

Statistical Properties and Estimation of Inverted Topp-Leone Distribution

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Received: 30 Jul. 2019, Revised: 21 Feb. 2020, Accepted: 26 Feb. 2020.

Published online: 1 Jul. 2020.

Abstract: A new probability distribution, i.e. the inverted Topp-Leone distribution is proposed. Some of its statistical properties such as; quantile function, mode, moments, probability weighted moments, incomplete moments, stress-strength model, moments of residual life function, Rényi entropy, and stochastic ordering are provided. Maximum likelihood estimation method based on complete, Type I, and Type II censored samples is considered. Simulation issues are provided to assess the results of the study. Moreover, the results are applied to a real data set.

Keywords: Topp-Leone distribution, maximum likelihood estimation, Type I censored samples, Type II censored samples.

1 Introduction

The inverted (or inverse) distributions are beneficial to explore additional properties of the phenomenon. The applicability of inverted distributions can be found in studies based on econometrics, biological and engineering sciences, survey sampling, medical applications and life testing problems.

Considerable researchers addressed an inverted distributions and their applications; for examples, the inverse Weibull distribution was introduced by Keller and Kamath [1], Abd EL-Kader et al. [2] proposed inverted Pareto Type I distribution and investigated its properties, inverted Pareto Type II distribution was proposed by AL-Dayian[3], Aljuaid[4] discussed estimation of the parameters of exponentiated inverted Weibull distribution, inverse Lindley distribution was handled by Sharma et al.[5], Abd AL-Fattah et al. [6] introduced inverted Kumaraswamy distribution and explored its properties, Tahir et al. [7] introduced inverted Nadarajah-Haghighi distribution, Hassan and Abd-Allah [8] investigated inverse power Lomax distribution, and Hassan and Mohamed [9] addressed inverted exponentiated Lomax distribution.

The Topp-Leone (TL) distribution with finite support has been proposed by Topp and Leone [10] as an attractive model in reliability studies. TL distribution has J-shaped in its density function and a bathtub shaped in its hazard function. Due to the importance of TL distribution, various authors have conducted relevant studies.

The probability density function (pdf) of the TL distribution is given by:

$$g(t; \theta) = 2\theta t^{\theta-1}(1-t)(2-t)^{\theta-1}, \quad 0 \leq t \leq 1, \theta > 0, \quad (1)$$

and the cumulative distribution function (cdf) related to (1) is given by

$$G(t; \theta) = t^\theta (2-t)^\theta. \quad (2)$$

Due to the importance of inverted distributions we provide an inverse form of the TL distribution defined on the domain $(0, \infty)$, study its properties and discuss its parameter estimation based on censored samples.

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Definition: A random variable X is said to have the inverted TL (ITL) distribution if we use the transformation $X = \frac{1}{T} - 1$, where T has a TL distribution with pdf (1). The cdf of X has the ITL distribution with shape parameter $\theta > 0$ denoted by $X \sim \text{ITL}(\theta)$ and is defined by:

$$F_{ITL}(x; \theta) = 1 - \left\{ \frac{(1+2x)^\theta}{(1+x)^{2\theta}} \right\}; \quad x \geq 0, \theta > 0. \tag{3}$$

Using the transformation $X = \frac{1}{T} - 1$ is better than using the transformation $X = \frac{1}{T}$ because it lets the pdf and its domain to be more flexible to model positive real data. The pdf related to (3) is given by:

$$f_{ITL}(x; \theta) = 2\theta x (1+x)^{-2\theta-1} (1+2x)^{\theta-1}; \quad x, \theta > 0. \tag{4}$$

The survival function and hazard rate function (hrf) of the ITL distribution are

$$S_{ITL}(x; \theta) = (1+x)^{-2\theta} (1+2x)^\theta,$$

and,

$$h_{ITL}(x; \theta) = 2\theta x [(1+x)(1+2x)]^{-1}.$$

The pdf curves show that the ITL distribution has a long right tail (see Fig.1). The hrf plots of the ITL distribution at dissimilar values of the parameter values are shown in Fig. 2. From the graph, we observe that the value of the hrf increases at the initial stage and approaches zero as x gets larger.

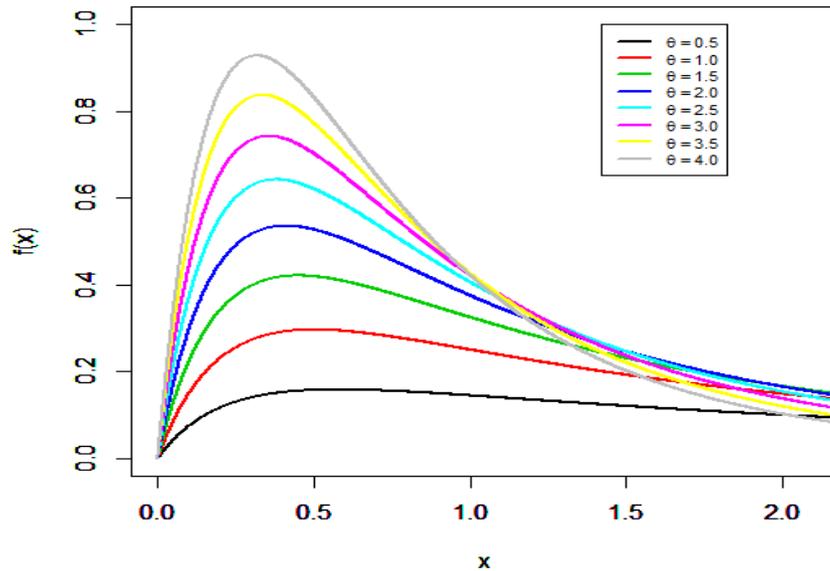


Fig.1: Plots of the pdf of the ITL distribution at various values of θ

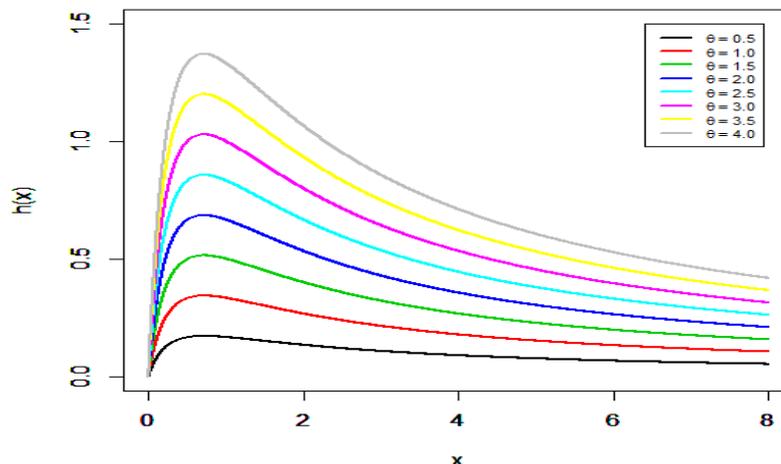


Fig.2: Plots of the hrf of the ITL distribution at various values of θ

The present paper is organized as follows: The essential properties of the ITL distribution are derived in Section 2. Section 3 gives maximum likelihood (ML) estimators which are discussed via complete, Type I and Type II censored samples. Simulation study is addressed in Section 4 to clarify the application forms of the separate results. Section 5 involves application to a real data set. The last section is devoted to conclusion.

2 Main Properties of the ITL Distribution

Here, we provide some main properties of the ITL distribution including, quantile function, mode, probability weighted moments, Rènyi entropy, moments, stress- strength model, incomplete moments, residual life function and stochastic ordering.

2.1 Quantile Function and Mode

The quantile function of the ITL, say $x = Q(x) = F^{-1}(u)$ is derived by inverting (3) as follows:

$$x = Q(x) = F^{-1}(u), \quad 0 < u < 1.$$

Solving $F(x) = u$ for x gives the u^{th} quantile for the ITL random variable as follows

$$x_u = \frac{-2((1-u)^{\frac{1}{\theta}} - 1) + \sqrt{4((1-u)^{\frac{1}{\theta}} - 1)^2 - 4(1-u)^{\frac{1}{\theta}}((1-u)^{\frac{1}{\theta}} - 1)}}{2(1-u)^{\frac{1}{\theta}}}. \quad (5)$$

In particular, the first quartile, say Q_1 , the second quartile, say Q_2 , and the third quartile, say Q_3 are obtained by setting $u=0.25, 0.5, 0.75$, respectively, in (5).

Furthermore, we consider the density function of the ITL distribution given in (4). The mode of the ITL is the maxima of the pdf and it is obtained by equating $df_{ITL}(x; \theta)/dx$ with zero, as follows:

$$\frac{df_{ITL}(x; \theta)}{dx} = 2\theta(1+x)^{-2\theta-2}(1+2x)^{\theta-2}(1-2\theta x^2 - 2x^2) = 0,$$

then solving this equation for x , we obtain the mode value as $x_{mode} = 1/\sqrt{2(\theta + 1)}$.

2.2 Moments

Moments have an important role in any statistical analysis. They can be used to describe important characteristics and shapes of distribution, such as spread and dispersion measured by mean and variance as well as peakedness of the distribution measured by kurtosis, and to investigate the symmetry of the shape of the distribution measured by skewness.

The m^{th} moment about the origin of random variable X has the ITL model is obtained from pdf (4) as follows:

$$\begin{aligned} \mu'_m &= E(X^m) = 2\theta \int_0^\infty x^{m+1} (1+x)^{-2\theta-1} (1+2x)^{\theta-1} dx \\ &= 2\theta \int_0^\infty x^{m+1} (1+x)^{-2\theta-1} (1+x+x)^{\theta-1} dx \\ &= 2\theta \int_0^\infty x^{m+1} (1+x)^{-\theta-2} [1+(x/1+x)]^{\theta-1} dx. \end{aligned} \tag{6}$$

The generalized binomial expansion, for $\theta > 0$ is real non integer and $|z| < 1$ is

$$(1+z)^{\theta-1} = \sum_{j=0}^\infty \binom{\theta-1}{j} z^j. \tag{7}$$

Employing, the binomial theorem (7) in (6), where θ is real non integer, we have

$$\mu'_m = \sum_{j=0}^\infty 2\theta \binom{\theta-1}{j} \beta(m+j+2, \theta-m); \quad \theta > m, m=1,2,3,\dots$$

where $\beta(.,.)$ is the beta function. Furthermore, the m^{th} central moment of X is given by

$$\mu_m = E(X - \mu'_1)^m = \sum_{i=0}^m \binom{m}{i} (-\mu'_1)^i (\mu'_{m-i}).$$

In particular, the measure of symmetry is the skewness which describes the symmetry of the distribution and defined as

$$SK = \frac{\mu'_3 - 3\mu'_2\mu'_1 + 2\mu_1^3}{[\mu'_2 - \mu_1^2]^{\frac{3}{2}}}.$$

The measure of peakedness is the kurtosis which describes the peakedness of the distribution and defined as follows:

$$KU = \frac{\mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu_1^2 - 3\mu_1^4}{[\mu'_2 - \mu_1^2]^2}.$$

Table1 illustrates, mean, variance, skewness and kurtosis of the ITL distribution for some values of θ . It shows that both values of the mean and variance of the ITL distribution decrease as the values of θ increase. It is also noticeable that both the skewness and the kurtosis are decreasing functions of θ .

Table 1. Mean variance, skewness, and kurtosis of ITL distribution for various values of θ

θ	Mean	Variance	SK	KU
4.5	0.838	0.703	4.574	108.697
5	0.758	0.517	3.804	52.153
6.5	0.601	0.262	2.756	21.179
7	0.564	0.22	2.569	18.04

2.3 Probability Weighted Moments

The probability weighted moments (PWM) are frequently utilized to obtain estimators of the parameters and quantiles of distributions (see Greenwood et al. [11]). The PWM of the ITL distribution is given by

$$\zeta_{r,s} = \int_0^\infty x^r f_{ITL}(x; \theta) [F_{ITL}(x; \theta)]^s dx, \tag{8}$$

where, r and s are positive integers. To obtain $\zeta_{r,s}$ of the ITL distribution, we use binomial expansion for $[F_{ITL}(x; \theta)]^s$ as follows:

$$[F_{ITL}(x; \theta)]^s = \sum_{w=0}^s (-1)^w \binom{s}{w} (1+x)^{-\theta w} [1 + (x/1+x)]^{\theta w}. \tag{9}$$

The PWM of the ITL distribution is obtained by substituting Equations (4) and (9) into (8) as follows:

$$\zeta_{r,s} = 2\theta \sum_{w=0}^s (-1)^w \binom{s}{w} \int_0^\infty x^{r+1} (1+x)^{-\theta-\theta w-2} [1 + (x/1+x)]^{\theta+\theta w-1} dx. \tag{10}$$

Then, applying the binomial theorem (7) in (10), we obtain

$$\zeta_{r,s} = 2\theta \sum_{l=0}^\infty \sum_{w=0}^s (-1)^w \binom{s}{w} \binom{\theta + \theta w - 1}{l} \beta(r+l+2, \theta w + \theta - r).$$

2.4 Incomplete Moments

Incomplete moments of a distribution are used to measure inequality. For example, the main application of the first incomplete moment refers to the Bonferroni and Lorenz curves. The m^{th} incomplete moment, say, $\omega_m(t)$ is obtained as follows:

$$\omega_m(t) = \int_0^t x^m f_{ITL}(x; \theta) dx = 2\theta \sum_{b=0}^\infty \binom{\theta-1}{b} \beta(m+b+2, \theta-m, t/1+t),$$

where $\beta(m+b+2, \theta-m, t/1+t)$ is the incomplete beta function.

In addition, for lifetime models, the m^{th} conditional moment of the ITL distribution is obtained as follows:

$$E(X^m | X > t) = E(X^m) - 2\theta \sum_{b=0}^\infty \binom{\theta-1}{b} \beta(m+b+2, \theta-m, t/1+t).$$

2.5 Moments of Residual Life

Given that a component survives up to time $t \geq 0$, the residual life is the period beyond until the time of failure, and defined by the conditional random variable $X - t | X > t$. Therefore, the n^{th} moment of the residual lifetime, of the ITL distribution is given by

$$\begin{aligned} \pi_n(t) &= \frac{1}{S_{ITL}(t; \theta)} \int_t^\infty (x-t)^n f_{ITL}(x; \theta) dx \\ &= \frac{2\theta}{S_{ITL}(t; \theta)} \sum_{d=0}^n \sum_{j=0}^\infty (-1)^{n-d} \binom{n}{d} \binom{\theta-1}{j} t^{n-d} \beta(\theta-d, d+j+2, 1/1+t). \end{aligned}$$

The mean residual life (MRL) is an important application of the moments of residual lifetime function. The MRL function represents the expected additional life length for an item which is alive at age t . Hence, for $n=1$, $\pi_1(t)$ is the MRL of the ITL distribution.

2.6 Rènyi Entropy

Entropy usually measures variation or uncertainty of a random variable X . Also, it measures the randomness of systems. Entropy is extensively applied in physics and molecular imaging of tumors. The Rènyi entropy of order δ , where $\delta > 0$ and $\delta \neq 1$, for the ITL distribution is derived as follows:

$$\begin{aligned} En_{\delta}(X) &= (1 - \delta)^{-1} \log \left\{ \int_0^{\infty} f_{ITL}(x; \theta)^{\delta} dx \right\} \\ &= (1 - \delta)^{-1} \log \left\{ \int_0^{\infty} (2\theta x)^{\delta} (1+x)^{-\theta\delta-2\delta} \left(1 + \frac{x}{1+x}\right)^{\theta\delta-\delta} dx \right\}. \end{aligned}$$

Using the binomial expansion, we obtain

$$En_{\delta}(X) = \frac{1}{1 - \delta} \log \sum_{z=0}^{\infty} (2\theta)^{\delta} \binom{\theta\delta - \delta}{z} \beta(\delta+z+1, \theta\delta+\delta-1).$$

2.7 Stochastic Ordering

A widely studied concept between probability distributions is the stochastic ordering which is an important tool in reliability theory and other fields to study comparative behavior between random variables. Suppose that X_i has the ITL distribution with parameter θ_i for $i = 1, 2$. Let $F_i(x; \theta)$ denote the cumulative distribution function and $f_i(x; \theta)$ denote the probability density function of X_i . We could say that X_1 is stochastically smaller than X_2 with respect to likelihood ratio order (denoted by $X_1 \leq_{lr} X_2$), if $f_1(x; \theta_1)/f_2(x; \theta_2)$ is a decreasing function for all values of x .

Let $X_1 \sim ITL(\theta_1)$ and $X_2 \sim ITL(\theta_2)$, here we will show that the ITL distributions are ordered with respect to likelihood ratio ordering when appropriate assumptions are satisfied. The density ratio is

$$\frac{f_1(x; \theta_1)}{f_2(x; \theta_2)} = (\theta_1/\theta_2)(1+x)^{2(\theta_2-\theta_1)}(1+2x)^{\theta_1-\theta_2},$$

then

$$\frac{d}{dx} \log \left[\frac{f_1(x; \theta_1)}{f_2(x; \theta_2)} \right] = \frac{2x(\theta_2 - \theta_1)}{(1+x)(1+2x)}.$$

If $\theta_1 > \theta_2$, we obtain $\frac{d}{dx} \left[\frac{f_1(x; \theta_1)}{f_2(x; \theta_2)} \right] < 0$ for all $x \geq 0$. Thus $\frac{d}{dx} \left[\frac{f_1(x; \theta_1)}{f_2(x; \theta_2)} \right]$ is decreasing in x and hence $X_1 \leq_{lr} X_2$.

Moreover, X_1 is said to be smaller than X_2 in other different orderings as stochastic order (denoted by $X_1 \leq_{st} X_2$), hazard rate order (denoted by $X_1 \leq_{hr} X_2$), and reversed hazard rate order (denoted by $X_1 \leq_{rhr} X_2$). These stochastic orders are related to each other as shown by Shaked and Shanthikumar [12] and the following implications hold:

$$X_1 \leq_{rhr} X \Leftarrow X_1 \leq_{lr} X_2 \Rightarrow X_1 \leq_{hr} X_2 \Rightarrow X_1 \leq_{st} X_2.$$

2.8 Stress Strength Reliability

Stress-strength (S-S) model is extensively used in reliability estimation. S-S model has many applications in physics and engineering including strength failure, structures, deterioration of rocket motors, and static fatigue of ceramic component. In the S-S model, reliability R measures the reliability of the component that has strength X_1 when it is subjected to random stress X_2 . The component fails if the applied stress exceeds its strength, then $R = P(X_2 < X_1)$. Let X_1 and X_2 be two independent random variables with ITL (θ_1) and ITL (θ_2) distributions respectively. Then, the S-S reliability of the ITL distribution is obtained as

$$R = \int_0^\infty f_{ITL_1}(x; \theta_1) F_{ITL_2}(x; \theta_2) dx. \tag{11}$$

Substituting (3) and (4) in (11), then R is as follows:

$$R = 1 - \sum_{a=0}^\infty 2 \theta_1 \binom{\theta_1 + \theta_2 - 1}{a} \beta(a+2, \theta_1 + \theta_2).$$

3 Parameter Estimation

In life testing and reliability studies, one may not continue the experiment until the last failure since the waiting time for the final failure is unbounded (see Muenz and Green [13]). The resulting data from such experiments are referred to as censored data. Type I censoring (TIC) is carried out when the study stops after a prefixed time period τ occurs. Type II censoring (TIIC) is conducted when the study stops after a prefixed failure k occurs. In TIC, the experimental time τ is fixed, but the number of failures is a random variable. While in TIIC, the number of failures k is fixed, but the experimental time is a random variable.

In this section, estimator of the ITL model parameter is obtained using the ML estimation method under complete, TIC, and TIIC schemes.

3.1 ML Estimator Based on Complete Sample

Suppose that X_1, X_2, \dots, X_n is a simple random sample drawn from the ITL distribution with pdf (4), the log-likelihood of a sample of size n of the ITL distribution is given by:

$$\ln l = n \ln 2 + n \ln \theta + \sum_{i=1}^n \ln(x_i) + (-2\theta - 1) \sum_{i=1}^n \ln(1 + x_i) + (\theta - 1) \sum_{i=1}^n \ln(1 + 2x_i).$$

The derivative of the log-likelihood function with respect to θ is given, as follows:

$$\frac{d \ln l}{d \theta} = \frac{n}{\theta} - 2 \sum_{i=1}^n \ln(1 + x_i) + \sum_{i=1}^n \ln(1 + 2x_i).$$

Then the ML estimator of θ , say $\hat{\theta}$, is obtained by equating first derivative of the log-likelihood function with zero as follows:

$$\hat{\theta} = \frac{n}{2 \sum_{i=1}^n \ln(1 + x_i) - \sum_{i=1}^n \ln(1 + 2x_i)}.$$

3.2 ML Estimator Based on TIC

We assume that the lifetime of n items from ITL distribution is put on a test, while the test ends at fixed time τ , before all n items have failed. The log-likelihood function for the ITL distribution based on TIC is as follows:

$$\begin{aligned} \ln l_1 = & \ln(\Gamma(n+1)/\Gamma(n-k+1)) + k \ln(2\theta) + \sum_{i=1}^k \ln x_{(i)} - (2\theta+1) \sum_{i=1}^k \ln(1+x_{(i)}) + \\ & (\theta-1) \sum_{i=1}^k \ln(1+2x_{(i)}) - 2\theta(n-k) \ln(1+\tau) + \theta(n-k) \ln(1+2\tau). \end{aligned}$$

The derivative of the log-likelihood function with respect to the unknown parameter is given, as follows:

$$\frac{d \ln l_1}{d \theta} = \frac{k}{\theta} - 2 \sum_{i=1}^k \ln (1+x_{(i)}) + \sum_{i=1}^k \ln (1+2x_{(i)}) - 2(n-k) \ln(1+\tau) + (n-k) \ln(1+2\tau).$$

Then the ML estimator of θ , say $\hat{\theta}_1$, is obtained by equating the first derivative of the log-likelihood function with zero as follows:

$$\hat{\theta}_1 = \frac{k}{2 \sum_{i=1}^k \ln (1+x_{(i)}) - \sum_{i=1}^k \ln (1+2x_{(i)}) + 2(n-k) \ln(1+\tau) - (n-k) \ln(1+2\tau)}.$$

3.3 ML Estimator Based on TIIC

We assume that the lifetime of n items from ITL distribution puts on a test, while the test ends at fixed value of failures k . Here, the number of failures observed is fixed and duration of the life test is random in contrast to TIC. Let $X_{(1)} < X_{(2)} < \dots < X_{(k)}$ be the observed lifetimes of the first k failed items. The log-likelihood function, via TIIC scheme, will be as follows:

$$\begin{aligned} \ln l_2 = & \ln(\Gamma(n+1)/\Gamma(n-k+1)) + k \ln(2\theta) + \sum_{i=1}^k \ln x_{(i)} - (2\theta+1) \sum_{i=1}^k \ln (1+x_{(i)}) \\ & + (\theta-1) \sum_{i=1}^k \ln (1+2x_{(i)}) - 2\theta(n-k) \ln(1+x_{(k)}) + \theta(n-k) \ln(1+2x_{(k)}). \end{aligned}$$

Based on TIIC, the ML estimator of θ , say $\hat{\theta}_2$, is obtained by equating the first derivative of the log-likelihood function with zero as follows:

$$\hat{\theta}_2 = \frac{k}{2 \sum_{i=1}^k \ln (1+x_{(i)}) - \sum_{i=1}^k \ln(1+2x_{(i)}) + 2(n-k) \ln(1+x_{(k)}) - (n-k) \ln(1+2x_{(k)})}.$$

If $k=n$, the TIIC is corresponding to complete samples.

4 Simulation Study

Here, simulation studies are discussed to define the efficiency of the ML estimates for distinct sample size. We generate $N = 10000$ random samples of size $n = 50, 100$, and 500 from the ITL distribution for specific values of the parameter $\theta = 0.5, 1, 1.5, 2.5$, and 3 . In TIC, two ended times are picked as $\tau = 5$ and 15 . Also, in case of the TIIC, the numbers of failure times (k) are picked using the following censoring levels; 70%, 90% and 100% (complete sample). The ML estimates of the parameter are obtained in complete, TIC and TIIC. The biases of the estimated parameters and their mean square errors (MSEs) are computed. The numerical results are obtained via MATHCAD 14, and listed in Tables 2 and 3 for the whole sampling schemes.

Table 2: ML estimates, biases, and MSEs for ITL based on TIC

Sample size (<i>n</i>)	Actual (θ)	τ	$\hat{\theta}$	Bias ($\hat{\theta}$)	MSE($\hat{\theta}$)	
50	0.5	5	0.3906	-0.1093	0.0165	
		15	0.4540	-0.0459	0.0074	
	1	5	0.7879	-0.2120	0.0558	
		15	0.9698	-0.0301	0.0168	
	1.5	5	1.151	-0.3480	0.138	
		15	1.3891	-0.11086	0.0326	
	2.5	5	1.7083	-0.7916	0.6465	
		15	1.8776	-0.6223	0.4009	
	3	5	1.901	-1.098	1.224	
		15	2.0169	-0.9830	0.9773	
	100	0.5	5	0.3909	-0.1090	0.0142
			15	0.4513	-0.0486	0.0050
1		5	0.7884	-0.2115	0.0504	
		15	0.9653	-0.0346	0.0093	
1.5		5	1.1496	-0.3503	0.1310	
		15	1.3850	-0.1149	0.0233	
2.5		5	1.7014	-0.7985	0.6475	
		15	1.8733	-0.6269	0.3994	
3		5	1.8969	-1.1030	1.2259	
		15	2.0157	-0.9842	0.9744	
500		0.5	5	0.3903	-0.1096	0.0124
			15	0.4495	-0.0504	0.0030
	1	5	0.7860	-0.2139	0.0468	
		15	0.9610	-0.0389	0.0030	
	1.5	5	1.1468	-0.3531	0.1264	
		15	1.3807	-0.1192	0.0163	
	2.5	5	1.700	-0.7998	0.6417	
		15	1.8699	-0.6300	0.3983	
	3	5	1.8949	-1.1050	1.2229	
		15	2.0129	-0.9870	0.9754	

Table 3: ML estimates, biases and MSEs for ITL based on complete and THIC (* Indicate that the value multiply 10^{-4})

Sample size (n)	Actual (θ)	k	$\hat{\theta}$	Bias ($\hat{\theta}$)	MSE($\hat{\theta}$)
50	0.5	50	0.506	0.006358	0.4043*
		35	0.512	0.012	1.346*
		45	0.507	0.007499	0.5623*
	1	50	1.013	0.013	1.617*
		35	1.023	0.023	5.383*
		45	1.015	0.015	2.249*
	1.5	50	1.519	0.019	3.638*
		35	1.535	0.035	0.1211
		45	1.522	0.022	5.061*
	2.5	50	2.532	0.032	0.1011
		35	2.558	0.058	0.3364
		45	2.537	0.037	0.1406
3	50	3.038	0.038	0.1455	
	35	3.07	0.07	0.4845	
	45	3.045	0.045	0.2024	
100	0.5	100	0.505	0.004554	0.2074*
		70	0.506	0.00619	0.3832*
		90	0.506	0.005526	0.3054*
	1	100	1.009	0.009108	0.8296*
		70	1.012	0.012	1.533*
		90	1.011	0.011	1.222*
	1.5	100	1.514	0.014	1.867*
		70	1.519	0.019	3.448*
		90	1.517	0.017	2.749*
	2.5	100	2.523	0.023	5.185*
		70	2.531	0.031	9.579*
		90	2.528	0.028	7.636*
3	100	3.027	0.027	7.466*	
	70	3.037	0.037	0.1379	
	90	3.033	0.033	0.011	
500	0.5	500	0.5	0.0004423	0.001957*
		70	0.501	0.001155	0.01335*
		90	0.501	0.0008514	0.007248*
	1	500	1.001	0.0008847	0.007826*
		350	1.002	0.002311	0.0534*
		450	1.002	0.001703	0.02899*
	1.5	500	1.501	0.001327	0.01761*
		350	1.503	0.003466	0.1201*
		450	1.503	0.002554	0.06523*
	2.5	500	2.502	0.002212	0.04891*
		350	2.506	0.005777	0.3337*
		450	2.504	0.004257	0.1812*
3	500	3.003	0.002654	0.07044*	
	350	3.007	0.006932	0.4806*	
	450	3.005	0.005108	0.2609*	

5 Applications to Real Data

The importance of the ITL model is clarified via analysis to one data set. The failure times of 50 devices data set are considered (see Aarset [14]). We fit the ITL distribution and other three competing models, namely; an inverse Lindley (IL), inverse Rayleigh distribution (IR) and inverse exponential (IE) distributions. The probability densities of IL, IR and IE distributions are given, respectively, by:

$$f_{IL}(x; \theta) = \frac{\theta^2(1+x)}{(\theta+1)x^3} e^{-\frac{\theta}{x}}; x, \theta > 0, \quad f_{IR}(x; \theta) = \frac{2\theta}{x^3} e^{-\frac{\theta}{x^2}}; x, \theta > 0,$$

and

$$f_{IE}(x; \theta) = \frac{\theta}{x^2} e^{-\frac{\theta}{x}}; x, \theta > 0.$$

The data set is

The failure times of Aarset data	0.1, 0.2, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86.
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The ML estimates of the parameter as well as their standard errors (SEs) are presented in Table 4. Measures, such as minus log-likelihood (-LogL), Kolmogorov-Smirnov (KS) test statistic, Akaike information criterion (AIC), corrected AIC (CAIC), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC) are also listed in Table 4.

Table 4. Analytical results of the ITL model and other competing models for data set

Model	ML Estimates (SE)	-LogL	AIC	BIC	CAIC	HQIC	KS
ITL	$\hat{\alpha} = 0.378(0.0534)$	268.339	538.677	538.376	538.76	539.405	0.3169
IL	$\hat{\alpha} = 2.846(0.334)$	324.041	650.082	649.781	650.166	650.811	0.6288
IR	$\hat{\alpha} = 0.383(0.0542)$	525.139	1052.278	1051.977	1052.361	1053.006	0.8094
IE	$\hat{\alpha} = 2.259(0.3195)$	317.149	636.298	635.997	636.381	637.026	0.622

Based on Table 4, we notice that the ITL distribution yields the best fit. Thus, it may be preferred as the most appropriate model for explaining the considered data set. Further information is presented in Fig.3. Also, PP-plots are shown in Fig.4 for the real data.

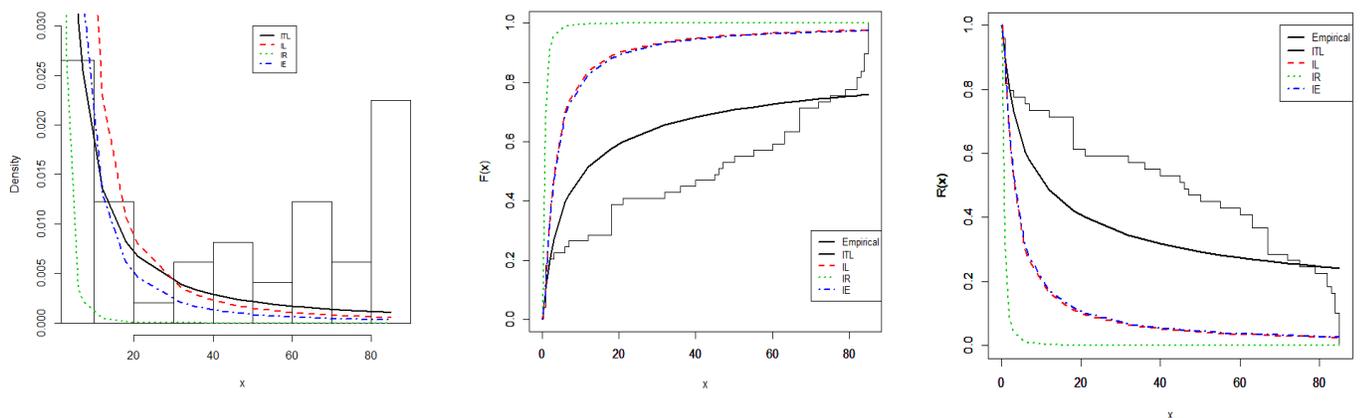


Fig. 3: Estimated pdf, cdf and survival function of the ITL, IL, IR, and IE for Aarset data

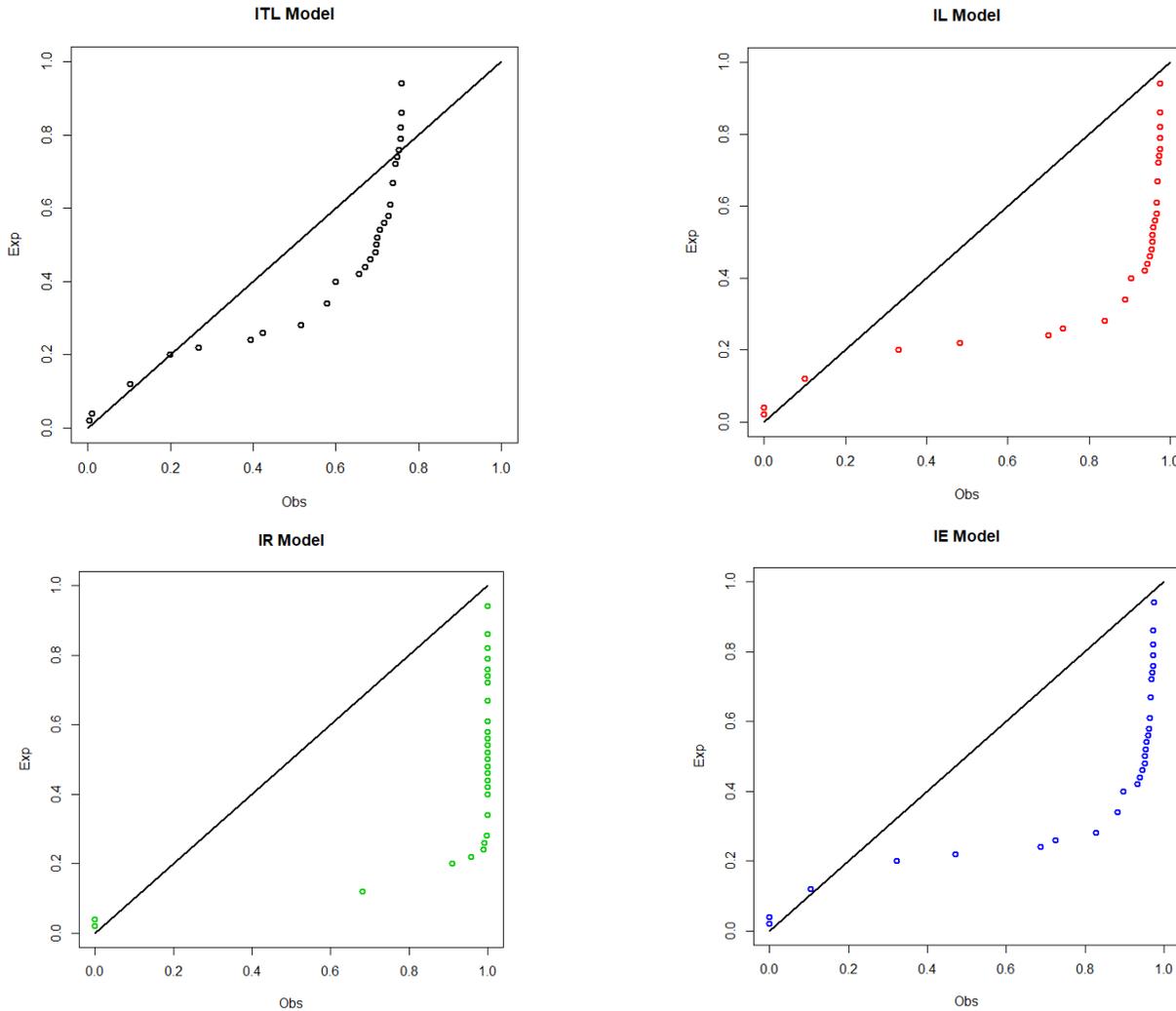


Fig. 4: PP plots of the ITL, IL, IR, and IE for Aarset data

According to the above-mentioned figures, the ITL distribution gives better fits then, we believe that the recommended model may be a useful alternative model for some statistical research.

6 Conclusions

We offer a new probability distribution, i.e. the inverted Topp-Leone. Some of its main properties, such as quantile function, mode, moments, probability weighted moments, incomplete moments, Rényi entropy, stress- strength model, residual life function and stochastic ordering are obtained. The maximum likelihood method of estimation is used in case of complete, Type I censored, and Type II censored samples. Application to a real data shows that the suggested distribution may be the most appropriate model.

Acknowledgement: The authors are grateful to the Editor in Chief and anonymous referee for the insightful and constructive comments that improved this paper.

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