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Strongly log-Convex Functions

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Abstract: Some new concepts of the strongly log convex functions are considered in this paper. Properties of the strongly convex functions are investigated under suitable conditions. The minimum of the differentiable strongly *log*-convex functions is characterized by variational inequality, which is itself an interesting problem. Some important special cases are discussed. It is proved that the parallelogram laws for inner product spaces can be obtained as applications of strongly log-affine functions as a novel application. Results obtained in this paper can be viewed as refinement and improvement of previously known results.

Keywords: Convex functions, generalized convexity, log-convex functions, parallelogram laws, variational inequalities.

1 Introduction

Convexity theory is a branch of mathematical sciences with a wide range of applications in industry, physical, social, regional and engineering sciences. In recent years, several new generalizations of convex functions have been introduced using novel and innovative ideas to tackle difficult problems, which arise in various fields of pure and applied sciences. Exponentially convex(concave) functions can be considered as a significant extension of the convex functions and have important applications in information theory, big data analysis, machine learning and statistic, see, for example, [1–8] and the references therein. Alirazaie and Mathur [2], and Noor et al [7–9] have derived several results for these exponentially convex functions. Related to the exponentially convex functions, we have log-convex functions. It is known that log-convex functions are convex functions, but the converse is not true. For example. the function e^x is a log-convex function, but not convex. Hypergeometric functions including Gamma and Beta functions are log-convex functions, which have important applications in several branches of pure and applied sciences. A different class of exponentially convex functions was investigated by Awan et al. [10]. Noor et al [11] have considered another equivalent formulation of log-convex functions to discuss several new hidden aspects.

Strongly convex functions were introduced and studied by Polyak [12]. For the applications of strongly convex functions, see [13–18, 20–22]. Motivated and inspired by the ongoing research in generalized convexity, we consider strongly log-convex function, which is the main motivation of this paper. Several new concepts of monotonicity are introduced. We establish the relationship between these classes and derive some new results under some mild conditions. We have shown that new log-parallelogram laws can be obtained from strongly log-affine functions, which can be used to characterize the inner product spaces. This fact can be viewed itself an elegant and interesting applications of the strongly log-affine functions. As special cases, one can obtain various new and refined versions of known results. It is expected that the ideas and techniques of this paper may stimulate further research in this field.

2 Formulations and basic facts

Let K be a nonempty closed set in a real Hilbert space H. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ be the inner product and norm, respectively.

Definition 1. [15, 16, 18] *A set K in H is said to be a convex set, if*

$$u + t(v - u) \in K, \qquad \forall u, v \in K, t \in [0,1].$$

We now introduce some new classes of strongly log-convex functions and strongly log-affine functions.

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Definition 2. *A function F on the convex set K is said to be strongly* log- *convex, if there exists a constant $\mu > 0$, such that*

$$\log F(u+t(v-u)) \leq (1-t)\log F(u)+t\log F(v) -\mu t(1-t)\|v-u)\|^2, \forall u,v \in K, t\in[0,1].$$

We would like to mention that, if $\mu = 0$, then Definition 2 reduces to

Definition 3. *[11] A function F on the convex set K is said to be* log- *convex, if*

$$\log F(u+t(v-u)) \leq (1-t)\log F(u)+t\log F(v), \forall u,v \in K, t\in[0,1].$$

For the properties of log-convex functions, see Noor et al. [11].

A function F is said to strongly log-concave, if and only if, $-F$ is strongly log-convex function.
If $t=\frac{1}{2}$, then

$$\log F\left(\frac{u+v}{2}\right) \leq \frac{\log F(u)+\log F(v)}{2} -\mu\frac{1}{4}\|v-u\|^2, \forall u,v \in K.$$

The function F is said to be strongly $J-$log-convex function.

Note that if a functions is both strongly log-convex and strongly log-concave, then it is strongly log-affine function. Consequently, we have new concepts.

Definition 4. *A function F is said to be a strongly* log-*affine function, if there exists a constant $\mu > 0$, such that*

$$\log F(u+t(v-u)) = (1-t)\log F(u)+t\log F(v) -\mu t(1-t)\|v-u\|^2, \forall u,v \in K, t\in[0,1].$$

If $t=\frac{1}{2}$, then Definition 4 reduces to:

Definition 5.*A function F is called strongly quadratic* log-*equation, if there exists a constant $\mu > 0$, such that*

$$\log F\left(\frac{u+v}{2}\right) = \frac{\log F(u)+\log F(v)}{2} -\mu\frac{1}{4}\|v-u\|^2, \forall u,v \in K, t\in[0,1].$$

Definition 6. *A function F on the convex set K is said to be strongly* log-*quasiconvex function, if there exists a constant $\mu > 0$ such that*

$$\log F(u+t(v-u) \leq \max\{\log F(u),\log F(v)\} -\mu t(1-t)\|v-u\|^2, \forall u,v \in K, t\in[0,1].$$

Definition 7. *A function F on the convex set K is said to be strongly* log-*convex , if there exists a constant $\mu > 0$ such that*

$$\log F(u+t(v-u)) \leq (\log F(u))^{1-t}(\log F(v))^t -\mu t(1-t)\|v-u\|^2, \forall u,v \in K, t\in[0,1],$$

where $F(\cdot) > 0$.

From the above definitions, we have

$$\begin{aligned}
&\log F(u+t(v-u))\\
&\leq (\log F(u))^{1-t}(\log F(v))^t -\mu t(1-t)\|v-u\|^2\\
&\leq (1-t)\log F(u)+t\log F(v) -\mu t(1-t)\|v-u\|^2\\
&\leq \max\{\log F(u),\log F(v)\} -\mu t(1-t)\|v-u\|^2.
\end{aligned}$$

This shows that every strongly log-convex function is a strongly log-convex function and every strongly log-convex function is a strongly log-quasiconvex function. However, the converse is not true.

Definition 8. *An operator $T: K \to H$ is said to be:*

1. *strongly monotone, iff, there exists a constant $\alpha > 0$ such that*
$$\langle Tu-Tv, u-v\rangle \geq \alpha\|v-u\|^2, \forall u,v \in K.$$
2. *strongly pseudomonotone, iff, there exists a constant $\nu > 0$ such that*
$$\langle Tu, v-u\rangle + \nu\|v-u\|^2 \geq 0$$
$$\Rightarrow$$
$$\langle Tv, v-u\rangle \geq 0, \forall u,v \in K.$$
3. *strongly relaxed pseudomonotone, iff, there exists a constant $\mu > 0$ such that*
$$\langle Tu, v-u\rangle \geq 0$$
$$\Rightarrow$$
$$-\langle Tv, u-v\rangle + \mu\|v-u\|^2 \geq 0, \forall u,v \in K.$$

Definition 9. *A differentiable function F on the convex set K is said to be strongly* log-*pseudoconvex function, if and only if, if there exists a constant $\mu > 0$ such that*

$$\langle\frac{F'(u)}{F(u)}, v-u\rangle + \mu\|v-u\|^2 \geq 0 \Rightarrow F(v) \geq F(u), \forall u,v \in K.$$

3 Main results

In this section, we consider some basic properties of strongly log-convex functions.

Theorem 1. *Let F be a differentiable function on the convex set K. Then the function F is strongly* log-*convex function, if and only if,*

$$F(v)-F(u) \geq \langle\frac{F'(u)}{F(u)}, v-u\rangle + \mu\|v-u\|^2, \forall v,u \in K. \tag{1}$$

Proof. Let F be a strongly log-convex function on the convex set K. Then

$$F(u+t(v-u)) \leq (1-t)F(u)+tF(v) -\mu t(1-t)\|v-u\|^2, \forall u,v \in K,$$

which can be written as

$$F(v)-F(u) \geq \{\frac{F(u+t(v-u)-F(u)}{t}\} + (1-t)\mu\|v-u\|^2, \forall u,v \in K.$$

Taking the limit in the above inequality as $t \to 0,$, we have

$$F(v) - F(u) \geq \langle \frac{F'(u)}{F(u)}, v-u)\rangle + \mu\|v-u\|^2, \forall u, v \in K.$$

which is (1), the required result.

Conversely, let (1) hold. Then, $\forall u, v \in K, t \in [0,1]$, $v_t = u + t(v-u) \in K$, we have

$$\begin{aligned}\log F(v) - \log F(v_t) &\geq \langle \frac{F'(v_t)}{F(v_t)}, v - v_t)\rangle + \mu\|v - v_t)\|^2 \\ &= (1-t)\langle \frac{F'(v_t)}{F(v_t)}, v-u\rangle \\ &\quad + \mu(1-t)^2\|v-u\|^2, \forall u, v \in K. \quad (2)\end{aligned}$$

In a similar way, we have

$$\begin{aligned}\log F(u) - \log F(v_t) &\geq \langle \frac{F'(v_t)}{F(v_t)}, u - v_t)\rangle + \mu\|u - v_t)\|^2 \\ &= -t\langle \frac{F'(v_t)}{F(v_t)}, v-u\rangle + \mu t^2\|v-u\|^2. \quad (3)\end{aligned}$$

Multiplying (2) by t and (3) by $(1-t)$ and adding the resultant, we have

$$\begin{aligned}\log F(u + t\eta(v,u)) &\leq (1-t)\log F(u) + t\log F(v) \\ &\quad -\mu t(1-t)\|v-u\|^2, \forall u, v \in K,\end{aligned}$$

showing that F is a strongly log-convex function.

Theorem 2. *Let F be a differentiable strongly* log-*convex function on the convex set K. Then*

$$\langle \frac{F'(u)}{F(u)} - \frac{F'(v)}{F(v)}, u - v\rangle \geq \mu\|v-u\|^2, \quad \forall u, v \in K. \quad (4)$$

*Proof.*Let F be a differentiable strongly log-convex function on the convex set K. Then, from Theorem 1. we have

$$\log F(v) - \log F(u) \geq \langle \frac{F'(u)}{F(u)}, v-u\rangle + \mu\|v-u\|^2, \forall u, v \in K. \quad (5)$$

Changing the role of u and v in (5), we have

$$\log F(u) - \log F(v) \geq \langle \frac{F'(v)}{F(v)}, u-v)\rangle + \mu\|u-v\|^p, \forall u, v \in K. \quad (6)$$

Adding (5) and (6), we have

$$\langle \frac{F'(u)}{F(u)} - \frac{F'(v)}{F(v)}, u-v\rangle \geq 2\mu\|v-u\|^2, \forall u, v \in K. \quad (7)$$

which shows that $F'(.)$ is a strongly monotone operator.

Theorem 3.*If the differential operator $F'(.)$ of a differentiable strongly* log-*convex function F is strongly monotone operator, then*

$$\log F(v) - \log F(u) \geq \langle \frac{F'(u)}{F(u)}, v-u\rangle + \mu\|v-u\|^2, \forall u, v \in K. \quad (8)$$

*Proof.*Let F' be a strongly monotone operator. Then, from (7), we have

$$\langle \frac{F'(v)}{F(v)}, u-v\rangle \geq \langle \frac{F'(u)}{F(u)}, u-v)\rangle + \mu\|v-u\|^2. \quad \forall u, v \in K. \quad (9)$$

Since K is an convex set, $\forall u, v \in K$, $t \in [0,1]$, $v_t = u + t(v-u) \in K$. Taking $v = v_t$ in (9, we have

$$\begin{aligned}\langle \frac{F'(v_t)}{F(v_t)}, u - v_t\rangle &\leq \langle \frac{F'(u)}{F(u)}, u - v_t\rangle - 2\mu\|v_t - u\|^2 \\ &= -t\langle \frac{F'(u)}{F(u)}, v-u\rangle - 2\mu t^2\|v-u\|^2,\end{aligned}$$

which implies that

$$\langle \frac{F'(v_t)}{F(v_t)}, v-u\rangle \geq \langle \frac{F'(u)}{F(u)}, v-u\rangle + 2\mu t\|v-u\|^2. \quad (10)$$

Consider the auxiliary function

$$\zeta(t) = F(u + t(v-u), \forall u, v \in K,$$

from which, we have

$$\zeta(1) = F(v), \quad \zeta(0) = F(u).$$

Then, from (10), we have

$$\zeta'(t) = \langle \frac{F'(v_t)}{F(v_t)}, v-u\rangle \geq \langle \frac{F'(u)}{F(u)}, v-u\rangle + 2\mu t\|v-u\|^2. \quad (11)$$

Integrating (11) between 0 and 1, we have

$$\zeta(1) - \zeta(0) = \int_0^1 \zeta'(t)dt \geq \langle \frac{F'(u)}{F(u)}, v-u\rangle + \mu\|v-u\|^2.$$

Thus it follows that

$$F(v) - F(u) \geq \langle \frac{F'(u)}{F(u)}, v-u\rangle + \mu\|v-u\|^2, \forall u, v \in K,$$

which is the required (8).

Theorem 3 can be viewed as a converse of Theorem 2.

We now give a necessary condition for strongly log-pseudoconvex function.

Theorem 4. *Let $\frac{F'(u)}{F(u)}$ be a relaxed pseudomonotone operator. Then F is a strongly* log-*pseudoconnvex function.*

Proof. Let $\frac{F'(u)}{F(u)}$ be a strongly relaxed pseudomonotone operator. Then, $\forall u, v \in K$,

$$\langle \frac{F'(u)}{F(u)}, v-u\rangle \geq 0.$$

implies that

$$\langle \frac{F'(v)}{F(v)}, v-u\rangle \geq \mu\|v-u\|^2, \forall u, v \in K. \quad (12)$$

Since K is an convex set, $\forall u, v \in K$, $t \in [0,1]$, $v_t = u + t(v-u) \in K$.
Taking $v = v_t$ in (12), we have

$$\langle \frac{F'(v_t)}{F(v_t)}, v_t - u\rangle \geq \mu t^{p-1}\|v_t - u\|^2. \quad (13)$$

Consider the auxiliary function

$$\zeta(t) = \log F(u+t(v-u)) = \log F(v_t), \quad \forall u,v \in K, t \in [0,1],$$

which is differentiable. Then, using (13), we have

$$\zeta'(t) = \langle \frac{F'(v_t)}{F(v_t)}, v-u)\rangle \geq \mu t\|v-u\|^2.$$

Integrating the above relation between 0 to 1, we have

$$\zeta(1) - \zeta(0) = \int_0^1 \zeta'(t)dt \geq \frac{\mu}{2}\|\xi(v,u)\|^2,$$

that is,

$$\log F(v) - \log F(u) \geq \frac{\mu}{2}\|v-u\|^2), \forall u,v \in K,$$

showing that F is a strongly log-pseudoconvex function.

Definition 10. *A function F is said to be sharply strongly* log-*pseudoconvex, if there exists a constant $\mu > 0$ such that*

$$\langle \frac{F'(u)}{F(u)}, v-u\rangle \geq 0$$
$$\Rightarrow$$
$$\log F(v) \geq \log F(v+t(u-v)) + \mu t(1-t)\|v-u\|^2, \forall u,v \in K.$$

Theorem 5. *Let F be a sharply strongly* log-*pseudoconvex function on K with a constant $\mu > 0$. Then*

$$\langle \frac{F'(v)}{F(v)}, v-u\rangle \geq \mu\|v-u\|^2, \forall u,v \in K.$$

Proof. Let F be a sharply strongly log-pseudoconvex function on K. Then

$$\log F(v) \geq \log F(v+t(u-v)) + \mu t(1-t)\|v-u\|^2, \quad \forall u,v \in K, t \in [0,1],$$

from which, we have

$$\frac{\log F(v+t(u-v)) - \log F(v)}{t} + \mu(1-t)\|v-u\|^2 \geq 0.$$

Taking limit in the above inequality, as $t \to 0$, we have

$$\langle \frac{F'(v)}{F(v)}, v-u\rangle \geq \mu\|\xi(v,u)\|^2, \forall u,v \in K,$$

the required result.

Definition 11. *A function F is said to be a pseudoconvex function with respect to a strictly positive bifunction $B(.,.)$ such that*

$$\log F(v) < \log F(u)$$
$$\Rightarrow$$
$$\log F(u+t(v-u)) < \log F(u) + t(t-1)B(v,u),$$
$$\forall u,v \in K, t \in [0,1].$$

Theorem 6. *If the function F is strongly* log-*convex function such that $\log F(v) < \log F(u)$, then F is strongly* log-*pseudoconvex function.*

Proof. Since $\log F(v) < \log F(u)$ and F is strongly log-convex function, then
$\forall u,v \in K, \quad t \in [0,1]$, we have

$$\begin{aligned}
&\log F(u+t(v-u)) \\
&\leq \log F(u) + t(\log F(v) - \log F(u)) - \mu t(1-t)\|v-u\|^2 \\
&< \log F(u) + t(a-t)(\log F(v) - \log F(u)) - \mu t(1-t)\|v-u\|^2 \\
&= \log F(u) + t(t-1)(\log F(u) - \log F(v)) - \mu t(1-t)\|v-u\|^2 \\
&< F(u) + t(t-1)B(u,v) - \mu t(1-t)\|v-u\|^2, \forall u,v \in K,
\end{aligned}$$

where $B(u,v) = \log F(u) - \log F(v) > 0$, which is the required result.

We now discuss the optimality for the differentiable strongly log-convex functions, which is the main motivation of our next result.

Theorem 7. *Let F be a differentiable strongly* log-*convex function with modulus $\mu > 0$. If $u \in K$ is the minimum of the function F, then*

$$\log F(v) - \log F(u) \geq \mu\|v-u\|^2, \quad \forall u,v \in K. \tag{14}$$

Proof. Let $u \in K$ be a minimum of the function F. Then

$$F(u) \leq F(v), \forall v \in K,$$

from which, it follows that

$$\log F(u) \leq \log F(v), \forall v \in K, \tag{15}$$

Since K is a convex set, so, $\forall u,v \in K, \quad t \in [0,1]$,

$$v_t = (1-t)u + tv \in K.$$

Taking $v = v_t$ in (15), we have

$$0 \leq \lim_{t \to 0}\{\frac{\log F(u+t(v-u)) - \log F(u)}{t}\} = \langle \frac{F'(u)}{F(u)}, v-u\rangle. \tag{16}$$

Since F is differentiable strongly log-convex function, so

$$\log F(u+t(v-u)) \leq \log F(u) + t(F(v) - F(u)) - \mu t(1-t)\|v-u\|^2, \forall u,v \in K,$$

from which, using (16), we have

$$\begin{aligned}
\log F(v) - \log F(u) &\geq \lim_{t \to 0}\{\frac{\log F(u+t(v-u)) - \log F(u)}{t}\} \\
&\quad + \mu\|v-u\|^2 \\
&= \langle \frac{F'(u)}{F(u)}, v-u\rangle + \mu\|v-u\|^2,
\end{aligned}$$

the required result (14).

Remark. We would like to mention that, if $u \in K$ satisfies the inequality

$$\langle (\log F(u))', v-u\rangle = \langle \frac{F'(u)}{F(u)}, v-u\rangle + \mu\|v-u\|^2 \geq 0, \quad \forall u,v \in K, \tag{17}$$

then $u \in K$ is the minimum of the function F. The inequality (17) is called the log-variational inequality. It is an interesting problem for future research

Theorem 8. *Let f be a strongly* log-*affine function. Then F is a strongly convex function, if and only if, $H = F - f$ is a convex function.*

Proof. Let f be a strongly log-affine function, Then

$$\log f((1-t)u+tv) = (1-t)\log f(u) + t\log f(v) - \mu t(1-t)\|v-u\|^2, \forall u,v \in K. \quad (18)$$

From the higher order strongly convexity of F, we have

$$\log F((1-t)u+tv) \leq (1-t)\log F(u) + t\log F(v) - \mu t(1-t)\|v-u\|^2, \forall u,v \in K.$$

From (18) and (19), we have

$$\log F((1-t)u+tv) - \log f((1-t)u+tv) \leq (1-t)(\log F(u) - \log f(u)) + t(\log F(v) - \log f(v)), \quad (19)$$

from which it follows that

$$\begin{aligned} &\log H((1-t)u+tv) \\ &= \log F((1-t)u+tv) - \log f((1-t)u+tv) \\ &\leq (1-t)\log F(u) + t\log F(v) - (1-t)\log f(u) - t\log f(v) \\ &= (1-t)(\log F(u) - \log f(u)) + t(\log F(v) - \log f(v)), \end{aligned}$$

which show that $H = F - f$ is a log-convex function.The inverse implication is obvious.

It is worth mentioning that the strongly log-convex function is also strongly Wright log-convex function. From the definition 2, we have

$$\begin{aligned} &\log F(u+t(v-u)) + \log F(v+t(u-v)) \\ &= \log F(u+t(v-u))F(v+t(u-v)) \\ &\leq \log F(u) + \log F(v) - 2\mu t(1-t)\|v-u\|^2 \\ &= \log F(u)F(v) - 2\mu t(1-t)\|v-u\|^2 \end{aligned}$$

Consequently, we have

$$\begin{aligned} &F(u+t(v-u))F(v+t(u-v)) \\ &= F(u)F(v) exp\{-2\mu t(1-t)\|v-u\|^2\}, \end{aligned}$$

which is called multiplicative strongly Wright log-convex function. One can studies the properties and applications of the strongly Wright log-convex functions in optimization and mathematical programming.

4 Applications

In this section, we show that the characterizations of uniformly Banach spaces involving the notion of strongly log-affine are given.

From Definition 4, we have

$$\begin{aligned} &\|\log F(u+t(v-u))\|^2 \\ &= (1-t)\|\log F(u)\|^2 + t\|\log F(v)\|^2 \\ &- \mu t(1-t)\|v-u\|^2, \forall u,v \in K, t \in [0,1]. \end{aligned} \quad (20)$$

Taking $t = \frac{1}{2}$ in (20), we have

$$\|\log F(\frac{u+v}{2})\|^2 + \mu\frac{1}{4}\|v-u\|^2 = \frac{1}{2}\|\log F(u)\|^2 + \frac{1}{2}\|\log F(v)\|^2, \forall u,v \in K,$$

which is called the log-parallelogram for the inner product spaces.

Setting $\log F(u) = \|u\|^2$ in (21), we obtain

$$\|u+v\|^2 + \mu\|v-u\|^2 = 2\{\|u\|^2 + \|v\|^2\}, \forall u,v \in K, \quad (21)$$

which is well known parallelogram for the inner product spaces.

5 Conclusion

In this paper, we have introduced and studied a new class of convex functions, which is called strongly log-convex function. It is shown that several new classes of log-convex functions can be obtained as special cases of these strongly log-convex functions. We have studied the basic properties of these functions. It is shown that optimality conditions of the differentiable strongly lg-convex functions can be characterized by a class of variational inequalities, which is called *log* variational inequality. To investigate the applications and approximate solutions of *log* variational inequalities is an open interesting problem for future research. We have shown that one can derive a new log-parallelogram laws from strongly log-affine functions. These log-parallelogram laws can be used to characterize the inner product spaces. The interested readers may explore the applications and other properties of the strongly log-convex functions in various fields of pure and applied sciences. This is an interesting direction of future research.

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Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

References

[1] S. N. Bernstein, Sur les fonctions absolument monotones, *Acta Math.* **52**, 1-66(1929).
[2] G. Alirezaei and R. Mazhar, On exponentially concave functions and their impact in information theory, *J. Inform. Theory Appl.* **9(5)**, 265-274(2018).

[3] T. Antczak, On (p,r)-invex sets and functions, *J. Math. Anal. Appl.* **263**, 355-379(2001).
[4] M. Avriel, r-Convex functions. *Math. Program.,* **2**, 309-323(1972).
[5] S.Pal and T. K. Wong, On exponentially concave functions and a new information geometry, *Annals. Prob. 46(2)*, 1070-1113(2018).
[6] Y. X. Zhao, S. Y. Wang and L. Coladas Uria, Characterizations of r-Convex Func-tions, *J Optim. Theory Appl.* **145**, 186-195(2010).
[7] M. A. Noor and K. I. Noor, Strongly exponentially convex functions, *U.P.B. Bull Sci. Appl. Math. Series A,* **81(4)**. 75-84(2019).
[8] M. A. Noor and K. I. Noor, Strongly exponentially convex functions and their properties, *J. Adv. Math. Stud.,* **9(2)**, 177-185(2019).
[9] M. A. Noor and K. I. Noor, On exponentially convexc functions, *J. Orisa Math. Soc.* **38(1-2)**, 33-51(2019).
[10] M. U. Awan, M. A. Noor and K. I. Noor, Hermite-Hadamard inequalities for exponentially convex functions, *Appl. Math. Inform. Sci.,* **12(2)**, 405-409(2018).
[11] M. A. Noor, K. I. Noor and M. U. Awan, New prospective of log-convex functions, *Appl. Math. Inform. Sci.* **14(5)**, (2020).
[12] B. T. Polyak, Existence theorems and convergence of minimizing sequences in extremum problems with restrictions, *Soviet Math. Dokl.,* **7**, 2-75(1966).
[13] M. A. Noor and K. I. Noor, On generalized strongly convex functions involving bifunction, *Appl. Math. Inform. Sci.,* **13(3)**, 411-416(2019).
[14] B. B. Mohsen, M. A. Noor, K.I. Noor and M. Postolache, Strongly convex functions of higher order involving bifunction, *Mathematics,* **7(11)**(2019):1028.
[15] G. Cristescu, L. Lupsa, *Non-Connected Convexities and Applications,* Kluwer Academic Publisher, Dordrechet, 2002.
[16] C. F. Niculescu and L. E. Persson, *Convex Functions and Their Applications.* Springer-Verlag, New York, 2006
[17] C. F. Niculescu and L. E. Persson, *Convex Functions and Their Applications.* Springer-Verlag, New York, 2018
[18] J. Pecaric, F. Proschan and Y. L. Tong, *Convex Functions, Partial Orderings and Statistical Applications,* Academic Press, New York, 1992
[19] M. A. Noor, New approximation schemes for general variational inequalities, *J. Math. Anal. Appl.* **251**, 217-229(2000).
[20] M. A. Noor, Some developments in general variational inequalities, *Appl. Math. Comput.* **152**, 199-277 (2004).
[21] D. L. Zu and P. Marcotte, Co-coercivity and its role in the convergence of iterative schemes for solving variational inequalities. *SIAM Journal on Optimization,* **6(3)**, 714-726(1996).
[22] M. A. Noor and K. I. Noor, Higher order general convex functions and general variational inequalities, *Canadian J. Appl. Math.* **3(1)**, 1-17(2021).