^{k+1} + \right.\right.\right.$$

$$\left.\left.\left.(-1)^k h^{(k)}(R_2)(R_2-t)^{k+1} \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\|D_{*R_1}^{\nu} h\right\|_{L_1([R_1,R_2])}, \left\|D_{R_2-}^{\nu} h\right\|_{L_1([R_1,R_2])} \right\}}{\Gamma(\nu+1)} \cdot$$

$$\left[(t-R_1)^{\nu} + (R_2-t)^{\nu} \right], \tag{88}$$

$\forall\, t \in [R_1, R_2]$,

ii) when $\nu = 1$, *from (88), we have*

$$\left| \int_A f(y)\,dy - [h(R_1)(t - R_1) + h(R_2)(R_2 - t)] \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\| h' \right\|_{L_1([R_1,R_2])} (R_2 - R_1), \tag{89}$$

$\forall\, t \in [R_1, R_2]$,

iii) from (89), we obtain ($\nu = 1$ *case)*

$$\left| \int_A f(y)\,dy - (R_2 - R_1)(h(R_1) + h(R_2)) \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\| h' \right\|_{L_1([R_1,R_2])} (R_2 - R_1), \tag{90}$$

iv) at $t = \frac{R_1 + R_2}{2}$, $\nu > 1$, *the right hand side of (88) is minimized, and we get:*

$$\left| \int_A f(y)\,dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(R_2 - R_1)^{k+1}}{2^k} \right.\right.$$

$$\left.\left. \left[h^{(k)}(R_1) + (-1)^k h^{(k)}(R_2) \right] \right\} \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| D_{*R_1}^{\nu} h \right\|_{L_1([R_1,R_2])}, \left\| D_{R_2-}^{\nu} h \right\|_{L_1([R_1,R_2])} \right\}}{\Gamma(\nu + 1)} \frac{(R_2 - R_1)^{\nu}}{2^{\nu - 2}}, \tag{91}$$

v) if $h^{(k)}(R_1) = h^{(k)}(R_2) = 0$, *for all* $k = 0, 1, \ldots, n-1$, $\nu > 1$, *from (91) we obtain*

$$\left| \int_A f(y)\,dy \right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \cdot$$

$$\max\left\{ \left\| D_{*R_1}^{\nu} h \right\|_{L_1([R_1,R_2])}, \left\| D_{R_2-}^{\nu} h \right\|_{L_1([R_1,R_2])} \right\} \frac{(R_2 - R_1)^{\nu}}{\Gamma(\nu + 1)\, 2^{\nu - 2}}, \tag{92}$$

which is a sharp inequality,

vi) more generally, for $j = 0, 1, 2, \ldots, N \in \mathbb{N}$, *it holds*

$$\left| \int_A f(y)\,dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{R_2 - R_1}{N} \right)^{k+1} \right.\right.$$

$$\left.\left. \left[j^{k+1} h^{(k)}(R_1) + (-1)^k (N - j)^{k+1} h^{(k)}(R_2) \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| D_{*R_1}^{\nu} h \right\|_{L_1([R_1,R_2])}, \left\| D_{R_2-}^{\nu} h \right\|_{L_1([R_1,R_2])} \right\}}{\Gamma(\nu + 1)}$$

$$\left(\frac{R_2 - R_1}{N} \right)^{\nu} \left[j^{\nu} + (N - j)^{\nu} \right], \tag{93}$$

vii) if $h^{(k)}(R_1) = h^{(k)}(R_2) = 0$, $k = 1,...,n-1$, from (93) we get:

$$\left|\int_A f(y)\,dy - \left(\frac{R_2-R_1}{N}\right)[jh(R_1)+(N-j)h(R_2)]\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|$$

$$\leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D^{\nu}_{*R_1}h\right\|_{L_1([R_1,R_2])},\left\|D^{\nu}_{R_2-}h\right\|_{L_1([R_1,R_2])}\right\}}{\Gamma(\nu+1)}$$

$$\left(\frac{R_2-R_1}{N}\right)^{\nu}\left[j^{\nu}+(N-j)^{\nu}\right], \tag{94}$$

$j = 0,1,2,...,N$,

viii) when $N = 2$ and $j = 1$, (94) turns to

$$\left|\int_A f(y)\,dy - (R_2-R_1)(h(R_1)+h(R_2))\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D^{\nu}_{*R_1}h\right\|_{L_1([R_1,R_2])},\left\|D^{\nu}_{R_2-}h\right\|_{L_1([R_1,R_2])}\right\}}{\Gamma(\nu+1)}\frac{(R_2-R_1)^{\nu}}{2^{\nu-2}}. \tag{95}$$

Proof. By Theorem 3 and (74). See in the 3. Appendix the general proving method in this article.

We give

Corollary 2. *(to Theorem 11) Let the radial $f : \overline{B(0,R)} \to \mathbb{R}$. Let $\nu \geq 1$, $n = \lceil \nu \rceil$, and $h \in AC^n([0,R])$. We assume that $D^{\nu}_{*0}h$, $D^{\nu}_{R-}h \in L_1([0,R])$. Then*

i)

$$\left|\int_{B(0,R)} f(y)\,dy - \left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[h^{(k)}(0)\,t^{k+1}+\right.\right.\right.$$

$$\left.\left.\left.(-1)^k h^{(k)}(R)(R-t)^{k+1}\right]\right\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D^{\nu}_{*0}h\right\|_{L_1([0,R])},\left\|D^{\nu}_{R-}h\right\|_{L_1([0,R])}\right\}}{\Gamma(\nu+1)}\cdot$$

$$\left[t^{\nu}+(R-t)^{\nu}\right], \tag{96}$$

$\forall\, t \in [0,R]$,

ii) when $\nu = 1$, from (96), we have

$$\left|\int_{B(0,R)} f(y)\,dy - h(R)(R-t)\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left\|h'\right\|_{L_1([0,R])}R, \tag{97}$$

$\forall\, t \in [0,R]$,

iii) from (97), we obtain ($\nu = 1$ case)

$$\left|\int_{B(0,R)} f(y)\,dy - Rh(R)\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left\|h'\right\|_{L_1([0,R])}R, \tag{98}$$

iv) at $t=\frac{R}{2}$, $\nu>1$, *the right hand side of (96) is minimized, and we get:*

$$\left|\int_{B(0,R)}f(y)\,dy-\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{R^{k+1}}{2^k}\right.\right.$$

$$\left.\left.\left[h^{(k)}(0)+(-1)^k h^{(k)}(R)\right]\right\}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{*0}^{\nu}h\right\|_{L_1([0,R])},\left\|D_{R-}^{\nu}h\right\|_{L_1([0,R])}\right\}}{\Gamma(\nu+1)}\frac{R^{\nu}}{2^{\nu-2}}, \tag{99}$$

v) if $h^{(k)}(0)=h^{(k)}(R)=0$, *for all* $k=0,1,\ldots,n-1$, *from (99) we obtain*

$$\left|\int_{B(0,R)}f(y)\,dy\right|\leq\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\cdot$$

$$\max\left\{\left\|D_{*0}^{\nu}h\right\|_{L_1([0,R])},\left\|D_{R-}^{\nu}h\right\|_{L_1([0,R])}\right\}\frac{R^{\nu}}{\Gamma(\nu+1)\,2^{\nu-2}}, \tag{100}$$

which is a sharp inequality.

vi) more generally, for $j=0,1,2,\ldots,N\in\mathbb{N}$, *it holds*

$$\left|\int_{B(0,R)}f(y)\,dy-\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{R}{N}\right)^{k+1}\right.\right.$$

$$\left.\left.\left[j^{k+1}h^{(k)}(0)+(-1)^k(N-j)^{k+1}h^{(k)}(R)\right]\right\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{*0}^{\nu}h\right\|_{L_1([0,R])},\left\|D_{R-}^{\nu}h\right\|_{L_1([0,R])}\right\}}{\Gamma(\nu+1)}$$

$$\left(\frac{R}{N}\right)^{\nu}\left[j^{\nu}+(N-j)^{\nu}\right], \tag{101}$$

vii) if $h^{(k)}(0)=h^{(k)}(R)=0$, $k=1,\ldots,n-1$, *from (101) we get:*

$$\left|\int_{B(0,R)}f(y)\,dy-\left(\frac{R}{N}\right)(N-j)\,h(R)\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|$$

$$\leq\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{*0}^{\nu}h\right\|_{L_1([0,R])},\left\|D_{R-}^{\nu}h\right\|_{L_1([0,R])}\right\}}{\Gamma(\nu+1)}$$

$$\left(\frac{R}{N}\right)^{\nu}\left[j^{\nu}+(N-j)^{\nu}\right], \tag{102}$$

$j=0,1,2,\ldots,N$,

viii) when $N=2$ *and* $j=1$, *(102) turns to*

$$\left|\int_{B(0,R)}f(y)\,dy-Rh(R)\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{*0}^{\nu}h\right\|_{L_1([0,R])},\left\|D_{R-}^{\nu}h\right\|_{L_1([0,R])}\right\}}{\Gamma(\nu+1)}\frac{R^{\nu}}{2^{\nu-2}}. \tag{103}$$

Proof. Based on Theorem 11, just set there $R_1 = 0$, $R_2 = R$, the assumptions now are on $B(0,R)$, and use (73).

We continue with

Theorem 12. *Let the radial $f : \overline{A} \to \mathbb{R}$, and $p,q > 1 : \frac{1}{p}+\frac{1}{q} = 1$, $\nu > \frac{1}{q}$. Let $n = \lceil \nu \rceil$, and $h \in AC^n([R_1,R_2])$ (i.e. $h^{(n-1)}$ is absolutely continuous on $[R_1,R_2]$). We assume that $D^{\nu}_{*R_1}h$, $D^{\nu}_{R_2-}h \in L_q([R_1,R_2])$. Then*

i)

$$\left| \int_A f(y)\,dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[h^{(k)}(R_1)(t-R_1)^{k+1} + \right.\right.\right.$$

$$\left.\left.\left. (-1)^k h^{(k)}(R_2)(R_2-t)^{k+1} \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| D^{\nu}_{*R_1}h \right\|_{L_q([R_1,R_2])}, \left\| D^{\nu}_{R_2-}h \right\|_{L_q([R_1,R_2])} \right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \cdot$$

$$\left[(t-R_1)^{\nu+\frac{1}{p}} + (R_2-t)^{\nu+\frac{1}{p}} \right], \tag{104}$$

$\forall\, t \in [R_1,R_2]$,

ii) at $t = \frac{R_1+R_2}{2}$, the right hand side of (104) is minimized, and we get:

$$\left| \int_A f(y)\,dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(R_2-R_1)^{k+1}}{2^k} \right.\right.$$

$$\left.\left. \left[h^{(k)}(R_1) + (-1)^k h^{(k)}(R_2) \right] \right\} \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| D^{\nu}_{*R_1}h \right\|_{L_q([R_1,R_2])}, \left\| D^{\nu}_{R_2-}h \right\|_{L_q([R_1,R_2])} \right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \frac{(R_2-R_1)^{\nu+\frac{1}{p}}}{2^{\nu-1-\frac{1}{q}}}, \tag{105}$$

iii) if $h^{(k)}(R_1) = h^{(k)}(R_2) = 0$, for all $k = 0,1,\ldots,n-1$, we obtain

$$\left| \int_A f(y)\,dy \right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right) 2^{\nu-1-\frac{1}{q}}} \cdot$$

$$\max\left\{ \left\| D^{\nu}_{*R_1}h \right\|_{L_q([R_1,R_2])}, \left\| D^{\nu}_{R_2-}h \right\|_{L_q([R_1,R_2])} \right\} \cdot$$

$$\frac{(R_2-R_1)^{\nu+\frac{1}{p}}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}, \tag{106}$$

which is a sharp inequality,

iv) more generally, for $j = 0,1,2,\ldots,N \in \mathbb{N}$, it holds

$$\left| \int_A f(y)\,dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{R_2-R_1}{N} \right)^{k+1} \right.\right.$$

$$\left.\left. \left[j^{k+1} h^{(k)}(R_1) + (-1)^k (N-j)^{k+1} h^{(k)}(R_2) \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{*R_1}^{\nu}h\right\|_{L_q([R_1,R_2])},\left\|D_{R_2-}^{\nu}h\right\|_{L_q([R_1,R_2])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}$$

$$\left(\frac{R_2-R_1}{N}\right)^{\nu+\frac{1}{p}}\left[j^{\nu+\frac{1}{p}}+(N-j)^{\nu+\frac{1}{p}}\right], \tag{107}$$

v) if $h^{(k)}(R_1)=h^{(k)}(R_2)=0$, $k=1,...,n-1$, from (107) we get:

$$\left|\int_A f(y)\,dy-\left(\frac{R_2-R_1}{N}\right)[jh(R_1)+(N-j)h(R_2)]\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|$$

$$\leq\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{*R_1}^{\nu}h\right\|_{L_q([R_1,R_2])},\left\|D_{R_2-}^{\nu}h\right\|_{L_q([R_1,R_2])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}$$

$$\left(\frac{R_2-R_1}{N}\right)^{\nu+\frac{1}{p}}\left[j^{\nu+\frac{1}{p}}+(N-j)^{\nu+\frac{1}{p}}\right], \tag{108}$$

$j=0,1,2,...,N$,

vi) when $N=2$ and $j=1$, (108) turns to

$$\left|\int_A f(y)\,dy-(R_2-R_1)(h(R_1)+h(R_2))\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{*R_1}^{\nu}h\right\|_{L_q([R_1,R_2])},\left\|D_{R_2-}^{\nu}h\right\|_{L_q([R_1,R_2])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\frac{(R_2-R_1)^{\nu+\frac{1}{p}}}{2^{\nu-1-\frac{1}{q}}}, \tag{109}$$

vii) when $1/q<\nu\leq 1$, inequality (109) is again valid without any boundary conditions.

*Proof.*By Theorem 4 and (74). See in the 3. Appendix the general proving method in this article.

We give

Corollary 3.*(to Theorem 12) Let the radial $f:\overline{B(0,R)}\to\mathbb{R}$. Let $\nu>0$, $n=\lceil\nu\rceil$, and $h\in AC^n([0,R])$; $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\nu>\frac{1}{q}$. We assume that $D_{*0}^{\nu}h$, $D_{R-}^{\nu}h\in L_q([0,R])$. Then*

i)

$$\left|\int_{B(0,R)} f(y)\,dy-\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[h^{(k)}(0)\,t^{k+1}+\right.\right.\right.$$

$$\left.\left.\left.(-1)^k h^{(k)}(R)(R-t)^{k+1}\right]\right\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{*0}^{\nu}h\right\|_{L_q([0,R])},\left\|D_{R-}^{\nu}h\right\|_{L_q([0,R])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\cdot \tag{110}$$

$$\left[t^{\nu+\frac{1}{p}}+(R-t)^{\nu+\frac{1}{p}}\right],$$

$\forall\, t\in[0,R]$,

ii) at $t=\frac{R}{2}$, the right hand side of (110) is minimized, and we get:

$$\left|\int_{B(0,R)} f(y)\,dy-\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{R^{k+1}}{2^k}\right.\right.$$

$$\left[h^{(k)}(0)+(-1)^k h^{(k)}(R)\right]\Bigg\} \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\Bigg| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{\left\|D_{*0}^{\nu}h\right\|_{L_q([0,R])}, \left\|D_{R-}^{\nu}h\right\|_{L_q([0,R])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \frac{R^{\nu+\frac{1}{p}}}{2^{\nu-1-\frac{1}{q}}}, \tag{111}$$

iii) if $h^{(k)}(0)=h^{(k)}(R)=0$*, for all* $k=0,1,\ldots,n-1$*, we obtain*

$$\left|\int_{B(0,R)} f(y)\,dy\right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)2^{\nu-1-\frac{1}{q}}}\cdot$$

$$\max\left\{\left\|D_{*0}^{\nu}h\right\|_{L_q([0,R])}, \left\|D_{R-}^{\nu}h\right\|_{L_q([0,R])}\right\} \frac{R^{\nu+\frac{1}{p}}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}, \tag{112}$$

which is a sharp inequality.

iv) more generally, for $j=0,1,2,\ldots,N\in\mathbb{N}$*, it holds*

$$\Bigg|\int_{B(0,R)} f(y)\,dy - \Bigg\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{R}{N}\right)^{k+1}$$

$$\left[j^{k+1}h^{(k)}(0)+(-1)^k(N-j)^{k+1}h^{(k)}(R)\right]\Bigg\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\Bigg| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{\left\|D_{*0}^{\nu}h\right\|_{L_q([0,R])}, \left\|D_{R-}^{\nu}h\right\|_{L_q([0,R])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}$$

$$\left(\frac{R}{N}\right)^{\nu+\frac{1}{p}}\left[j^{\nu+\frac{1}{p}}+(N-j)^{\nu+\frac{1}{p}}\right], \tag{113}$$

v) if $h^{(k)}(0)=h^{(k)}(R)=0$*,* $k=1,\ldots,n-1$*, from (113) we get:*

$$\left|\int_{B(0,R)} f(y)\,dy - \left(\frac{R}{N}\right)(N-j)\,h(R)\,\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|$$

$$\leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{\left\|D_{*0}^{\nu}h\right\|_{L_q([0,R])}, \left\|D_{R-}^{\nu}h\right\|_{L_q([0,R])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}$$

$$\left(\frac{R}{N}\right)^{\nu+\frac{1}{p}}\left[j^{\nu+\frac{1}{p}}+(N-j)^{\nu+\frac{1}{p}}\right], \tag{114}$$

$j=0,1,2,\ldots,N$,

vi) when $N=2$ *and* $j=1$*, (114) turns to*

$$\left|\int_{B(0,R)} f(y)\,dy - Rh(R)\,\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{\left\|D_{*0}^{\nu}h\right\|_{L_q([0,R])}, \left\|D_{R-}^{\nu}h\right\|_{L_q([0,R])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \frac{R^{\nu+\frac{1}{p}}}{2^{\nu-1-\frac{1}{q}}}, \tag{115}$$

vii) when $1/q<\nu\leq 1$*, inequality (115) is again valid without any boundary conditions.*

Proof. Based on Theorem 12, just set there $R_1 = 0$, $R_2 = R$, the assumptions now are on $B(0,R)$, and use (73).

We continue with multivariate Canavati type fractional Iyengar type inequalities:

Theorem 13. *Let the radial $f : \overline{A} \to \mathbb{R}$. Let $\nu \geq 1$, $n = [\nu]$, and $h \in C^{\nu}_{R_1+}([R_1,R_2]) \cap C^{\nu}_{R_2-}([R_1,R_2])$. Then*

i)

$$\left| \int_A f(y)\,dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[h^{(k)}(R_1)(t-R_1)^{k+1} + (-1)^k h^{(k)}(R_2)(R_2-t)^{k+1} \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| D^{\nu}_{R_1+}h \right\|_{\infty,[R_1,R_2]}, \left\| \overline{D}^{\nu}_{R_2-}h \right\|_{\infty,[R_1,R_2]} \right\}}{\Gamma(\nu+2)} \cdot \left[(t-R_1)^{\nu+1} + (R_2-t)^{\nu+1} \right], \tag{116}$$

$\forall\, t \in [R_1,R_2]$,

ii) at $t = \frac{R_1+R_2}{2}$, the right hand side of (116) is minimized, and we get:

$$\left| \int_A f(y)\,dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(R_2-R_1)^{k+1}}{2^k} \left[h^{(k)}(R_1) + (-1)^k h^{(k)}(R_2) \right] \right\} \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| D^{\nu}_{R_1+}h \right\|_{\infty,[R_1,R_2]}, \left\| \overline{D}^{\nu}_{R_2-}h \right\|_{\infty,[R_1,R_2]} \right\}}{\Gamma(\nu+2)} \frac{(R_2-R_1)^{\nu+1}}{2^{\nu-1}}, \tag{117}$$

iii) if $h^{(k)}(R_1) = h^{(k)}(R_2) = 0$, for all $k = 0,1,\ldots,n-1$, we obtain

$$\left| \int_A f(y)\,dy \right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \cdot \max\left\{ \left\| D^{\nu}_{R_1+}h \right\|_{\infty,[R_1,R_2]}, \left\| \overline{D}^{\nu}_{R_2-}h \right\|_{\infty,[R_1,R_2]} \right\} \frac{(R_2-R_1)^{\nu+1}}{\Gamma(\nu+2)\,2^{\nu-1}}, \tag{118}$$

which is a sharp inequality,

iv) more generally, for $j = 0,1,2,\ldots,N \in \mathbb{N}$, it holds

$$\left| \int_A f(y)\,dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{R_2-R_1}{N} \right)^{k+1} \left[j^{k+1} h^{(k)}(R_1) + (-1)^k (N-j)^{k+1} h^{(k)}(R_2) \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| D^{\nu}_{R_1+}h \right\|_{\infty,[R_1,R_2]}, \left\| \overline{D}^{\nu}_{R_2-}h \right\|_{\infty,[R_1,R_2]} \right\}}{\Gamma(\nu+2)} \left(\frac{R_2-R_1}{N} \right)^{\nu+1} \left[j^{\nu+1} + (N-j)^{\nu+1} \right], \tag{119}$$

v) if $h^{(k)}(R_1) = h^{(k)}(R_2) = 0$*,* $k = 1, ..., n-1$*, from (119) we get:*

$$\left| \int_A f(y)\,dy - \left(\frac{R_2 - R_1}{N}\right)[jh(R_1) + (N-j)h(R_2)] \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right|$$

$$\leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| D^{\nu}_{R_1+} h \right\|_{\infty,[R_1,R_2]}, \left\| \overline{D}^{\nu}_{R_2-} h \right\|_{\infty,[R_1,R_2]} \right\}}{\Gamma(\nu+2)}$$

$$\left(\frac{R_2 - R_1}{N}\right)^{\nu+1} \left[j^{\nu+1} + (N-j)^{\nu+1} \right], \tag{120}$$

$j = 0, 1, 2, ..., N,$

vi) when $N = 2$ *and* $j = 1$*, (120) turns to*

$$\left| \int_A f(y)\,dy - (R_2 - R_1)(h(R_1) + h(R_2)) \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| D^{\nu}_{R_1+} h \right\|_{\infty,[R_1,R_2]}, \left\| \overline{D}^{\nu}_{R_2-} h \right\|_{\infty,[R_1,R_2]} \right\}}{\Gamma(\nu+2)} \frac{(R_2 - R_1)^{\nu+1}}{2^{\nu-1}}. \tag{121}$$

Proof. By Theorem 5 and (74). See in the 3. Appendix the general proving method in this article.

We give

Corollary 4. *(to Theorem 13) Let the radial* $f : \overline{B(0,R)} \to \mathbb{R}$*. Let* $\nu \geq 1$*,* $n = [\nu]$*, and* $h \in C^{\nu}_{0+}([0,R]) \cap C^{\nu}_{R-}([0,R])$*. Then*

i)

$$\left| \int_{B(0,R)} f(y)\,dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[h^{(k)}(0)\, t^{k+1} + \right. \right. \right.$$

$$\left. \left. \left. (-1)^k h^{(k)}(R)(R-t)^{k+1} \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| D^{\nu}_{0+} h \right\|_{\infty,[0,R]}, \left\| \overline{D}^{\nu}_{R-} h \right\|_{\infty,[0,R]} \right\}}{\Gamma(\nu+2)} \cdot$$

$$\left[t^{\nu+1} + (R-t)^{\nu+1} \right], \tag{122}$$

$\forall\, t \in [0,R],$

ii) at $t = \frac{R}{2}$*, the right hand side of (122) is minimized, and we get:*

$$\left| \int_{B(0,R)} f(y)\,dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{R^{k+1}}{2^k} \right. \right.$$

$$\left. \left. \left[h^{(k)}(0) + (-1)^k h^{(k)}(R) \right] \right\} \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| D^{\nu}_{0+} h \right\|_{\infty,[0,R]}, \left\| \overline{D}^{\nu}_{R-} h \right\|_{\infty,[0,R]} \right\}}{\Gamma(\nu+2)} \frac{R^{\nu+1}}{2^{\nu-1}}, \tag{123}$$

iii) if $h^{(k)}(0) = h^{(k)}(R) = 0$*, for all* $k = 0, 1, ..., n-1$*, we obtain*

$$\left| \int_{B(0,R)} f(y)\,dy \right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \cdot$$

$$\max\left\{\left\|D_{0+}^{\nu}h\right\|_{\infty,[0,R]},\left\|\overline{D}_{R-}^{\nu}h\right\|_{\infty,[0,R]}\right\}\frac{R^{\nu+1}}{\Gamma(\nu+2)2^{\nu-1}}, \tag{124}$$

which is a sharp inequality.

iv) more generally, for $j=0,1,2,...,N\in\mathbb{N}$*, it holds*

$$\left|\int_{B(0,R)}f(y)\,dy-\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{R}{N}\right)^{k+1}\right.\right.$$

$$\left.\left.\left[j^{k+1}h^{(k)}(0)+(-1)^{k}(N-j)^{k+1}h^{(k)}(R)\right]\right\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{0+}^{\nu}h\right\|_{\infty,[0,R]},\left\|\overline{D}_{R-}^{\nu}h\right\|_{\infty,[0,R]}\right\}}{\Gamma(\nu+2)}$$

$$\left(\frac{R}{N}\right)^{\nu+1}\left[j^{\nu+1}+(N-j)^{\nu+1}\right], \tag{125}$$

v) if $h^{(k)}(0)=h^{(k)}(R)=0$*,* $k=1,...,n-1$*, from (125) we get:*

$$\left|\int_{B(0,R)}f(y)\,dy-\left(\frac{R}{N}\right)(N-j)\,h(R)\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|$$

$$\leq\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{0+}^{\nu}h\right\|_{\infty,[0,R]},\left\|\overline{D}_{R-}^{\nu}h\right\|_{\infty,[0,R]}\right\}}{\Gamma(\nu+2)} \tag{126}$$

$$\left(\frac{R}{N}\right)^{\nu+1}\left[j^{\nu+1}+(N-j)^{\nu+1}\right],$$

$j=0,1,2,...,N,$

vi) when $N=2$ *and* $j=1$*, (126) turns to*

$$\left|\int_{B(0,R)}f(y)\,dy-Rh(R)\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{0+}^{\nu}h\right\|_{\infty,[0,R]},\left\|\overline{D}_{R-}^{\nu}h\right\|_{\infty,[0,R]}\right\}}{\Gamma(\nu+2)}\frac{R^{\nu+1}}{2^{\nu-1}}. \tag{127}$$

Proof. Based on Theorem 13, just set there $R_1=0$, $R_2=R$, the assumptions now are on $B(0,R)$, and use (73).

We continue with

Theorem 14. *Let the radial* $f:\overline{A}\to\mathbb{R}$*. Let* $\nu\geq1$*,* $n=[\nu]$*, and* $h\in C_{R_1+}^{\nu}([R_1,R_2])\cap C_{R_2-}^{\nu}([R_1,R_2])$*. Then*

i)

$$\left|\int_{A}f(y)\,dy-\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[h^{(k)}(R_1)(t-R_1)^{k+1}+\right.\right.\right.$$

$$\left.\left.\left.(-1)^{k}h^{(k)}(R_2)(R_2-t)^{k+1}\right]\right\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{R_1+}^{\nu}h\right\|_{L_1([R_1,R_2])},\left\|\overline{D}_{R_2-}^{\nu}h\right\|_{L_1([R_1,R_2])}\right\}}{\Gamma(\nu+1)}. \tag{128}$$

$$\left[(t-R_1)^{\nu}+(R_2-t)^{\nu}\right],$$

$\forall\, t\in[R_1,R_2]$,

ii) when $\nu=1$, *from (128), we have*

$$\left|\int_A f(y)\,dy-\left[h(R_1)(t-R_1)+h(R_2)(R_2-t)\right]\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left\|h'\right\|_{L_1([R_1,R_2])}(R_2-R_1), \tag{129}$$

$\forall\, t\in[R_1,R_2]$,

iii) from (129), we obtain ($\nu=1$ *case)*

$$\left|\int_A f(y)\,dy-(R_2-R_1)(h(R_1)+h(R_2))\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left\|h'\right\|_{L_1([R_1,R_2])}(R_2-R_1), \tag{130}$$

iv) at $t=\frac{R_1+R_2}{2}$, $\nu>1$, *the right hand side of (128) is minimized, and we get:*

$$\left|\int_A f(y)\,dy-\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{(R_2-R_1)^{k+1}}{2^k}\right.\right.$$

$$\left.\left.\left[h^{(k)}(R_1)+(-1)^k h^{(k)}(R_2)\right]\right\}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D^{\nu}_{R_1+}h\right\|_{L_1([R_1,R_2])},\left\|\overline{D}^{\nu}_{R_2-}h\right\|_{L_1([R_1,R_2])}\right\}}{\Gamma(\nu+1)}\frac{(R_2-R_1)^{\nu}}{2^{\nu-2}}, \tag{131}$$

v) if $h^{(k)}(R_1)=h^{(k)}(R_2)=0$, *for all* $k=0,1,...,n-1$, $\nu>1$, *from (131) we obtain*

$$\left|\int_A f(y)\,dy\right|\leq\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\cdot$$

$$\max\left\{\left\|D^{\nu}_{R_1+}h\right\|_{L_1([R_1,R_2])},\left\|\overline{D}^{\nu}_{R_2-}h\right\|_{L_1([R_1,R_2])}\right\}\frac{(R_2-R_1)^{\nu}}{\Gamma(\nu+1)\,2^{\nu-2}}, \tag{132}$$

which is a sharp inequality,

vi) more generally, for $j=0,1,2,...,N\in\mathbb{N}$, *it holds*

$$\left|\int_A f(y)\,dy-\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{R_2-R_1}{N}\right)^{k+1}\right.\right.$$

$$\left.\left.\left[j^{k+1}h^{(k)}(R_1)+(-1)^k(N-j)^{k+1}h^{(k)}(R_2)\right]\right\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D^{\nu}_{R_1+}h\right\|_{L_1([R_1,R_2])},\left\|\overline{D}^{\nu}_{R_2-}h\right\|_{L_1([R_1,R_2])}\right\}}{\Gamma(\nu+1)}$$

$$\left(\frac{R_2-R_1}{N}\right)^{\nu}\left[j^{\nu}+(N-j)^{\nu}\right], \tag{133}$$

vii) if $h^{(k)}(R_1) = h^{(k)}(R_2) = 0$, $k = 1, ..., n-1$, *from (133) we get:*

$$\left| \int_A f(y)\,dy - \left(\frac{R_2 - R_1}{N}\right) [jh(R_1) + (N-j)h(R_2)] \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right|$$

$$\leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| D_{R_1+}^{\nu} h \right\|_{L_1([R_1,R_2])}, \left\| \overline{D}_{R_2-}^{\nu} h \right\|_{L_1([R_1,R_2])} \right\}}{\Gamma(\nu+1)}$$

$$\left(\frac{R_2 - R_1}{N}\right)^{\nu} \left[j^{\nu} + (N-j)^{\nu} \right], \tag{134}$$

$j = 0, 1, 2, ..., N$,
viii) when $N = 2$ *and* $j = 1$, *(134) turns to*

$$\left| \int_A f(y)\,dy - (R_2 - R_1)(h(R_1) + h(R_2)) \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| D_{R_1+}^{\nu} h \right\|_{L_1([R_1,R_2])}, \left\| \overline{D}_{R_2-}^{\nu} h \right\|_{L_1([R_1,R_2])} \right\}}{\Gamma(\nu+1)} \frac{(R_2 - R_1)^{\nu}}{2^{\nu-2}}. \tag{135}$$

Proof. By Theorem 6 and (74). See in the 3. Appendix the general proving method in this article.

We give

Corollary 5. *(to Theorem 14) Let the radial* $f : \overline{B(0,R)} \to \mathbb{R}$. *Let* $\nu \geq 1$, $n = [\nu]$, *and* $h \in C_{0+}^{\nu}([0,R]) \cap C_{R-}^{\nu}([0,R])$. *Then*
i)

$$\left| \int_{B(0,R)} f(y)\,dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[h^{(k)}(0)\, t^{k+1} + \right.\right.\right.$$

$$\left.\left.\left. (-1)^k h^{(k)}(R)(R-t)^{k+1} \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| D_{0+}^{\nu} h \right\|_{L_1([0,R])}, \left\| \overline{D}_{R-}^{\nu} h \right\|_{L_1([0,R])} \right\}}{\Gamma(\nu+1)}.$$

$$\left[t^{\nu} + (R-t)^{\nu} \right], \tag{136}$$

$\forall\, t \in [0,R]$,
ii) when $\nu = 1$, *from (136), we have*

$$\left| \int_{B(0,R)} f(y)\,dy - h(R)(R-t) \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\| h' \right\|_{L_1([0,R])} R, \tag{137}$$

$\forall\, t \in [0,R]$,
iii) from (137), we obtain ($\nu = 1$ *case)*

$$\left| \int_{B(0,R)} f(y)\,dy - Rh(R) \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left\|h'\right\|_{L_1([0,R])}R, \tag{138}$$

iv) at $t=\frac{R}{2}$, $\nu>1$, *the right hand side of (136) is minimized, and we get:*

$$\left|\int_{B(0,R)} f(y)\,dy-\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{R^{k+1}}{2^k}\right.\right.$$

$$\left.\left.\left[h^{(k)}(0)+(-1)^k h^{(k)}(R)\right]\right\}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{0+}^{\nu}h\right\|_{L_1([0,R])},\left\|\overline{D}_{R-}^{\nu}h\right\|_{L_1([0,R])}\right\}}{\Gamma(\nu+1)}\frac{R^{\nu}}{2^{\nu-2}}, \tag{139}$$

v) if $h^{(k)}(0)=h^{(k)}(R)=0$, *for all* $k=0,1,...,n-1$, $\nu>1$, *from (139) we obtain*

$$\left|\int_{B(0,R)} f(y)\,dy\right|\leq\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\cdot$$

$$\max\left\{\left\|D_{0+}^{\nu}h\right\|_{L_1([0,R])},\left\|\overline{D}_{R-}^{\nu}\right\|_{L_1([0,R])}\right\}\frac{R^{\nu}}{\Gamma(\nu+1)\,2^{\nu-2}}, \tag{140}$$

which is a sharp inequality.

vi) more generally, for $j=0,1,2,...,N\in\mathbb{N}$, *it holds*

$$\left|\int_{B(0,R)} f(y)\,dy-\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{R}{N}\right)^{k+1}\right.\right.$$

$$\left.\left.\left[j^{k+1}h^{(k)}(0)+(-1)^k(N-j)^{k+1}h^{(k)}(R)\right]\right\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{0+}^{\nu}h\right\|_{L_1([0,R])},\left\|\overline{D}_{R-}^{\nu}h\right\|_{L_1([0,R])}\right\}}{\Gamma(\nu+1)}$$

$$\left(\frac{R}{N}\right)^{\nu}\left[j^{\nu}+(N-j)^{\nu}\right], \tag{141}$$

vii) if $h^{(k)}(0)=h^{(k)}(R)=0$, $k=1,...,n-1$, *from (141) we get:*

$$\left|\int_{B(0,R)} f(y)\,dy-\left(\frac{R}{N}\right)(N-j)\,h(R)\,\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|$$

$$\leq\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{0+}^{\nu}h\right\|_{L_1([0,R])},\left\|\overline{D}_{R-}^{\nu}h\right\|_{L_1([0,R])}\right\}}{\Gamma(\nu+1)}$$

$$\left(\frac{R}{N}\right)^{\nu}\left[j^{\nu}+(N-j)^{\nu}\right], \tag{142}$$

$j=0,1,2,...,N,$

viii) when $N=2$ *and* $j=1$, *(142) turns to*

$$\left|\int_{B(0,R)} f(y)\,dy-Rh(R)\,\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{0+}^{\nu}h\right\|_{L_1([0,R])},\left\|\overline{D}_{R-}^{\nu}h\right\|_{L_1([0,R])}\right\}}{\Gamma(\nu+1)}\frac{R^{\nu}}{2^{\nu-2}}. \tag{143}$$

Proof. Based on Theorem 14, just set there $R_1=0$, $R_2=R$, the assumptions now are on $B(0,R)$, and use (73).

We continue with

Theorem 15. *Let the radial $f:\overline{A}\to\mathbb{R}$. Let $\nu\geq 1$, $n=[\nu]$, and $h\in C_{R_1+}^{\nu}([R_1,R_2])\cap C_{R_2-}^{\nu}([R_1,R_2])$. Here $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Then*

i)

$$\left|\int_A f(y)\,dy-\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[h^{(k)}(R_1)(t-R_1)^{k+1}+\right.\right.\right.$$

$$\left.\left.\left.(-1)^k h^{(k)}(R_2)(R_2-t)^{k+1}\right]\right\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{R_1+}^{\nu}h\right\|_{L_q([R_1,R_2])},\left\|\overline{D}_{R_2-}^{\nu}h\right\|_{L_q([R_1,R_2])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\cdot$$

$$\left[(t-R_1)^{\nu+\frac{1}{p}}+(R_2-t)^{\nu+\frac{1}{p}}\right], \tag{144}$$

$\forall\, t\in[R_1,R_2]$,

ii) at $t=\frac{R_1+R_2}{2}$, the right hand side of (144) is minimized, and we get:

$$\left|\int_A f(y)\,dy-\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{(R_2-R_1)^{k+1}}{2^k}\right.\right.$$

$$\left.\left.\left[h^{(k)}(R_1)+(-1)^k h^{(k)}(R_2)\right]\right\}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{R_1+}^{\nu}h\right\|_{L_q([R_1,R_2])},\left\|\overline{D}_{R_2-}^{\nu}h\right\|_{L_q([R_1,R_2])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\frac{(R_2-R_1)^{\nu+\frac{1}{p}}}{2^{\nu-1-\frac{1}{q}}}, \tag{145}$$

iii) if $h^{(k)}(R_1)=h^{(k)}(R_2)=0$, for all $k=0,1,\ldots,n-1$, we obtain

$$\left|\int_A f(y)\,dy\right|\leq\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)2^{\nu-1-\frac{1}{q}}}\cdot$$

$$\max\left\{\left\|D_{R_1+}^{\nu}h\right\|_{L_q([R_1,R_2])},\left\|\overline{D}_{R_2-}^{\nu}h\right\|_{L_q([R_1,R_2])}\right\}\cdot$$

$$\frac{(R_2-R_1)^{\nu+\frac{1}{p}}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}, \tag{146}$$

which is a sharp inequality,

iv) more generally, for $j=0,1,2,\ldots,N\in\mathbb{N}$, it holds

$$\left|\int_A f(y)\,dy-\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{R_2-R_1}{N}\right)^{k+1}\right.\right.$$

$$\left[j^{k+1}h^{(k)}(R_1)+(-1)^k(N-j)^{k+1}h^{(k)}(R_2)\right]\Bigg\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\Bigg|\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{R_1+}^{\nu}h\right\|_{L_q([R_1,R_2])},\left\|\overline{D}_{R_2-}^{\nu}h\right\|_{L_q([R_1,R_2])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}$$

$$\left(\frac{R_2-R_1}{N}\right)^{\nu+\frac{1}{p}}\left[j^{\nu+\frac{1}{p}}+(N-j)^{\nu+\frac{1}{p}}\right], \tag{147}$$

v) if $h^{(k)}(R_1)=h^{(k)}(R_2)=0$, $k=1,...,n-1$, from (147) we get:

$$\left|\int_A f(y)\,dy-\left(\frac{R_2-R_1}{N}\right)[jh(R_1)+(N-j)h(R_2)]\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|$$

$$\leq\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{R_1+}^{\nu}h\right\|_{L_q([R_1,R_2])},\left\|\overline{D}_{R_2-}^{\nu}h\right\|_{L_q([R_1,R_2])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}$$

$$\left(\frac{R_2-R_1}{N}\right)^{\nu+\frac{1}{p}}\left[j^{\nu+\frac{1}{p}}+(N-j)^{\nu+\frac{1}{p}}\right], \tag{148}$$

$j=0,1,2,...,N$,

vi) when $N=2$ and $j=1$, (148) turns to

$$\left|\int_A f(y)\,dy-(R_2-R_1)(h(R_1)+h(R_2))\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{R_1+}^{\nu}h\right\|_{L_q([R_1,R_2])},\left\|\overline{D}_{R_2-}^{\nu}h\right\|_{L_q([R_1,R_2])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\frac{(R_2-R_1)^{\nu+\frac{1}{p}}}{2^{\nu-1-\frac{1}{q}}}. \tag{149}$$

*Proof.*By Theorem 7 and (74). See in the 3. Appendix the general proving method in this article.

We give

Corollary 6.*(to Theorem 15) Let the radial $f:\overline{B(0,R)}\to\mathbb{R}$. Let $\nu\geq1$, $n=[\nu]$, and $h\in C_{0+}^{\nu}([0,R])\cap C_{R-}^{\nu}([0,R])$. Here $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Then*

i)

$$\left|\int_{B(0,R)}f(y)\,dy-\Bigg\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[h^{(k)}(0)t^{k+1}+\right.\right.$$

$$\left.\left.(-1)^k h^{(k)}(R)(R-t)^{k+1}\right]\Bigg\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{0+}^{\nu}h\right\|_{L_q([0,R])},\left\|\overline{D}_{R-}^{\nu}h\right\|_{L_q([0,R])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\cdot \tag{150}$$

$$\left[t^{\nu+\frac{1}{p}}+(R-t)^{\nu+\frac{1}{p}}\right],$$

$\forall\, t\in[0,R]$,

ii) at $t=\frac{R}{2}$*, the right hand side of (150) is minimized, and we get:*

$$\left|\int_{B(0,R)} f(y)\,dy-\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{R^{k+1}}{2^k}\right.\right.$$

$$\left.\left.\left[h^{(k)}(0)+(-1)^k h^{(k)}(R)\right]\right\}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{0+}^{\nu}h\right\|_{L_q([0,R])},\left\|\overline{D}_{R-}^{\nu}h\right\|_{L_q([0,R])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\frac{R^{\nu+\frac{1}{p}}}{2^{\nu-1-\frac{1}{q}}}, \tag{151}$$

iii) if $h^{(k)}(0)=h^{(k)}(R)=0$*, for all* $k=0,1,...,n-1$*, we obtain*

$$\left|\int_{B(0,R)} f(y)\,dy\right|\leq\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)2^{\nu-1-\frac{1}{q}}}\cdot$$

$$\max\left\{\left\|D_{0+}^{\nu}h\right\|_{L_q([0,R])},\left\|\overline{D}_{R-}^{\nu}h\right\|_{L_q([0,R])}\right\}\frac{R^{\nu+\frac{1}{p}}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}, \tag{152}$$

which is a sharp inequality,

iv) more generally, for $j=0,1,2,...,N\in\mathbb{N}$*, it holds*

$$\left|\int_{B(0,R)} f(y)\,dy-\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{R}{N}\right)^{k+1}\right.\right.$$

$$\left.\left.\left[j^{k+1}h^{(k)}(0)+(-1)^k(N-j)^{k+1}h^{(k)}(R)\right]\right\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{0+}^{\nu}h\right\|_{L_q([0,R])},\left\|\overline{D}_{R-}^{\nu}h\right\|_{L_q([0,R])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}$$

$$\left(\frac{R}{N}\right)^{\nu+\frac{1}{p}}\left[j^{\nu+\frac{1}{p}}+(N-j)^{\nu+\frac{1}{p}}\right], \tag{153}$$

v) if $h^{(k)}(0)=h^{(k)}(R)=0$*,* $k=1,...,n-1$*, from (153) we get:*

$$\left|\int_{B(0,R)} f(y)\,dy-\left(\frac{R}{N}\right)(N-j)\,h(R)\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|$$

$$\leq\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{0+}^{\nu}h\right\|_{L_q([0,R])},\left\|\overline{D}_{R-}^{\nu}h\right\|_{L_q([0,R])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}$$

$$\left(\frac{R}{N}\right)^{\nu+\frac{1}{p}}\left[j^{\nu+\frac{1}{p}}+(N-j)^{\nu+\frac{1}{p}}\right], \tag{154}$$

$j=0,1,2,...,N$,

vi) when $N=2$ *and* $j=1$*, (154) turns to*

$$\left|\int_{B(0,R)} f(y)\,dy-Rh(R)\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{\left\|D_{0+}^{\nu}h\right\|_{L_q([0,R])}, \left\|\overline{D}_{R-}^{\nu}h\right\|_{L_q([0,R])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \frac{R^{\nu+\frac{1}{p}}}{2^{\nu-1-\frac{1}{q}}}. \tag{155}$$

Proof. Based on Theorem 15, just set there $R_1 = 0$, $R_2 = R$, the assumptions now are on $B(0,R)$, and use (73).

If $g \in C^{n+1}([R_1,R_2])$, $0 \le R_1 < R_2$, then $h(s) = g(s)s^{N-1} \in C^{n+1}([R_1,R_2])$, $n \in \mathbb{N}$, $N \ge 2$.
Next we present multivariate Conformable fractional Iyengar type inequalities:

Theorem 16. *Let $\alpha \in (n,n+1]$ and $g \in C^{n+1}([R_1,R_2])$, $0 < R_1 < R_2$, $n \in \mathbb{N}$; $\beta = \alpha - n$. Then*
i)

$$\left|\int_A f(y)\,dy - \left\{\sum_{k=0}^{n} \frac{1}{(k+1)!}\left[h^{(k)}(R_1)(z-R_1)^{k+1} + \right.\right.\right.$$

$$\left.\left.\left.(-1)^k h^{(k)}(R_2)(R_2-z)^{k+1}\right]\right\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right| \le$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\Gamma(\beta)\max\left\{\left\|\mathbf{T}_{\alpha}^{R_1}(h)\right\|_{\infty,[R_1,R_2]}, \left\|{}_{\alpha}^{R_2}\mathbf{T}(h)\right\|_{\infty,[R_1,R_2]}\right\}}{\Gamma(\alpha+2)}\cdot$$

$$\left[(z-R_1)^{\alpha+1} + (R_2-z)^{\alpha+1}\right], \tag{156}$$

$\forall z \in [R_1,R_2]$,
ii) at $z = \frac{R_1+R_2}{2}$, the right hand side of (156) is minimized, and we get:

$$\left|\int_A f(y)\,dy - \left\{\sum_{k=0}^{n} \frac{1}{(k+1)!}\frac{(R_2-R_1)^{k+1}}{2^k}\right.\right.$$

$$\left.\left.\left[h^{(k)}(R_1) + (-1)^k h^{(k)}(R_2)\right]\right\}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right| \le$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\Gamma(\beta)\max\left\{\left\|\mathbf{T}_{\alpha}^{R_1}(h)\right\|_{\infty,[R_1,R_2]}, \left\|{}_{\alpha}^{R_2}\mathbf{T}(h)\right\|_{\infty,[R_1,R_2]}\right\}}{\Gamma(\alpha+2)} \frac{(R_2-R_1)^{\alpha+1}}{2^{\alpha-1}}, \tag{157}$$

iii) if $h^{(k)}(R_1) = h^{(k)}(R_2) = 0$, for all $k = 0,1,\dots,n$, we obtain

$$\left|\int_A f(y)\,dy\right| \le \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\cdot$$

$$\frac{\Gamma(\beta)\max\left\{\left\|\mathbf{T}_{\alpha}^{R_1}(h)\right\|_{\infty,[R_1,R_2]}, \left\|{}_{\alpha}^{R_2}\mathbf{T}(h)\right\|_{\infty,[R_1,R_2]}\right\}}{\Gamma(\alpha+2)} \frac{(R_2-R_1)^{\alpha+1}}{2^{\alpha-1}}, \tag{158}$$

which is a sharp inequality,
iv) more generally, for $j = 0,1,2,\dots,N \in \mathbb{N}$, it holds

$$\left|\int_A f(y)\,dy - \left\{\sum_{k=0}^{n} \frac{1}{(k+1)!}\left(\frac{R_2-R_1}{N}\right)^{k+1}\right.\right.$$

$$\left.\left.\left[h^{(k)}(R_1)\,j^{k+1} + (-1)^k h^{(k)}(R_2)(N-j)^{k+1}\right]\right\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right| \le$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\Gamma(\beta)\max\left\{\left\|\mathbf{T}_{\alpha}^{R_1}(h)\right\|_{\infty,[R_1,R_2]},\left\|{}_{\alpha}^{R_2}\mathbf{T}(h)\right\|_{\infty,[R_1,R_2]}\right\}}{\Gamma(\alpha+2)}$$

$$\left(\frac{R_2-R_1}{N}\right)^{\alpha+1}\left[j^{\alpha+1}+(N-j)^{\alpha+1}\right], \quad (159)$$

v) if $h^{(k)}(R_1)=h^{(k)}(R_2)=0$, $k=1,...,n$, from (159) we get:

$$\left|\int_A f(y)\,dy-\left(\frac{R_2-R_1}{N}\right)[jh(R_1)+(N-j)h(R_2)]\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|$$

$$\leq\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\Gamma(\beta)\max\left\{\left\|\mathbf{T}_{\alpha}^{R_1}(h)\right\|_{\infty,[R_1,R_2]},\left\|{}_{\alpha}^{R_2}\mathbf{T}(h)\right\|_{\infty,[R_1,R_2]}\right\}}{\Gamma(\alpha+2)}$$

$$\left(\frac{R_2-R_1}{N}\right)^{\alpha+1}\left[j^{\alpha+1}+(N-j)^{\alpha+1}\right], \quad (160)$$

$j=0,1,2,...,N$,

vi) when $N=2$ and $j=1$, (160) turns to

$$\left|\int_A f(y)\,dy-(R_2-R_1)(h(R_1)+h(R_2))\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\Gamma(\beta)\max\left\{\left\|\mathbf{T}_{\alpha}^{R_1}(h)\right\|_{\infty,[R_1,R_2]},\left\|{}_{\alpha}^{R_2}\mathbf{T}(h)\right\|_{\infty,[R_1,R_2]}\right\}}{\Gamma(\alpha+2)}\frac{(R_2-R_1)^{\alpha+1}}{2^{\alpha-1}}, \quad (161)$$

Proof. By Theorem 8 and as in our other multivariate results.

We continue with

Corollary 7. *Let $\alpha\in(n,n+1]$ and $g\in C^{n+1}([0,R])$, $R>0$, $n\in\mathbb{N}$; $\beta=\alpha-n$. Then*
i)

$$\left|\int_{B(0,R)} f(y)\,dy-\left\{\sum_{k=0}^{n}\frac{1}{(k+1)!}\left[h^{(k)}(0)z^{k+1}+\right.\right.\right.$$

$$\left.\left.\left.(-1)^k h^{(k)}(R)(R-z)^{k+1}\right]\right\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\Gamma(\beta)\max\left\{\left\|\mathbf{T}_{\alpha}^{0}(h)\right\|_{\infty,[0,R]},\left\|{}_{\alpha}^{R}\mathbf{T}(h)\right\|_{\infty,[0,R]}\right\}}{\Gamma(\alpha+2)}\cdot$$

$$\left[z^{\alpha+1}+(R-z)^{\alpha+1}\right], \quad (162)$$

$\forall\, z\in[0,R]$,

ii) at $z=\frac{R}{2}$, the right hand side of (162) is minimized, and we get:

$$\left|\int_{B(0,R)} f(y)\,dy-\left\{\sum_{k=0}^{n}\frac{1}{(k+1)!}\frac{R^{k+1}}{2^k}\right.\right.$$

$$\left.\left.\left[h^{(k)}(0)+(-1)^k h^{(k)}(R)\right]\right\}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\Gamma(\beta)\max\left\{\left\|\mathbf{T}_{\alpha}^{0}(h)\right\|_{\infty,[0,R]},\left\|{}_{\alpha}^{R}\mathbf{T}(h)\right\|_{\infty,[0,R]}\right\}}{\Gamma(\alpha+2)}\frac{R^{\alpha+1}}{2^{\alpha-1}}, \tag{163}$$

iii) assuming $h^{(k)}(0)=h^{(k)}(R)=0$, for all $k=0,1,...,n$, we obtain

$$\left|\int_{B(0,R)}f(y)\,dy\right|\leq\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\cdot$$

$$\frac{\Gamma(\beta)\max\left\{\left\|\mathbf{T}_{\alpha}^{0}(h)\right\|_{\infty,[0,R]},\left\|{}_{\alpha}^{R}\mathbf{T}(h)\right\|_{\infty,[0,R]}\right\}}{\Gamma(\alpha+2)}\frac{R^{\alpha+1}}{2^{\alpha-1}}, \tag{164}$$

which is a sharp inequality.

iv) more generally, for $j=0,1,2,...,N\in\mathbb{N}$, it holds

$$\left|\int_{B(0,R)}f(y)\,dy-\left\{\sum_{k=0}^{n}\frac{1}{(k+1)!}\left(\frac{R}{N}\right)^{k+1}\right.\right.$$

$$\left.\left.\left[h^{(k)}(0)\,j^{k+1}+(-1)^{k}h^{(k)}(R)(N-j)^{k+1}\right]\right\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\Gamma(\beta)\max\left\{\left\|\mathbf{T}_{\alpha}^{0}(h)\right\|_{\infty,[0,R]},\left\|{}_{\alpha}^{R}\mathbf{T}(h)\right\|_{\infty,[0,R]}\right\}}{\Gamma(\alpha+2)}$$

$$\left(\frac{R}{N}\right)^{\alpha+1}\left[j^{\alpha+1}+(N-j)^{\alpha+1}\right], \tag{165}$$

v) if $h^{(k)}(0)=h^{(k)}(R)=0$, $k=1,...,n$, from (165) we get:

$$\left|\int_{B(0,R)}f(y)\,dy-\left(\frac{R}{N}\right)(N-j)\,h(R)\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|$$

$$\leq\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\Gamma(\beta)\max\left\{\left\|\mathbf{T}_{\alpha}^{0}(h)\right\|_{\infty,[0,R]},\left\|{}_{\alpha}^{R}\mathbf{T}(h)\right\|_{\infty,[0,R]}\right\}}{\Gamma(\alpha+2)}$$

$$\left(\frac{R}{N}\right)^{\alpha+1}\left[j^{\alpha+1}+(N-j)^{\alpha+1}\right], \tag{166}$$

$j=0,1,2,...,N$,

vi) when $N=2$ and $j=1$, (166) turns to

$$\left|\int_{B(0,R)}f(y)\,dy-Rh(R)\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\Gamma(\beta)\max\left\{\left\|\mathbf{T}_{\alpha}^{0}(h)\right\|_{\infty,[0,R]},\left\|{}_{\alpha}^{R}\mathbf{T}(h)\right\|_{\infty,[0,R]}\right\}}{\Gamma(\alpha+2)}\frac{R^{\alpha+1}}{2^{\alpha-1}}, \tag{167}$$

*Proof.*By Theorem 16, just set there $R_1=0$, $R_2=R$, the assumptions now are on $B(0,R)$, and use (73).

We continue with L_p results.

Theorem 17. *Let $\alpha \in (n, n+1]$ and $g \in C^{n+1}([R_1, R_2])$, $0 < R_1 < R_2$, $n \in \mathbb{N}$; $\beta = \alpha - n$. Let also $p_1, p_2, p_3 > 1 : \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, with $\beta > \frac{1}{p_1} + \frac{1}{p_3}$. Then*

i)

$$\left| \int_A f(y)\, dy - \left\{ \sum_{k=0}^{n} \frac{1}{(k+1)!} \left[h^{(k)}(R_1)(z-R_1)^{k+1} + (-1)^k h^{(k)}(R_2)(R_2-z)^{k+1} \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| \mathbf{T}_{\alpha}^{R_1}(h) \right\|_{p_3,[R_1,R_2]}, \left\| {}_{\alpha}^{R_2}\mathbf{T}(h) \right\|_{p_3,[R_1,R_2]} \right\}}{n!\,(p_1 n+1)^{\frac{1}{p_1}} (p_2(\beta-1)+1)^{\frac{1}{p_2}} \left(\alpha + \frac{1}{p_1} + \frac{1}{p_2} \right)} \cdot \left[(z-R_1)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}} + (R_2-z)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}} \right], \quad (168)$$

$\forall\, z \in [R_1, R_2]$,

ii) at $z = \frac{R_1+R_2}{2}$, the right hand side of (168) is minimized, and we get:

$$\left| \int_A f(y)\, dy - \left\{ \sum_{k=0}^{n} \frac{1}{(k+1)!} \frac{(R_2-R_1)^{k+1}}{2^k} \left[h^{(k)}(R_1) + (-1)^k h^{(k)}(R_2) \right] \right\} \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| \mathbf{T}_{\alpha}^{R_1}(h) \right\|_{p_3,[R_1,R_2]}, \left\| {}_{\alpha}^{R_2}\mathbf{T}(h) \right\|_{p_3,[R_1,R_2]} \right\}}{n!\,(p_1 n+1)^{\frac{1}{p_1}} (p_2(\beta-1)+1)^{\frac{1}{p_2}} \left(\alpha + \frac{1}{p_1} + \frac{1}{p_2} \right)} \frac{(R_2-R_1)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}}{2^{\alpha-1-\frac{1}{p_3}}}, \quad (169)$$

iii) assuming $h^{(k)}(R_1) = h^{(k)}(R_2) = 0$, for all $k = 0, 1, \ldots, n$, we obtain

$$\left| \int_A f(y)\, dy \right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \cdot \frac{\max\left\{ \left\| \mathbf{T}_{\alpha}^{R_1}(h) \right\|_{p_3,[R_1,R_2]}, \left\| {}_{\alpha}^{R_2}\mathbf{T}(h) \right\|_{p_3,[R_1,R_2]} \right\}}{n!\,(p_1 n+1)^{\frac{1}{p_1}} (p_2(\beta-1)+1)^{\frac{1}{p_2}} \left(\alpha + \frac{1}{p_1} + \frac{1}{p_2} \right)} \frac{(R_2-R_1)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}}{2^{\alpha-1-\frac{1}{p_3}}}, \quad (170)$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, \ldots, N \in \mathbb{N}$, it holds

$$\left| \int_A f(y)\, dy - \left\{ \sum_{k=0}^{n} \frac{1}{(k+1)!} \left(\frac{R_2-R_1}{N} \right)^{k+1} \left[h^{(k)}(R_1)\, j^{k+1} + (-1)^k h^{(k)}(R_2)(N-j)^{k+1} \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| \mathbf{T}_{\alpha}^{R_1}(h) \right\|_{p_3,[R_1,R_2]}, \left\| {}_{\alpha}^{R_2}\mathbf{T}(h) \right\|_{p_3,[R_1,R_2]} \right\}}{n!\,(p_1 n+1)^{\frac{1}{p_1}} (p_2(\beta-1)+1)^{\frac{1}{p_2}} \left(\alpha + \frac{1}{p_1} + \frac{1}{p_2} \right)} \left(\frac{R_2-R_1}{N} \right)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}} \left[j^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}} + (N-j)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}} \right], \quad (171)$$

v) if $h^{(k)}(R_1) = h^{(k)}(R_2) = 0$, $k = 1, ..., n$, from (171) we get:

$$\left| \int_A f(y)\,dy - \left(\frac{R_2 - R_1}{N}\right)[jh(R_1) + (N-j)h(R_2)]\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right|$$

$$\leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| \mathbf{T}_{\alpha}^{R_1}(h) \right\|_{p_3,[R_1,R_2]}, \left\| {}_{\alpha}^{R_2}\mathbf{T}(h) \right\|_{p_3,[R_1,R_2]} \right\}}{n!\,(p_1 n + 1)^{\frac{1}{p_1}}\,(p_2(\beta-1)+1)^{\frac{1}{p_2}}\left(\alpha + \frac{1}{p_1} + \frac{1}{p_2}\right)}$$

$$\left(\frac{R_2 - R_1}{N}\right)^{\alpha + \frac{1}{p_1} + \frac{1}{p_2}} \left[j^{\alpha + \frac{1}{p_1} + \frac{1}{p_2}} + (N-j)^{\alpha + \frac{1}{p_1} + \frac{1}{p_2}} \right], \tag{172}$$

$j = 0, 1, 2, ..., N,$

vi) when $N = 2$ and $j = 1$, (172) turns to

$$\left| \int_A f(y)\,dy - (R_2 - R_1)(h(R_1) + h(R_2))\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| \mathbf{T}_{\alpha}^{R_1}(h) \right\|_{p_3,[R_1,R_2]}, \left\| {}_{\alpha}^{R_2}\mathbf{T}(h) \right\|_{p_3,[R_1,R_2]} \right\}}{n!\,(p_1 n + 1)^{\frac{1}{p_1}}\,(p_2(\beta-1)+1)^{\frac{1}{p_2}}\left(\alpha + \frac{1}{p_1} + \frac{1}{p_2}\right)}$$

$$\frac{(R_2 - R_1)^{\alpha + \frac{1}{p_1} + \frac{1}{p_2}}}{2^{\alpha - 1 - \frac{1}{p_3}}}, \tag{173}$$

Proof. By Theorem 9 and as in our other multivariate results.

We continue with

Corollary 8. *Let $\alpha \in (n, n+1]$ and $g \in C^{n+1}([0,R])$, $R > 0$, $n \in \mathbb{N}$; $\beta = \alpha - n$. Let also $p_1, p_2, p_3 > 1 : \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, with $\beta > \frac{1}{p_1} + \frac{1}{p_3}$. Then*

i)

$$\left| \int_{B(0,R)} f(y)\,dy - \left\{ \sum_{k=0}^{n} \frac{1}{(k+1)!} \left[h^{(k)}(0)\,z^{k+1} + \right.\right.\right.$$

$$\left.\left.\left. (-1)^k h^{(k)}(R)(R-z)^{k+1} \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| \mathbf{T}_{\alpha}^{0}(h) \right\|_{p_3,[0,R]}, \left\| {}_{\alpha}^{R}\mathbf{T}(h) \right\|_{p_3,[0,R]} \right\}}{n!\,(p_1 n + 1)^{\frac{1}{p_1}}\,(p_2(\beta-1)+1)^{\frac{1}{p_2}}\left(\alpha + \frac{1}{p_1} + \frac{1}{p_2}\right)}.$$

$$\left[t^{\alpha + \frac{1}{p_1} + \frac{1}{p_2}} + (R-t)^{\alpha + \frac{1}{p_1} + \frac{1}{p_2}} \right], \tag{174}$$

$\forall\, z \in [0,R]$,

ii) at $z = \frac{R}{2}$, the right hand side of (174) is minimized, and we get:

$$\left| \int_{B(0,R)} f(y)\,dy - \left\{ \sum_{k=0}^{n} \frac{1}{(k+1)!} \frac{R^{k+1}}{2^k} \right.\right.$$

$$\left.\left. \left[h^{(k)}(0) + (-1)^k h^{(k)}(R) \right] \right\} \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|\mathbf{T}_{\alpha}^{0}(h)\right\|_{p_3,[0,R]},\left\|{}_{\alpha}^{R}\mathbf{T}(h)\right\|_{p_3,[0,R]}\right\}}{n!\left(p_1 n+1\right)^{\frac{1}{p_1}}\left(p_2(\beta-1)+1\right)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)}$$
$$\frac{R^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}}{2^{\alpha-1-\frac{1}{p_3}}}, \tag{175}$$

iii) assuming $h^{(k)}(0)=h^{(k)}(R)=0$, for all $k=0,1,\ldots,n$, we obtain

$$\left|\int_{B(0,R)} f(y)\,dy\right| \le \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\cdot$$
$$\frac{\max\left\{\left\|\mathbf{T}_{\alpha}^{0}(h)\right\|_{p_3,[0,R]},\left\|{}_{\alpha}^{R}\mathbf{T}(h)\right\|_{p_3,[0,R]}\right\}}{n!\left(p_1 n+1\right)^{\frac{1}{p_1}}\left(p_2(\beta-1)+1\right)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)}\frac{R^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}}{2^{\alpha-1-\frac{1}{p_3}}}, \tag{176}$$

which is a sharp inequality.

iv) more generally, for $j=0,1,2,\ldots,N\in\mathbb{N}$, it holds

$$\left|\int_{B(0,R)} f(y)\,dy-\left\{\sum_{k=0}^{n}\frac{1}{(k+1)!}\left(\frac{R}{N}\right)^{k+1}\right.\right.$$
$$\left.\left.\left[h^{(k)}(0)\,j^{k+1}+(-1)^{k}h^{(k)}(R)\,(N-j)^{k+1}\right]\right\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right| \le$$
$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|\mathbf{T}_{\alpha}^{0}(h)\right\|_{p_3,[0,R]},\left\|{}_{\alpha}^{R}\mathbf{T}(h)\right\|_{p_3,[0,R]}\right\}}{n!\left(p_1 n+1\right)^{\frac{1}{p_1}}\left(p_2(\beta-1)+1\right)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)}$$
$$\left(\frac{R}{N}\right)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}\left[j^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}+(N-j)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}\right], \tag{177}$$

v) if $h^{(k)}(0)=h^{(k)}(R)=0$, $k=1,\ldots,n$, from (177) we get:

$$\left|\int_{B(0,R)} f(y)\,dy-\left(\frac{R}{N}\right)(N-j)\,h(R)\,\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|$$
$$\le \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|\mathbf{T}_{\alpha}^{0}(h)\right\|_{p_3,[0,R]},\left\|{}_{\alpha}^{R}\mathbf{T}(h)\right\|_{p_3,[0,R]}\right\}}{n!\left(p_1 n+1\right)^{\frac{1}{p_1}}\left(p_2(\beta-1)+1\right)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)}$$
$$\left(\frac{R}{N}\right)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}\left[j^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}+(N-j)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}\right], \tag{178}$$

$j=0,1,2,\ldots,N,$

viii) when $N=2$ and $j=1$, (178) turns to

$$\left|\int_{B(0,R)} f(y)\,dy-Rh(R)\,\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right| \le$$
$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|\mathbf{T}_{\alpha}^{0}(h)\right\|_{p_3,[0,R]},\left\|{}_{\alpha}^{R}\mathbf{T}(h)\right\|_{p_3,[0,R]}\right\}}{n!\left(p_1 n+1\right)^{\frac{1}{p_1}}\left(p_2(\beta-1)+1\right)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)}\frac{R^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}}{2^{\alpha-1-\frac{1}{p_3}}}. \tag{179}$$

*Proof.*By Theorem 17, just set there $R_1=0$, $R_2=R$, the assumptions now are on $B(0,R)$, and use (73).

Our proving method follows next.

3 Appendix

Proof. **Detailed proof of Theorem 10** (serving as a model proof for this article).

We apply Theorem 2 (i) for h:

$$\left| \int_{R_1}^{R_2} h(s)\,ds - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[h^{(k)}(R_1)(t-R_1)^{k+1} + (-1)^k h^{(k)}(R_2)(R_2-t)^{k+1} \right] \right| \leq \frac{\max\left\{ \left\| D_{*R_1}^{\nu} h \right\|_{L_\infty([R_1,R_2])}, \left\| D_{R_2-}^{\nu} \right\|_{L_\infty([R_1,R_2])} \right\}}{\Gamma(\nu+2)} \cdot \left[(t-R_1)^{\nu+1} + (R_2-t)^{\nu+1} \right] =: \psi(t), \tag{180}$$

$\forall\, t \in [R_1, R_2]$.

Equivalently, we have that

$$-\psi(t) \leq \int_{R_1}^{R_2} h(s)\,ds - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[h^{(k)}(R_1)(t-R_1)^{k+1} + (-1)^k h^{(k)}(R_2)(R_2-t)^{k+1} \right] \leq \psi(t), \tag{181}$$

$\forall\, t \in [R_1, R_2]$.

That is

$$-\psi(t) \leq \int_{R_1}^{R_2} f(s\omega)\, s^{N-1} ds - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[h^{(k)}(R_1)(t-R_1)^{k+1} + (-1)^k h^{(k)}(R_2)(R_2-t)^{k+1} \right] \leq \psi(t), \tag{182}$$

$\forall\, t \in [R_1, R_2]$, and $\forall\, \omega \in S^{N-1}$.

Therefore it holds

$$-\psi(t) \int_{S^{N-1}} d\omega \leq \int_{S^{N-1}} \left(\int_{R_1}^{R_2} f(s\omega)\, s^{N-1} ds \right) d\omega - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[h^{(k)}(R_1)(t-R_1)^{k+1} + (-1)^k h^{(k)}(R_2)(R_2-t)^{k+1} \right] \right\} \int_{S^{N-1}} d\omega \leq \psi(t) \int_{S^{N-1}} d\omega, \quad \forall\, t \in [R_1, R_2], \tag{183}$$

which is (by (74))

$$-\psi(t) \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \leq \int_A f(y)\,dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[h^{(k)}(R_1)(t-R_1)^{k+1} + (-1)^k h^{(k)}(R_2)(R_2-t)^{k+1} \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \leq \psi(t) \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}, \quad \forall\, t \in [R_1, R_2]. \tag{184}$$

Consequently, we derive

$$\left| \int_A f(y)\,dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[h^{(k)}(R_1)(t-R_1)^{k+1} + \right. \right. \right.$$

$$(-1)^k h^{(k)}(R_2)(R_2-t)^{k+1}\Big]\Big\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\Bigg| \leq \psi(t)\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} =$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{\left\|D_{*R_1}^{\nu}h\right\|_{L_\infty([R_1,R_2])}, \left\|D_{R_2-}^{\nu}\right\|_{L_\infty([R_1,R_2])}\right\}}{\Gamma(\nu+2)} \cdot$$

$$\left[(t-R_1)^{\nu+1}+(R_2-t)^{\nu+1}\right], \quad \forall\, t \in [R_1,R_2], \tag{185}$$

proving Theorem 10 (i).

Next consider

$$\varphi(t) := (t-R_1)^{\nu+1}+(R_2-t)^{\nu+1}, \quad \forall\, t \in [R_1,R_2].$$

Then

$$\varphi'(t) = (\nu+1)\left[(t-R_1)^{\nu}-(R_2-t)^{\nu}\right] = 0,$$

and φ has the only critical number $t = \frac{R_1+R_2}{2}$. Hence $\varphi(t)$ has a minimum over $[R_1,R_2]$ which is $\varphi\left(\frac{R_1+R_2}{2}\right) = \frac{(R_2-R_1)^{\nu+1}}{2^{\nu}}$.

Consequently, it holds (by (185))

$$\left|\int_A f(y)\,dy - \left\{\sum_{k=0}^{n-1} \frac{1}{(k+1)!}\frac{(R_2-R_1)^{k+1}}{2^k}\right.\right.$$

$$\left.\left.\left[h^{(k)}(R_1)+(-1)^k h^{(k)}(R_2)\right]\right\}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{\left\|D_{*R_1}^{\nu}h\right\|_{L_\infty([R_1,R_2])}, \left\|D_{R_2-}^{\nu}\right\|_{L_\infty([R_1,R_2])}\right\}}{\Gamma(\nu+2)} \frac{(R_2-R_1)^{\nu+1}}{2^{\nu-1}}, \tag{186}$$

proving Theorem 10 (ii).

The rest of Theorem 10 is obvious or follows the same way as above.

The rest of the proofs of this article as similar are omitted.

Conflict of Interest

The authors declare that they have no conflict of interest.

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