Mathematical Sciences Letters
An International Journal

http://dx.doi.org/10.18576/msl/050114

Metric Dimension of Some Families of Graph

Gohar Ali[1,*], *Roohi Laila*[1] *and Murtaza Ali*[2]

[1] Department of Mathematics, Islamia College Peshawar, Pakistan
[2] Department of Basic Sciences University of Engineering and Technology Peshawar, Mardan Campus Mardan, Pakistan

Received: 23 Apr. 2015, Revised: 21 Dec. 2015, Accepted: 23 Dec. 2015
Published online: 1 Jan. 2016

Abstract: If G is a connected graph, the distance $d(x,y)$ between two vertices $x,y \in V(G)$ is the length of a shortest path between them. Let $d(x,y)$ denote the distance between vertices x and y of a connected graph G. If $d(z,x) \neq d(z,y)$, then z is said to resolve x and y and therefore z is called a resolving vertex for the vertices x and y. Let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered set of vertices of G and let v be a vertex of G. The representation $r(v|G)$ of v with respect to W is the k-tuple $(d(v,w_1), d(v,w_2), \ldots, d(v,w_k))$. If distinct vertices of G have distinct representations with respect to W, then W is called a resolving set or locating set for G. A resolving set of minimum cardinality is called a basis for G and this cardinality is the metric dimension of G, denoted by $dim(G)$. A family Γ of connected graphs is a family with constant metric dimension if $dim(G)$ is finite and does not depend upon the choice of G in Γ. In this paper, we find the constant metric dimension of $P_n(1,2,3)$ and M_n.

Keywords: Metric dimension, basis, resolving set

1 Notation and Preliminary Results

If G is a connected graph, the distance $d(u,v)$ between two vertices $u,v \in V(G)$ is the length of a shortest path between them. Let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered set of vertices of G and let v be a vertex of G. The representation of v with respect to W is denoted by $r(v|W)$ is the k-tuple $(d(v,w_1), d(v,w_2), \ldots, d(v,w_k))$. If distinct vertices of G have distinct representations with respect to W then W is called a resolving set or locating set for G [5]. A resolving set of minimum cardinality is called a metric basis for G and this cardinality is the metric dimension of G, denoted by $dim(G)$. The concepts of resolving set and metric basis have previously appeared in the literature (see [1]-[3],[6]-[13]). For a given ordered set of vertices $W = \{w_1, w_2, \ldots, w_k\}$ of a graph G, the i^{th} component of $r(v|W)$ is 0 if and only if $v = w_i$. Thus, to show that W is a resolving set it suffices to verify that $r(x|W) \neq r(y|W)$ for each pair of distinct vertices $x,y \in V(G)$. Motivated by the problem of uniquely determining the location of an intruder in a network, the concept of metric dimension was introduced by Slater in [11] and studied independently by Harary et al. [6]. Applications of this invariant to the navigation of robots in networks are discussed in [8] and applications to chemistry in [5] while applications to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures are given in [8].

The join of two graphs G and H is denoted by $G+H$, a fan is $f_n = K_1 + P_n$ for $n \geq 1$. Caceres et al. [4] found the metric dimension of fan f_n. The graph obtained from a wheel W_{2n} by deleting n alternating spokes, is called Jahangir graph denoted by J_{2n} with $n \geq 2$ (also known as gear graph). Tomescu et al. [15] compute the metric dimension of Jahangir graph J_{2n}. Also Tomescu et al. [15] computed the partition and connected dimension of wheel graph W_n. In [5] Chartrand et al. proved that a graph G has metric dimension 1 if and only $G \cong P_n$, hence paths on n vertices constitute a family of graphs with constant metric dimension. Similarly, Cycles with $n \geq 3$ vertices also constitute such a family of graphs as their metric dimension is 2. Since prisms D_n are the trivalent plane graphs obtained by the cartesian product of the path P_2 with a cycle C_n, hence they constitute a family of 3-regular graphs with constant metric dimension. Also Javaid et al. proved in [7] that the plane graph Antiprism A_n constitutes a family of regular graphs with constant metric dimension as $dim(A_n) = 3$ for every $n \geq 5$.
In this paper, we extend this study by considering the two different families of graphs with constant metric dimension.

* Corresponding author e-mail: gohar.ali@gmail.com

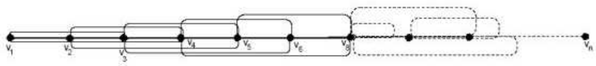


Fig. 1: Graph $P_n(1,2,3)$

The graph $P_n(1,2,3)$ is a graph with n vertices and $3(n-2)$ edges. The edge set of $P_n(1,2,3)$ as $E(P_n(1,2,3)) = \{v_iv_{i+1}, v_iv_{i+2}, v_iv_{i+3}\}$ where $i = 1,2,\dots,n$ and $n+1$ is taken modulo n.

2 Main Results

Theorem 1*The metric dimension of $G \cong P_n(1,2,3)$ for $n \geq 6$ and ($n \in$ **Z**) is constant and equal to* 3.

Proof: For any $W = \{v_1, v_2, v_3\} \subseteq V(P_n(1,2,3))$, we need to show that W is a resolving set for $P_n(1,2,3)$. The representations of the vertices of $V(P_n(1,2,3)) \setminus W$ with respect to W are as follows:
Case(1). If $n \equiv 0 \pmod 3$ where $n \geq 6$, then the representations of the vertices are as follows:

$r(v_{3i}|W) = (i,i,i-1)$ for $i = 1,2,\dots,\frac{n}{3}-1$.

$r(v_{3i+1}|W) = (i,i,i)$ for $i = 1,2,\dots,\frac{n}{3}-1$.

$r(v_{3i+2}|W) = (i+1,i,i-1)$ for $i = 1,2,\dots,\frac{n}{3}-1$.

Note that there are no two vertices having the same representations. Which implies that $dim(P_n(1,2,3)) \leq 3$. To prove the theorem, it is sufficient to show that $dim(P_n(1,2,3)) \geq 3$. By contradiction assume that there exists a resolving set W' with $|W'| = 2$ we have the following possibilities:
Let us suppose that $W' = \{v_3, v_t\}$ where $t \equiv 0 \pmod 3$ and $t \leq n$. Then $r(v_4|\{v_3, v_t\}) = (1,(t-3)/3)$ which is a contradiction.

(1) Let us suppose that $W' = \{v_4, v_t\}$ where $t \equiv 1 \pmod 3$ and $t \leq n$.
Then $r(v_1|\{v_4, v_t\}) = (1,(t-1)/3)$ which is a contradiction.

(2) Let us suppose that $W' = \{v_5, v_t\}$ where $t \equiv 2 \pmod 3$ and $t \leq n$.
Then $r(v_2)|\{v_5, v_t\} = (1,(t-2)/3)$ which is a contradiction.

(3) Let us suppose that $W' = \{v_3, v_t\}$ where $t \equiv 1 \pmod 3$ and $t \leq n$.
Then $r(v_1|\{v_3, v_t\}) = (1,(t-1)/3)$ which is a contradiction.

(4) Let us suppose that $W' = \{v_3, v_t\}$ where $t \equiv 2 \pmod 3$ and $t \leq n$.
Then $r(v_2|\{v_3, v_t\}) = r(v_4|\{v_3, v_t\}) = (1,(t-2)/3)$ which is a contradiction.

(5) Let us suppose that $W' = \{v_4, v_t\}$ where $t \equiv 2 \pmod 3$ and $t \leq n$.
Then $r(v_2|\{v_4, v_t\}) = r(v_3|\{v_4, v_t\}) = (1,(t-3)/3)$ which is a contradiction.
Hence from above it follows that there is no resolving set with two vertices for $V(P_n(1,2,3))$ Therefore $dim(P_n(1,2,3)) = 3$ in this case.

Case(2). If $n \equiv 1 \pmod 3$ where $n \geq 6$, then the representations of the vertices are as follows:

$r(v_{3i}|W) = (i,i,i-1)$ for $i = 1,2,\dots,\frac{n-1}{3}-1$.

$r(v_{3i+1}|W) = (i,i,i)$ for $i = 1,2,\dots,\frac{n-1}{3}$.

$r(v_{3i+2}|W) = (i+1,i,i-1)$ for $i = 1,2,\dots,\frac{n-1}{3}-1$.

Note that there are no two vertices having the same representations. So $dim(P_n(1,2,3)) \leq 3$. For $dim(P_n(1,2,3)) \geq 3$ we proceeding on the same line as above in case(1) we get there is no resolving set W' with $|W'| = 2$, so we have $dim(P_n(1,2,3)) = 3$ in this case.

Case(3). If $n \equiv 2 \pmod 3$ where $n \geq 6$, then the representations of the vertices are as follows:

$r(v_{3i}|W) = (i,i,i-1)$ for $i = 1,2,\dots,\frac{n-2}{3}-1$.

$r(v_{3i+1}|W) = (i,i,i)$ for $i = 1,2,\dots,\frac{n-2}{3}$.

$r(v_{3i+2}|W) = (i+1,i,i-1)$ for $i = 1,2,\dots,\frac{n-2}{3}$.

Note that there are no two vertices having the same representations, so $dim(P_n(1,2,3)) \leq 3$. For $dim(P_n(1,2,3)) \geq 3$ we proceeding on the same line as above in case(1) we get there is no resolving set W' with $|W'| = 2$, so we have $dim(P_n(1,2,3)) = 3$ in this case.

Metric dimension for the graph M_n.
The graph M_n is constructed from the graphs study by M. Bača in [2,3]. The vertex set of M_n consists on three types of vertices. $E(M_n) = \{a_i; b_i; c_i : 1 \leq i \leq n\}$ such that $V(M_n) = \{a_i\} \cup \{b_i\} \cup \{c_i\}$ where $deg(a_i) = 2$, $deg(b_i) = 5$ and $deg(c_i) = 1$. The edge set is $E(M_n) = \{b_ib_{i+1}, b_ia_i, b_ia_{i-1}, b_ic_i\}$ where $n+1$ is taken modulu n.

Theorem 2*The metric dimension of $G \cong M_n$ for $n \geq 6$($n \in \mathbb{Z}$) is constant and equal to 3.*

Proof: We distinguish the following two cases.
Case(1). If $n = 2k$, $k \geq 3$ and k is an integer, for any $W = \{c_1, a_1, c_{k+1}\} \subset V(M_n)$. We show that W is a

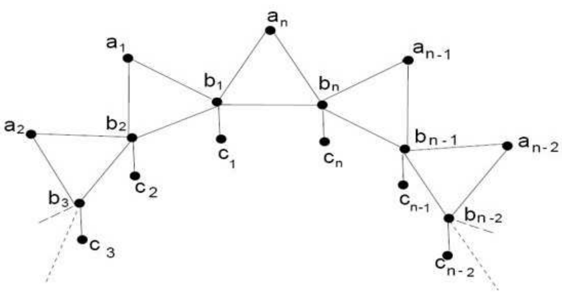


Fig. 2: Graph M_n

resolving set for $V(M_n)$. The representations of the vertices are as follows:

$$r(c_i|W)=\begin{cases}(i+1,i,3+k-i), & \text{for } ,2\le i\le k,\\ (2k-i+3,2k+3-i,i+1-k), & \text{for } k+2\le i\le n\end{cases}$$

$$r(a_i|W)=\begin{cases}(i+1,i,2+k-i), & \text{for } ,2\le i\le k,\\ (2k-i+2,2k+2-i,i+1-k), & \text{for } k+1\le i\le n\end{cases}$$

$$r(b_i|W)=\begin{cases}(1,1,k+i+3), & \text{for } ,i=1,\\ (i,i-1,k+2-i), & \text{for } 2\le i\le k+1,\\ (2k+2-i,2k+2-i,i-k), & \text{for } k+2\le i\le n.\end{cases}$$

We observe that there are no two vertices having the same representations implying that $dim(M_n)\le 3$. Now we show that $dim(M_n)\ge 3$, by proving that there is no resolving set W' with $|W'|=2$ then we have the following possibilities:

(1) W is the subset of $\{b_i : i=1,2,\ldots,n\}$, we suppose that one resolving vertex is b_1 and the other is $b_t, 2\le t\le k+1$ then for $2\le t\le k$ we have $r(c_1|\{b_1,b_t\}) = r(a_n|\{b_1,b_t\}) = (1,t)$ and for $t=k+1$ the representation is $r(a_k|\{b_1,b_t\}) = r(a_{k+1}|\{b_1,b_t\}) = (k,1)$ which is a contradiction.

(2) W is the subset of $\{c_i : i=1,2,\ldots,n\}$, we suppose that one resolving vertex is c_1 and the other is $c_t, 2\le t\le k+1$ then for $2\le t\le k$ we have $r(b_n|\{c_1,c_t\}) = r(a_n|\{c_1,c_t\}) = (2,t+1)$ and for $t=k+1$ the representation is $r(a_k|\{c_1,c_t\}) = r(a_{k+1}|\{c_1,c_t\}) = (k,1)$ which is a contradiction.

(3) W is the subset of $\{a_i : i=1,2,\ldots,n\}$, we suppose that one resolving vertex is a_1 and the other is $a_t, 2\le t\le k+1$ then for $2\le t\le k$ we have $r(c_1|\{a_1,a_t\}) = r(a_n|\{c_1,c_t\}) = (2,t+1)$ and for $t=k+1$ the representation is $r(b_{k+1}|\{a_1,a_t\}) = r(b_{k+2}|\{a_1,a_t\}) = (k,1)$ which is a contradiction.

(4) One vertex belong to $\{b_i\}\subset V(M_n)$ and the other vertex belong to $\{c_i\}\subset V(M_n)$, we suppose that one resolving vertex is b_1 and the other is $c_t, 2\le t\le k+1$. Then for $2\le t\le k$ we have $r(a_n|\{b_1,c_t\}) = r(b_n|\{b_1,c_t\}) = (1,t+1)$ and for $t=k+1$ the representation is $r(a_k|\{b_1,c_t\}) = r(a_{k+2}|\{b_1,c_t\}) = (k,1)$ which is a contradiction.

(5) One vertex belong to $\{b_i\}\subset V(M_n)$ and the other vertex belong to $\{a_i\}\subset V(M_n)$, we suppose that one resolving vertex is b_1 and the other is $a_t, 2\le t\le k+1$. Then for $2\le t\le k$ we have $r(a_n|\{b_1,a_t\}) = r(c_n|\{b_1,a_t\}) = (1,t+1)$ and for $t=k+1$ the representation is $r(b_k|\{b_1,a_t\}) = r(a_{k+2}|\{b_1,a_t\}) = (k-1,2)$ which is a contradiction.

(6) One vertex belong to $\{c_i\}\subset V(M_n)$ and the other vertex belong to $\{a_i\}\subset V(M_n)$, we suppose that one resolving vertex is c_1 and the other is $a_t, 2\le t\le k+1$. Then for $2\le t\le k$ we have $r(a_n|\{c_1,a_t\}) = r(b_n|\{c_1,a_t\}) = (2,t+1)$ and for $t=k+1$ the representation is $r(a_k|\{c_1,a_t\}) = r(c_{k+2}|\{c_1,a_t\}) = (k+1,2)$ which is a contradiction.
Hence, from above it follows that there is no resolving set with two vertices for $V(M_n)$ implying that $dim(M_n)=3$.

Case(2). Suppose $n=2k+1, k\ge 3$ $(k\in\mathbb{Z})$ for any $W=\{c_1,a_1,c_{k+1}\}\subset V(M_n)$, we will show that W is a resolving set for $V(M_n)$. For this we take the representations of vertices of $V(M_n)\setminus W$ with respect to $\{c_1,a_1,c_{k+1}\}$. The representations of the vertices are as follows:

$$r(c_i|W)=\begin{cases}(i+1,i,3+k-i), & \text{for } ,2\le i\le k,\\ (2k-i+4,2k+4-i,i-k), & \text{for } k+2\le i\le n.\end{cases}$$

$$r(a_i|W)=\begin{cases}(i+1,i,2+k-i), & \text{for } ,2\le i\le k,\\ (k+2,k+1,2), & \text{for } i=k+1,\\ (2k+3-i,2k+3-i,i-k), & \text{for } k+2\le i\le n.\end{cases}$$

$$r(b_i|W)=\begin{cases}(1,1,k+1), & \text{for } ,i=1,\\ (i,i-1,k+2-i), & \text{for } 2\le i\le k+1,\\ (2k+3-i,2k+3-i,i-k-1), & \text{for } k+2\le i\le n.\end{cases}$$

Proceeding on same line as in Case(1) we observe that there are no two vertices having the same representations. which implies that $dim(M_n)\le 3$, now we show that $dim(M_n)\ge 3$ for this consider that $dim(M_n)=2$ then there are the same possibilities as in Case(1) and contradiction can be deduced, which implies that

$dim(M_n) \geq 3$ in this case. Finally from Case(1) and Case(2) we get $dim(M_n) = 3$.

3 Conclusion

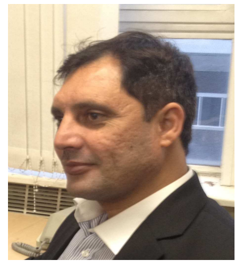
In this paper we found the metric dimension of $P_n(1,2,3)$ and M_n. Next we are thinking on the metric dimension of some special type of join of $P_n(1,2,3)$ and M_n.

Acknowledgement

The first author is supported by the Higher Eduction Commission of Pakistan (HEC). The authors thanks for the refrees due to helpful comments and suggestions, for the improment of our research work in this and upcoming work.

References

[1] Andreas Brandstadt, Van Bang Le, Jeremy P. Spinrad,Graph classes a survey, Saim, May 20,2004.
[2] M. Bača, On Magic labellings of convex polytopes, Annals Discrete Math. 51(1992), 13-16.
[3] M. Bača, Labelling of two classes of convex polytopes, Utilitas Math. 34(1998), 24-31.
[4] J. Caceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, D. R. Wood. On the metric dimension of some families of graphs. Electronic Notes in Disc. Math.22(2004), 129-133.
[5] G. Chartrand, L. Eroh, M. A. Johnson, O. R. Oellermann, Resolvability in graphs and metric dimension of a graph, Disc. Appl. Math. 105(200), 99-113.
[6] F. Harary, R. A. Melter, On the metric dimension of a graph, Ars Combin. 2(1976), 191-195.
[7] I. Javaid, M. T. Rahim, K. Ali, Families of regular graphs with constant metric dimension, Utilitas Math. 75(2008), 21-33.
[8] R. A. Melter, I. Tomescu, Metric bases in digital geometry, Computer Vision, Graphics, and Image Processing,25(1984), 113-121.
[9] Murtaza. Ali, Gohar Ali, Muhammad Imran, A.Q. Baig and Muhammad Kashif Shafiq. On metric dimension of MÖbious Ladder. Ars Combinatoria. 105(2012), 403-410.
[10] Murtaza. Ali, M. T. Rahim, U. Ali, G. Ali. On Cycle Related Graphs with Constant Metric Dimension, Open Journal of Discrete Mathematics, 2(2012), 21-23.
[11] P. J. Slater, Dominating and reference sets in graphs, J. Math. Phys. Sci. 22(1998), 445-455.
[12] I. Tomescu, I. Javaid, On the metric dimension of the Jahangir graph, Bull. Math. Soc. Sci. Math. Roumanie, 50(98), 371-376.
[13] I. Javaid, On the connected partition dimension of unicyclic graphs. J. Comb. Math. Comput. 65(2008), 71-77.
[14] I. Javaid, Shokat, Sara, On the patition dimension of some wheel related graph. J. Prim Res. Math.4(2008), 154-164.
[15] I. Tomescu, I. Javaid, Slamin, On the partition dimension and connected partition dimension wheels. Ars. Combinatoria. 84(2007), 311-317.



Gohar ALI received the PhD degree in Mathematics from Abdus Salam School of Mathematical Sciences (ASSMS) GC University Lahore Pakistan. His research interests are in Graph Labeling and Combinatorial Mathematics. He has published research articles in reputed international journals of mathematics. He is referee of mathematical journals.

Murtaza Ali Murtaza Ali is Assistant Professor of Mathematics at Department of Mathematics, University of Engineering and Technology Peshawar, Pakistan. His research interests are in the areas of metric graph theory, graph labeling and spectral graph theory. He has published research articles in reputed international journals of Mathematics and Informatics. He is referee of several international mathematical journals.

Roohi Laila Roohi Laila is a PhD student at Islamia College Peshawar Pakistan. She is working in the field of Graph Theory and Combinatorics, also she is working as lecturer of Mathematics at Bacha Khan University Charsadda.