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Estimation of $R = P(Y < X)$ using k-Upper Record Values from Kumaraswamy Distribution

Naser A. Abou-Elheggag[1], *Al-Wageh A. Farghal*[1,*] *and Hanan A. Ramadan*[2]

[1] Mathematics Department, Faculty of science, Sohag University, Sohag 82524, Egypt
[2] Mathematics Department, Faculty of science, South Valley university, Qena, Egypt

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Abstract: The problem of estimating the stress-strength $R = P(Y < X)$ when X and Y are two independent ordinary samples was considered by many authors. In this paper, the problem of estimation of $R = P(Y < X)$ when X and Y are two independent k-upper record values from the Kumaraswamy distribution is considered. Maximum likelihood (ML) and Bayes techniques are used for this purpose. The maximum likelihood estimator is used to construct both an exact confidence interval and percentile bootstrap confidence interval of the stress-strength. Bayes estimators have been developed under both symmetric (squared error) and asymmetric (LINEX) loss functions. Monte Carlo simulations are performed to compare the performances of the different methods.

Keywords: Stress–strength model; K-upper record; Kumaraswamy distribution; Maximum- likelihood estimator; Bootstrap confidence intervals; Bayes estimator; Exact confidence interval

1 Introduction

In the statistical literature $R = P(Y < X)$ is known as the stress-strength parameter. This problem arises in the classical stress-strength reliability where one is interested in assessing the proportion of the times the random strength X of a component exceeds the random stress Y to which the component is subjected. Also This problem arises in state where X and Y represent lifetimes of two devices and one wants to estimate the probability that one fails before the other. For example, if we have a failure voltage levels of two types of electrical cable insulation when specimens were subjected to an increasing voltage stress in a laboratory test, “we are interested” in finding the type of insulation that has longer life. Specifically, let X represent the life time of a type 1 insulation, and let Y represent the same for type 2 insulation. Then the lower confidence limit for $P(Y < X)$ with a value greater than 0.5 indicate the superiority of type 1 insulation in terms of longevity. Constantine and Karson [1] consider the case when X and Y are independent gamma random variables. Ahmed et al. [2] and Surles and Padgett [3] considered the estimation of R where X and Y are Burr-X random variable. Ghosh and Sun [4] considered the recent developments of Bayesian inference for stress-strength models. Kundu and Gupta [5] developed the inference procedures on R under classical and Bayesian frame work, when X and Y are independent generalized exponential distribution. Raqab et al. [6] considered the estimation (Bayes and modified ML) of R when X and Y are distributed as two independent three-parameter generalized exponential random variables with different shape parameters but having the same location and scale parameters. A recent account on inference about R when X and Y are exponentially distributed is given by Jiang and Wong [7]. Greco and Ventura [8] considered robust inference for the stress–strength reliability. Rezaei et al. [9] discussed the problem of estimation of R when X and Y are two independent generalized Pareto distributions with different parameters. Also, the problem of estimation of R involving two independent modified Weibull distributions is considered by Soliman et al. [10]. Baklizi [11] has considered a similar problem for the exponential distribution based on record values, using non-Bayesian approach. Nadar et al. [12] discussed the problem of estimation of R when X and Y are two independent Kumaraswamy’s distribution based on record values. As can be seen from the cited literature, the developments in this field covered a variety of data types including complete data, censored data as well as data with explanatory variables. However, there are many situations in which only observations more

* Corresponding author e-mail: alwageh_ahmed@yahoo.com

extreme than the current extreme value are recorded. If the observation is greater than all the preceding observations it is called an ”upper” record. On the other hand, if the observation is smaller than all the preceding observations then it is called a ” lower” record. Industrial stress testing is a natural example where only items that are weaker than all the observed failed items are destroyed, see for example Ahmadi and Arghami [13]. A growing interest in the inferences of records has arisen in the last two decades, but not much has been done in the Bayesian framework. Detailed discussions of record data and its applications can be found in Arnold et al. [14], Ahsanullah [15] and Soliman et al. [16]. An $k-$upper record values process is defined in terms of the k^{th} largest X yet seen. For a formal definition, we consider the definition in Arnold et al. [14], for the continuous case.
Let $T_{1,k} = k$, and for $n \geq 2$

$$T_{n\ ,\ k} = min\{j : j > T_{n-1,k}, X_j > X_{T_{n-1,k}-k+1:T_{n-1,k}}\},$$

where $X_{i:m}$ denotes the i^{th} order statistic in a sample of size m. The sequence of $k-$upper record values are then defined by $R_{n\ (k)} = X_{T_{n,k}-k+1:T_{n,k}}$ for $n \geq 1$. Note that for $k = 1$, an upper record value is recovered. See, Ahmadi et al. [17]. In this paper, it is shown how $k-$upper record data can be used to provide point estimation and confidence intervals for the stress-strength reliability model $R = P(Y < X)$ from the Kumaraswamy distribution with different parameters. We obtain the MLE of R and its exact distribution. The exact distribution is used to construct an exact confidence interval. Nadar and Kızılaslan [18] have considered a similar problem for Kumaraswamy's distribution based on record values. The interest in developing inference procedures for R arises because of its applications to a variety of fields.
A random variable X said to have a Kumaraswamy distribution, denoted by X $\sim$ Kw(a, b).
The distribution function (cdf) is

$$F(x;a,b) = 1-(1-x^a)^b, \qquad 0 < x < 1, \tag{1}$$

and hence the probability density function (pdf) given by

$$f(x;a,b) = abx^{a-1}(1-x^a)^{b-1}, \qquad 0 < x < 1, \tag{2}$$

where a>0and b>0 are the shape parameters. It is known that X is Kumaraswamy then -ln X is the two parameters generalized exponential distribution. Kumaraswamy [19] developed a more general (pdf) with hydrological applications, which is known as Kumaraswamy distribution. This distribution has been studied by many authors, see Sundar and Subbiah [20], Fletcher and Ponnamblam [21], Seifi et al. [22], Ponnambalam et al. [23], Ganji et al. [24] and Nadarajah [25]. The main aim of this paper is to focus on the estimation of $R = P(Y < X)$, where X and Y follow the Kumaraswamy distribution based on $k-$upper record data. In section 2, we obtain the maximum likelihood estimation (MLE) of R and its exact distribution. The exact distribution is used to construct an exact confidence interval. The parametric bootstrap percentile confidence interval of R is presented in section 3. Bayesian confidence intervals with exact confidenc interval of R are presented in section 4. Also, in Bayesian setting, the symmetric and asymmetric point estimations of R are obtained and discussed in section 5. In section 6, two numerical examples using simulated k-upper record data are illustrated and the results of different methods are discussed. The different methods have been compared using simulation study and their results have been reported in section 7. The conclusions is presented in the final section.

2 Maximum likelihood estimation

To formulate the present problem, let $X \sim Kw(a,b_1)$ and $Y \sim Kw(a,b_2)$ are two independently distributed . (here ”$\sim$ ”means follows or has the distribution). Therefore , be the stress strength reliability model define as:

$$\begin{aligned} R = p(Y < X) &= \int_{-\infty}^{\infty}\int_{-\infty}^{x} f(x,y)\,dydx \\ &= \int_{-\infty}^{\infty} G(x)\,dF(x) \\ &= \int_0^1 \left[1-(1-x^a)^{b_2}\right] ab_1x^{a-1}(1-x^a)^{b_1-1}dx, \end{aligned} \tag{3}$$

where $f(x,y)$ the joint probability density function, then

$$R = \frac{b_2}{b_1+b_2}. \tag{4}$$

Our interest is in estimating R under assumption that the available data for both X and Y are k-upper record values. Let $r = (r_1, ..., r_n)$ be the first n of k-upper record values from $Kw(a,b_1)$ and $s = (s_1, ..., s_m)$ Let be the first m of k-upper record values from $Kw(a,b_2)$, where a is known. The likelihood functions (for b_1 and b_2) based on the observed samples r and s are given in Arnold et al. [14], respectively, by

$$L_1(b_1|r) = k^n (1-F(r_n))^k \prod_{i=1}^{n} \frac{f(r_i)}{1-F(r_i)} = k^n a^n b_1^n (1-r_n^a)^{kb_1} \prod_{i=1}^{n} \frac{r_i^{a-1}}{(1-r_i^a)}, \tag{5}$$

and

$$L_2(b_2|s) = k^m (1-G(s_m))^k \prod_{j=1}^{m} \frac{g(s_j)}{1-G(s_j)} = k^m a^m b_2^m (1-s_m^a)^{kb_2} \prod_{j=1}^{m} \frac{s_j^{a-1}}{\left(1-s_j^a\right)}. \tag{6}$$

Thus the joint log-likelihood function (for 1 and 2) based on the observed samples r and s can be written as

$$\begin{aligned} l(b_1,b_2|r,s) = \log(L_1 \times L_2) &= n\log(k) + m\log(k) + n\log(a) + m\log(a) + n\log(b_1) + m\log(b_2) \\ &+ kb_1 \log(1-r_n^a) + kb_2 \log(1-s_m^a) + (a-1)\textstyle\sum_{i=1}^{n} \log(r_i) - \sum_{i=1}^{n} \log(1-r_i^a) \\ &+ (a-1)\textstyle\sum_{j=1}^{m} \log(s_j) - \sum_{j=1}^{m} \log\left(1-s_j^a\right) \end{aligned} \tag{7}$$

The MLE's of b_1 and b_2 say $\hat{b}_1$ and $\hat{b}_2$ respectively, can be obtained as the solutions of

$$\frac{\partial l(b_1,b_2|r,s)}{\partial b_1} = \frac{n}{b_1} + k\log(1-r_n^a) = 0, \tag{8}$$

and

$$\frac{\partial l(b_1,b_2|r,s)}{\partial b_2} = \frac{m}{b_2} + k\log(1-s_m^a) = 0. \tag{9}$$

From equations (8) and (9), we obtain

$$\hat{b}_1 = \frac{-n}{k\log(1-r_n^a)}, \tag{10}$$

and

$$\hat{b}_2 = \frac{-m}{k\log(1-s_m^a)}. \tag{11}$$

Since ML estimators are invariant, so the ML estimators of R becomes:

$$\hat{R}_{ML} = \frac{\hat{b}_2}{\hat{b}_1 + \hat{b}_2} = \frac{\log(1-r_n^a)}{\log(1-r_n^a) + \log(1-s_m^a)}. \tag{12}$$

To study the distribution of $\hat{R}_{ML}$ we need the distribution of $\hat{b}_1$ and $\hat{b}_2$. The (pdf) of r_n is given by (see, Arnold et al. [14])

$$\begin{aligned} f_{n,k}(r_n) &= \frac{k^n}{(n-1)!} (-\log(1-F(r_n)))^{n-1} (1-F(r_n))^{k-1} f(r_n) \\ &= \frac{(-1)^{n-1} ab^n k^n r_n^{a-1}}{(n-1)!} [\log(1-r_n^a)]^{n-1} (1-r_n^a)^{kb-1}. \end{aligned} \tag{13}$$

Consequently, the (pdf) of $z_1 = \hat{b}_1 = \frac{-n}{k\log(1-r_n^a)}$, is given by

$$f_{z_1}(Z_1) = \frac{(-1)^{2n-1} b_1^n n^n}{\Gamma(n) z_1^{n+1}} \exp\left(\frac{-nb_1}{z_1}\right), \ z_1 > 0, \tag{14}$$

similarly, the (pdf) of $z_2 = \hat{b}_2 = \frac{-m}{k\log(1-s_m^a)}$, is given by

$$f_{z_2}(z_2) = \frac{(-1)^{2m-1} b_2^m m^m}{\Gamma(m) z_2^{m+1}} \exp\left(\frac{-mb_2}{z_2}\right), \ z_2 > 0. \tag{15}$$

Now, we can write $\hat{R}_{ML}=\frac{z_2}{z_1+z_2}=\frac{1}{1+z_2/z_1}$. Therefore, using (14) and (15) we have $\frac{b_1z_2}{b_2z_1}\sim F_{2n,2m}$. Moreover, $Z_1$and $Z_{2.}$ are independently distributed. Thus $\frac{z_2}{z_1}\sim\frac{b_2F_{2n.2m}}{b_1}$, where $F_{2n,2m}$ is a scaled F distribution with 2n and 2m degrees of freedom. It follows that the distribution of $\hat{R}_{ML}$ is that

$$\hat{R}_{ML}\sim\left(1+\frac{b_2F_{2n.2m}}{b_1}\right)^{-1}. \tag{16}$$

Which can be obtained using simple transformation techniques. The $(1-\alpha)\,100\%$ confidence interval of R can be obtained as

$$\left\{\left(1+\frac{z_{1F_{\alpha/2,2n,2m}}}{z_2}\right)^{-1},\left(1+\frac{z_{1F_{(1-\alpha/2),2n,2m}}}{z_2}\right)^{-1}\right\}, \tag{17}$$

where $F_{1-(\alpha/2),2n,2m}$ and $F_{(\alpha/2),2n,2m}$ are the lower and upper $(\alpha/2)$th percentile points of a F distribution.

3 Bootstrap Confidence Intervals

It is well known that the record value are inpractice and the sample size are often very small. However, confidence intervals based on the asymptotic results do not perform very well for small sample size. So, in this section we obtain the confidence interval of based on the parameteric percentile boot strap method suggested by Efron [27] . When the available data are the two sets of k-upper record vlues, the algorithm for estimating the confidence interval of using-the parameteric percentile bootstrap method is illustrated below.

The following step are followed to obtain the k -upper record bootstrap samples from $Kw(a,b_1)$ and $Kw(a,b_2)$.

Step 1Based on the original k-upper record values $(r_1,......,r_n)$ and $(s_1,......,s_m)$, compute the MLEs $\hat{b}_1,\hat{b}_2$ and $\hat{R}_{ML}$ using 10, 11 and 12 respectively.

Step 2Use $\hat{b}_1$, $\hat{b}_2$ to generate a bootstrap sample of k-upper records $(r_1^*,......,r_n^*)$ and $(s_1^*,......,s_m^*)$ from the Kumaraswamy distribution and compute $\hat{R}^*$ using (12).

Step 3Repeat step 2, N times representing N bootstrap MLE's of R based on N different bootstrap samples.

Step 4Arrange $\hat{R}^*$ in an ascending order to obtain the bootstrap sample $\left[\varphi^{[1]},\varphi^{[2]},........,\varphi^{[N]}\right]$ where$\left(\varphi=\hat{R}^*\right)$.

Step 5Let $G(z)=p\left(\hat{R}^*\leq z\right)$ be the cumulative distribution of $\hat{R}^*$. Define $\hat{R}_{boot}=G^{-1}(z)$ for a given z . The approximate bootstrap $(1-\alpha)\,100\%$ confidence intervals of R is given by $\left(\hat{R}_{boot}\left(\frac{\alpha}{2}\right),\,\hat{R}_{boot}\left(1-\frac{\alpha}{2}\right)\right)$

4 Bayes Estimations for R

4.0.1 Exact confidenc interval

In this section we discuss the Bayesian estimator of the R. It is assumed here that the parameters b_1 and b_2 are independent and follow the gamma prior distributions. Therefore, the prior density function of b_1 and b_2 becomes

$$\pi_1(b_1)=\frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)}b_1^{\alpha_1-1}\exp(-\beta_1b_1)\qquad ,\, b_1>0, \tag{18}$$

and

$$\pi_2(b_2)=\frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)}b_2^{\alpha_2-1}\exp(-\beta_2b_2)\qquad ,\, b_2>0, \tag{19}$$

where, $\alpha_1,\alpha_2,\beta_1$ and β_2 are chosen to reflect prior knowledge about b_1 and b$_2$. From Eqs. (5- 6) and (18- 19), we can show that the posterior (pdf) of b_1 is given by

$$\pi_1^*(b_1|r)=\frac{b_1^{n+\alpha_1-1}\left[\beta_1-k\ln(1-r_n^a)\right]^{n+\alpha_1}\exp\left(-b_1\left(\beta_1-k\ln(1-r_n^a)\right)\right)}{\Gamma(n+\alpha_1)}, \tag{20}$$

and the posterior (pdf) of b_2 is given by

$$\pi_2^*(b_2|s)=\frac{b_2^{m+\alpha_2-1}\left[\beta_2-k\ln(1-s_m^a)\right]^{m+\alpha_2}\exp\left(-b_2\left(\beta_2-k\ln(1-s_m^a)\right)\right)}{\Gamma(m+\alpha_2)}. \tag{21}$$

Using the posteriors density of $b_1$and b_2 given in (20) and (21), the joint posterior of $b_1$and b_2 is given by

$$\begin{array}{l}\pi_3^*(b_1,b_2|r,s)=\frac{b_1^{n+\alpha_1-1}b_2^{m+\alpha_2-1}(1-r_n^a)^{kb_1}(1-s_m^a)^{kb_2}[\beta_1-k\ln(1-r_n^a)]^{n+\alpha_1}}{\Gamma(n+\alpha_1)\Gamma(m+\alpha_2)}\\ \times[\beta_2-k\ln(1-s_m^a)]^{m+\alpha_2}\exp(-\beta_1b_1-\beta_2b_2)\quad b_1,b_2>0.\end{array} \tag{22}$$

by using the following simple transformation technique

$$w=b_1+b_2\quad,\quad R=\frac{b_2}{b_1+b_2}=\frac{b_2}{w},$$

we can obtain the joint posterior distribution of R and w as

$$\begin{array}{l}\pi_3^*(R,w)=\frac{w^{(n+m+\alpha_1+\alpha_2)-1}R^{(m+\alpha_2)-1}(1-R)^{(n+\alpha_1)-1}(1-r_n^a)^{kw_1}[\beta_1-k\ln(1-r_n^a)]^{n+\alpha_1}}{\Gamma(n+\alpha_1)\Gamma(m+\alpha_2)}\\ \times\left[\frac{1-s_m^a}{1-r_n^a}\right]^{kw_1R_1}[\beta_2-k\ln(1-s_m^a)]^{m+\alpha_2}\exp(-((\beta_2-\beta_1)R+\beta_1)w),w>0.\end{array} \tag{23}$$

Using equation (23) with integrating out of w , the posterior distribution of R is given by

$$\pi_3^*(R|r,s)=QR^{(m+\alpha_2)-1}(1-R)^{(n+\alpha_1)-1}(1+Rd)^{-D}, \tag{24}$$

where

$$Q=\frac{\Gamma(n+m+\alpha_1+\alpha_2)}{\Gamma(n+\alpha_1)\Gamma(m+\alpha_2)}\left[\frac{\beta_2-k\ln(1-s_m^a)}{\beta_1-k\ln(1-r_n^a)}\right]^{m+\alpha_2},$$
$$D=n+m+\alpha_1+\alpha_2,$$
$$d=\frac{(\beta_2-\beta_1)-k\ln(1-s_m^a)+k\ln(1-r_n^a)}{\beta_1-k\ln(1-r_n^a)}.$$

It follows that

$$2b_1(\beta_1-k\ln(1-r_n^a))\sim\chi^2_{n+b_1}\text{ and }2b_2(\beta_2-k\ln(1-s_m^a))\sim\chi^2_{m+b_2}.$$

By the independence of two random quantities, we have

$$\frac{2b_1(\beta_1-k\ln(1-r_n^a))/(2(n+\alpha_1))}{2b_2(\beta_2-k\ln(1-s_m^a))/(2(m+\alpha_2))}\sim F_{2(n+\alpha_1),2(m+\alpha_2)},$$

hence,

$$\frac{b_1}{b_2}\sim\frac{(\beta_1-k\ln(1-r_n^a))(n+\alpha_1)}{(\beta_2-k\ln(1-s_m^a))(m+\alpha_2)}F_{2(n+\alpha_1),2(m+\alpha_2)}.$$

Using simple transformation techniques with $A=\frac{(\beta_1-k\ln(1-r_n^a))(n+\alpha_1)}{(\beta_2-k\ln(1-s_m^a))(m+\alpha_2)}$, we can obtain the $(1-\alpha)\,100\%$ confidence interval for R as follows

$$\left\{\left(1+AF_{\alpha/2,2(n+\alpha_1),2(m+\alpha_2)}\right)^{-1},\left(1+AF_{1-\alpha/2,2(n+\alpha_1),2(m+\alpha_2)}\right)^{-1}\right\}. \tag{25}$$

5 Bayes Point Estimations of R

Sometimes the use of symmetric loss function, namely squared error loss function (SEL), was found to be inappropriate, as for example, an overestimation of the reliability function is usually much more serious than an underestimation. In this case, an asymmetric loss function might be more appropriate. A number of asymmetric loss functions are proposed for use, among these, one of the most popular asymmetric loss functions is linear-exponential loss function (LINEX) which was introduced by Varian [28]. The LINEX loss function rises approximately exponentially on one side of zero and approximately linearly on the other side. Recently, many authors considered asymmetric loss functions in reliability and life testing, such as Basu and Ebrahimi [29], Soliman et al. [30] and Ren et al. [31]. Under the assumption that the minimal loss occurs at $\hat{R}=R$, the LINEX loss function for R can be expressed as

$$L(\Delta)\propto\exp(c\Delta)-c\Delta-1;\quad c\neq0. \tag{26}$$

Where $\Delta = (\hat{R} - R)$, $\hat{R}$ is an estimator of R. The sign and magnitude of the shape parameter c represents the direction and degree of asymmetry respectively (If $c > 0$), the overestimation is more serious than underestimation, and vice-versa. For c closed to zero, the LINEX loss is approximately squared error loss and therefore almost symmetric. For more details about $L(\Delta)$ see Zellner [32] and Calabria and Pulcini [**?**]. The posterior expectation under the LINEX loss function (26) is

$$E_R(L(\Delta)) = \exp(c\hat{R})E_R[\exp(-cR)] - c(\hat{R} - E_R(R)) - 1, \tag{27}$$

where $E_R(.)$ denotes the posterior expectation with respect to the posterior density of R. The Bayes estimator of R, denoted by $\hat{R}_{BL}$ under the LINEX loss function is the value $\hat{R}$ which minimizes (27), that is

$$\hat{R}_{BL} = \frac{-1}{c}\log\{E_R[\exp(-cR)]\}, \tag{28}$$

provided that the expectation $E_R[\exp(-cR)]$ exists and is finite.
In the following subsections, we obtain the Bayes point estimation of R relative to both a squared error loss function (SEL) and LINEX loss function.

Bayes estimation under a squared error loss function
Under squared error loss function $L(\hat{R}, R) = (\hat{R} - R)^2$, the Bayes estimate of R, denoted by $\hat{R}_{BS}$ is the posterior mean

$$\begin{aligned}\hat{R}_{BS} &= \int_0^1 QR^{(m+\alpha_2)}(1-R)^{(n+\alpha_1)-1}(1+Rd)^{-D}\\ &= Q\sum_{i=1}^{n+\alpha_1-1}(-1)^i\binom{n+\alpha_1-1}{i}\int_0^1 R_1^{E_i}(1+dR_1)^{-D}dR_1\\ &= Q\sum_{i=1}^{n+\alpha_1-1}\frac{(-1)^i\binom{n+\alpha_1-1}{i}}{(1+E_i)}\ {}_2F_1[D, E_i+1, E_i+2, -d],\end{aligned} \tag{29}$$

where $E_i = m + \alpha_2 + i$, and ${}_2F_1[a,b;c;z]$ is the hypergeometric function.
Bayes estimation under LINEX loss function
Relative to **LINEX** loss function, the Bayes estimate of R denoted by $\hat{R}_{BL}$ is given by

$$\begin{aligned}\hat{R}_{BL} &= \frac{-1}{c}\log\left[\int_0^1 \exp(-cR)\,QR^{(m+\alpha_2)-1}(1-R)^{(n+\alpha_1)-1}(1+dR)^{-D}dR\right]\\ &= \frac{-1}{c}\log\left[Q\sum_{j=0}^{\infty}\sum_{i=1}^{n+\alpha_1-1}\frac{(-1)^{i+j}\binom{n+\alpha_1-1}{i}}{j!}\times\int_0^1 R^{B_{i,j}-1}(1+dR)^{-D}dR\right]\\ &= \frac{-1}{c}\log\left[Q\sum_{j=0}^{\infty}\sum_{i=1}^{n+\alpha_1-1}\frac{(-1)^{i+j}\binom{n+\alpha_1-1}{i}}{j!}\ {}_2F_1[D, B_{i,j}, B_{i,j}+1, -d]\right],\end{aligned} \tag{30}$$

where $B_{i,j} = \alpha_2 + i + j + m$.

6 Data Analysis

In this section, we present two examples to illustrate the previous results and to compare the performance of the different estimation procedures.

Example1: In this example, we present a complete analysis using a simulated k-upper record data. The different estimators of R obtained in the above sections are computed as follows:
(1) For given values of the prior parameters α_1 and β_1 we generate b_1 from the prior distribution (18), and it is considered as the “actual” population value, and similarly for the prior parameters α_2 and β_2 we generate b_2 from the prior distribution (19). We obtained $b_1 = 3$ and $b_2 = 2$.
(2) We generate two sets of $k-$upper record values from Kumarswamy distribution using: $n = m = 5, k = 5, b_1 = 3$ and $b_2 = 2$ therefore $R = 0.4$.
From the following steps we can obtain the first n of $k-$upper record values.
(i) We generate data from Kumarswamy distribution of size k.
(ii) We arrange this data in ascending order, we have, $x_{1:1} > x_{1:2} > ... > x_{1:k}$, we will take the value as the number k say $x_{1:k}$
(iii) We generate another value from Kumarswamy distribution and insert to this data and descending sort we have

$x_{1:1} > x_{1:2} > ... > x_{1:k+1}$, and we will take the value as the number k.
(iv) Repeat step (iii), n times, we can get the first n of $k-$upper record values as $x_1, x_2, ..., x_n$.
(3) The data has been truncated after four decimal places. The $5-$upper record values from Y as: $0.0658, 0.1104, 0.2055, 0.2701, 0.3517$ and the $5-$upper record values from X as $0.0172, 0.0320, 0.1829, 0.1836, 0.2610$.
(4) Using equations (10) and (11), the MLE of b_1 and b_2 are : $\hat{b}_1 = 3.30598$ and $\hat{b}_2 = 2.30766$. Therefore, the ML estimates of R is $\hat{R}_{ML} = 0.411081$. The 95% exact confidence interval for R using (16) is $(0.1581, 0.7218)$.
(5) *Bootstrap method*: Based on the original data in step 3 and using the algorithm described in section (3), we compute the mean of $N = 1000$ bootstrap samples of $\hat{R}^*$ as a bootstrap point estimate, the result becomes $\hat{R}^*_{boot} = 0.3915$. The 95% bootstrap confidence interval (BCIs) of R is $(0.1710, 0.5874)$.
(6) *Bayes method*: For the case of informative priors, we use $\alpha_1 = 1, \alpha_2 = 2, \beta_1 = 1$ and $\beta_2 = 2$. The Bayes estimates of R relative to SEL and LINEX loss functions are computed using (29) and (30). The results are: $\hat{R}_{BS} = 0.421766$, the Bayes estimates under LINEX loss function are respectively $\hat{R}_{BL} = 0.3816$, $\hat{R}_{BL} = 0.4364$ and $\hat{R}_{BL} = 0.3944$ for $c = 5, -2$, and 0.0001.
(7) The 95% confidence interval of R using Bayes confidence interval in (25) is computed as $(0.3881, 0.8612)$. The posterior probability density function of R is plotted in Fig.1.

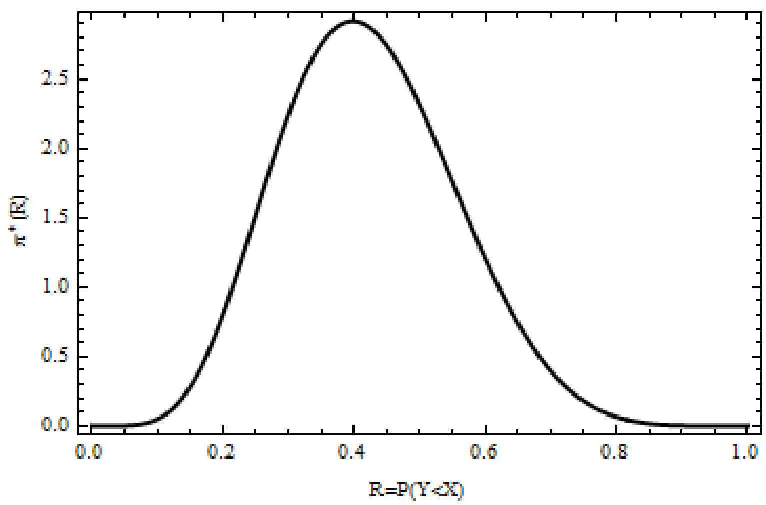

Fig. 1: Posterior probability density function of R for record data in Example 1

Example 2: Using the same steps in example 1, the sets of $2-$upper record values of sizes n and m are generated from Kumarswamy distribution using $b_1 = 1.5$ and $b_2 = 2$. Therefore, $R = 0.7273$ is assumed as the true value of R. For different values of n and m, we obtain the 95% confidence interval of R using the maximum likelihood method (ML), Bayes method (BS) and percentile bootstrap method ($Boot$) based on 1000 bootstrap samples. The results are recorded in Table 1.

Table (1) 0.95% confidence intervals of R, with lower L and upper U Limits.

m	n	$(.)_{Ml}$			$(.)_{Bs}$			$(.)_{Boot}$		
		L	U	length	L	U	length	L	U	length
5	5	0.2299	0.8049	0.5749	0.1574	0.6463	0.4889	0.2537	0.6905	0.4368
5	7	0.4001	0.8817	0.4816	0.0698	0.3819	0.3121	0.5080	0.8191	0.3111
5	10	0.2320	0.7412	0.5092	0.0622	0.3208	0.2585	0.3278	0.6377	0.3130
7	5	0.3348	0.8490	0.5142	0.2126	0.6992	0.4865	0.3443	0.7718	0.4275
7	7	0.4846	0.8930	0.4084	0.0959	0.4320	0.3361	0.5183	0.8377	0.3194
7	10	0.4201	0.8428	0.4227	0.0699	0.3156	0.2457	0.5016	0.7769	0.2753
10	5	0.2451	0.7548	0.5097	0.4397	0.8575	0.4179	0.2303	0.6530	0.4227
10	7	0.4807	0.8726	0.3920	0.1862	0.5914	0.4053	0.4879	0.8205	0.3327
10	10	0.4759	0.8465	0.3706	0.1105	0.3993	0.2889	0.5049	0.7912	0.2863

From Table 1, it is clear, as expected, that increasing the number of records on either variable results in shorter intervals.

7 Simulation study and comparisons

In this section, we present some results based on Monte Carlo simulations to compare the performance of the different estimators of the stress-strength reliability R. In this simulation, 1000 pairs of samples of $k-$upper record data $(k = 3, 5)$ were generated from kumarswamy distribution using case 1: $b_1 = 3, b_2 = 2$ and $a = 1$ with $R = 0.4$ and various combinations of sample sizes n and m , case 2: $b_1 = 1.5, b_2 = 1.7$ and $a = 2.1$ with $R = 0.5313$ and various combinations of sample sizes n and m . We obtain the MLE of R, $(.)_{Ml}$ using (12), the symmetric Bayes estimate of R, $(.)_{Bs}$ using (29), and the asymmetric Bayes estimate of R, $(.)_{Bl}$ using (30). The average estimation and the mean square error (MSE) of R for different methods are computed over 1000 replications. The results are reported in Table 2 and 4 for $3-$upper record data and in Tables 3 and 5 for $5-$upper record data. For all methods, It is clear as expected, when n and m increase, the MSE's decrease.

Table(2).The Means and (MSE) of R with $R = 0.4$, $k = 3$.

m	n	$(.)_{Ml}$	$(.)_{Bs}$	$(.)_{Bl}$			*Boot*
				$c2 = -2$	$c3 = 0.0001$	$c1 = 2$	
5	5	0.4248 (0.0253)	0.4275 (0.0083)	0.4442 (0.00972)	0.4275 (0.00834)	0.4113 (0.00744)	0.4068 (0.0224)
7	7	0.4037 (0.0150)	0.41027 (0.0059)	0.4231 (0.00647)	0.4103 (0.00587)	0.3979 (0.00556)	0.3911 (0.0139)
10	10	0.4164 (0.0126)	0.4171 (0.0058)	0.4268 (0.00639)	0.4172 (0.00582)	0.4078 (0.0055)	0.4069 (0.0118)
13	13	0.4124 (0.0078)	0.4143 (0.0047)	0.422 (0.00504)	0.4143 (0.00469)	0.4068 (0.00444)	0.4051 (0.0074)

Table(3).The Means and (MSE) of R with $R_1 = 0.4$, $k = 5$.

m	n	$(.)_{Ml}$	$(.)_{Bs}$	$(.)_{Bl}$			*Boot*
				$c = -2$	$c = 0.0001$	$c = 2$	
5	5	0.4221 (0.0198)	0.4258 (0.0065)	0.4427 (0.00783)	0.4258 (0.00649)	0.4095 (0.00569)	0.4042 0.0175
7	7	0.4014 (0.0126)	0.4099 (0.0054)	0.4228 (0.00602)	0.4099 (0.00542)	0.3974 (0.00512)	0.3880 0.0117
10	10	0.4053 (0.0116)	0.4090 (0.00532)	0.4185 (0.0060)	0.409 (0.00532)	0.3997 (0.00517)	0.3964 0.01107
13	13	0.4009 (0.0088)	0.4052 (0.0053)	0.4128 (0.00552)	0.4052 (0.00529)	0.3978 (0.00515)	0.3934 0.0085

Table(4).The Means and (MSE) of R with $R = 0.5313$, $k = 3$.

m	n	$(.)_{Ml}$	$(.)_{Bs}$	$(.)_{Bl}$			*Boot*
				$c_1 = -2$	$c_2 = 0.0001$	$c_3 = 2$	
5	5	0.5414 (0.02015)	0.5103 (0.00848)	0.5273 (0.00795)	0.5103 (0.00848)	0.4934 (0.00947)	0.5173 (0.01894)
7	7	0.5148 (0.01742)	0.4973 (0.00843)	0.5105 (0.00789)	0.4973 (0.00843)	0.4843 (0.0935)	0.4982 (0.01728)
10	10	0.5217 (0.01233)	0.5055 (0.00799)	0.5154 (0.00756)	0.5055 (0.007988)	0.4956 (0.00857)	0.5093 (0.01229)
13	13	0.5385 (0.00805)	0.5226 (0.00546)	0.5306 (0.00535)	0.5226 (0.005465)	0.5147 (0.00568)	0.5287 (0.00776)

Table(5).The Means and (MSE) of R with $R_1 = 0.5313$, $k = 5$.

m	n	$(.)_{Ml}$	$(.)_{Bs}$	$(.)_{Bl}$			*Boot*
				$c_1 = -2$	$c_2 = 0.0001$	$c_3 = 2$	
5	5	0.5264 (0.02049)	0.501 (0.00919)	0.518 (0.00839)	0.501 (0.00919)	0.4841 (0.01045)	0.5033 (0.01976)
7	7	0.527 (0.01674)	0.5047 (0.00883)	0.5179 (0.00828)	0.5047 (0.00883)	0.4916 (0.00967)	0.5101 (0.01635)
10	10	0.537 (0.01312)	0.5183 (0.00809)	0.5281 (0.00788)	0.5183 (0.00809)	0.5085 (0.00844)	0.5246 (0.01255)
13	13	0.5327 (0.00813)	0.5175 (0.005621)	0.5255 (0.00544)	0.5175 (0.00562)	0.5095 (0.00591)	0.523 (0.00802)

8 Conclusions

In this paper, we have addressed the problem of estimation of $R = P(Y < X)$ using k-upper record values from Kumarswamy distribution. It is shown how record data can be used to provide point estimation and confidence interval for R. We consider the maximum likelihood method, Bayesian method relative to symmetric and asymmetric loss functions and parametric bootstrap percentile method. The distribution of the MLE of R were used to construct exact confidence interval of R. In Bayesian approach, the posterior distribution of R is obtained in closed form and used to construct:

(i) exact Bayesian confidence interval for R.
(ii) symmetric and asymmetric Bayes point estimates of R.

Comparisons are made between the ML and the Bayes estimators based on simulation study. From results we can note that:

1. Generally, it appears that the MSE's of the Bayes estimates of R are smaller than MSE's of the ML estimates.
2. For all methods when n and m increase, the MSE's are reduce.
3. Tables (2, 3, 4 and 5) show that the Bayes estimators relative to asymmetric loss functions (LINEX) are sensitive to the value of the shape parameter c of the LINEX loss function. When c is close to zero, the MSEs of the Bayes estimators under LINEX loss function are very close to their corresponding MSEs under the squared error loss function.

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NASSER AMIN ABO- Elheggag is Assistant Professor of Mathematical Statistics at Mathematics Department Faculty of Science Sohag University Egypt. He received Ph. D. from Faculty of Science South Valley University Egypt in 1995. He received MSc from Faculty of Science, Assiut University in 1992. His areas of research where he has several Publications in the international journals and conferences include: Statistical inference, Theory of estimation, Bayesian inference, Order statistics, Records, Theory of reliability, censored.data, Life testing and Distribution theory.

Alwageh A. Farghal is a lecturer of Mathematical Statistics at Mathematics Department Faculty of Science Sohag University Egypt. He received Ph. D. from Faculty of Science Sohag University Egypt in 2016. His areas of research where he has several Publications in the international journals and conferences include: Statistical inference,Theory of estimation, Bayesian inference, Order statistics, Records, Theory of reliability,censored data, Life testing, Accelerated Life Tests and Distribution theory . He supervised for M. Sc. and Ph. D. students.

Hanan A. Ramadan is a Demonstrator at Mathematics Department Faculty of Science South Valley university, Qena, Egypt. She received Pre-Master Diploma in Mathematics (October, 2015), Maths. Department, Faculty of Science, South Valley university, Qena, Egypt. Her research interests include: Theory of reliability, Censored data, Life testing, distribution theory.