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Hadamard Integral Inequality for the Class of Geodesic Convex Functions

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Abstract: In this paper, we introduce the concept of (γ,η)-convex function by the inequality

$$f\big(\gamma_{x,y}(t)\big) \leq \eta_{f(x),f(y)}(t),$$

in which γ and η are two geodesic arcs. Then, we will find some refinements of Hadamard integral inequality for (γ,η)-convex functions in the case of Lebesgue and Sugeno integral.

Keywords: (γ,η)-convex function, geodesic arc, Hadamard inequality, Lebesgue integral, Sugeno integral

1 Introduction

In optimization theory, the concept of geodesic convex functions on Riemannian manifolds was introduced in 1980's instead of ordinary convex functions on a linear vector space, to establish the local-global property of a smooth nonlinear optimization problem with equality constraints, see [1,2]. In fact, in the definition of convexity, if the line segment is replaced by a geodesic arc, the concept of geodesic convexity is introduced.

The first characterization of geodesic convex functions with respect to the Riemannian metrics was elaborated in the case of a sub-manifold of $\mathbb{R}^n$ by using the tools of immersion. In this case, in order to check the geodesic convexity of a function on the feasible region, it is necessary and sufficient to state the positive semi-definiteness of the geodesic Hessian matrix in this domain.

Due to the importance of recognition the geometric structure in optimization problems, this concept may has extensive use and applications, see [3]. The class of geodesic convex functions with respect to the Riemannian metrics plays an important role in nonlinear optimization, e.g., in necessary and sufficient optimality criteria and in the connectedness of the solution set in linear and nonlinear complementarity systems [1,2].

In connection with the concept of geodesic convexity, Iqbal et al. [4] introduced the class of geodesic semi E-convex functions and discussed some of their properties. Kiliçman and Saleh [5] introduced the class of geodesic semi strongly E-convex functions and generalized geodesic semi strongly E-convex functions.

Sugeno integral in a kind of nonlinear integrals introduced by Sugeno [6] in order to represent and include the interactions between criteria of different phenomena. Sugeno integral is an idempotent, continuous and monotone operator. Most well-known integral inequalities have been proved for Sugeno integral, see [7, 8, 9, 10, 11, 12].

The Hadamard inequality is a classical integral inequality providing an upper bound for the mean value of a convex function $f:[a,b] \longrightarrow \mathbb{R}$,

$$\int_0^1 f\big((1-t)a+tb\big)dt \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

The above inequality should be reversed if f is concave. The purpose of this paper is to obtain a refinement of the inequality (1) for geodesic convex functions, upon the definition of Sugeno integral.

The paper is organized as follows. The definition of Sugeno integral and its properties and also the definitions of a geodesic path and geodesic convexity are presented

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in Section 2. In Section 3, a refinement of Hadamard inequality for geodesic convex functions in the case of Sugeno integral is considered. In Section 4, some applications regarding the obtained new results are given. A conclusion is given in Section 5.

2 Preliminaries

At the first, let us give the definitions of a geodesic path and geodesic convexity as follow.

Definition 1. ([2]). A geodesic is a $\mathbb{C}^\infty$ smooth path γ whose tangent is parallel along the path γ. Let M be a complete n-dimensional Riemannian manifold. For all $x,y \in M$, the mapping $\gamma_{x,y} : [0,1] \to M$ is a geodesic joining the points x and y if $\gamma_{x,y}(0) = y$ and $\gamma_{x,y}(1) = x$.

Definition 2. ([13]). Let M be a complete n-dimensional Riemannian manifold. A subset A of M is said to be totally convex if A contains every geodesic $\gamma_{x,y}$ of M whose endpoints x and y are in A. A real valued function $f : A \to \mathbb{R}$ is said to be geodesic convex if for all geodesic arcs $\gamma_{x,y}$ from x to y, we have

$$f\left(\gamma_{x,y}(t)\right) \le (1-t)f(y) + tf(x),$$

for all $x,y \in A$ and $t \in [0,1]$.

In what follows, let X be a non-empty set and Σ be a σ-algebra of subsets of X.

Definition 3.[14] *Let $\mu : \Sigma \longrightarrow [0,\infty)$ be a set function. We say that μ is a Sugeno measure if it satisfies*

1.$\mu(\emptyset) = 0$.
2.$E,F \in \Sigma$ and $E \subset F$ imply $\mu(E) \le \mu(F)$.
3.$E_n \in \Sigma$ $(n \in \mathbb{N})$, $E_1 \subset E_2 \subset \dots$, imply $\lim_{n\to\infty} \mu(E_n) = \mu(\bigcup_{n=1}^{\infty} E_n)$ (continuity from below).
4.$E_n \in \Sigma$ $(n \in \mathbb{N})$, $E_1 \supset E_2 \supset \dots$, $\mu(E_1) < \infty$, imply $\lim_{n\to\infty} \mu(E_n) = \mu(\bigcap_{n=1}^{\infty} E_n)$ (continuity from above).

The triple (X,Σ,μ) is called a sugeno measure space.

Let (X,Σ,μ) be a fuzzy measure space. By $\mathscr{F}_\mu(X)$ we denote the set

$$\mathscr{F}_\mu(X) = \{f : X \longrightarrow [0,\infty) : f \text{ is measurable w.r.t. } \Sigma\}.$$

For $f \in \mathscr{F}_\mu(X)$ and $\alpha > 0$, we denote by F_α and $F_{\tilde{\alpha}}$ the following sets

$$F_\alpha = \{x \in X : f(x) \ge \alpha\} \quad \text{and} \quad F_{\tilde{\alpha}} = \{x \in X : f(x) > \alpha\}.$$

Note that if $\alpha \le \beta$, then $F_\beta \subset F_\alpha$ and $F_{\tilde{\beta}} \subset F_{\tilde{\alpha}}$.

Definition 4. *[6,15,16] Let (X,Σ,μ) be a fuzzy measure space, $f \in \mathscr{F}_\mu(X)$ and $A \in \Sigma$, then the Sugeno integral of f on A with respect to the fuzzy measure μ is defined by*

$$\int_A f \mathrm{d}\mu = \bigvee_{\alpha \ge 0} \left(\alpha \wedge \mu(A \cap F_\alpha)\right),$$

where $\wedge$ is just the prototypical t-norm minimum and $\vee$ the prototypical t-conorm maximum. If $A = X$, then

$$\int_A f \mathrm{d}\mu = \bigvee_{\alpha \ge 0} \left(\alpha \wedge \mu(F_\alpha)\right).$$

The following properties of Sugeno integral are well known and can be found in [15,16].

Theorem 1. Let (X,Σ,μ) be a fuzzy measure space, $A,B \in \Sigma$ and $f,g \in \mathscr{F}_\mu(X)$ then

(F_1)$\int_A f \mathrm{d}\mu \le \mu(A)$.
(F_2)$\int_A k \mathrm{d}\mu = k \wedge \mu(A)$, k non-negative constant.
(F_3)If $f \le g$ on A then $\int_A f \mathrm{d}\mu \le \int_A g \mathrm{d}\mu$.
(F_4)If $A \subset B$ then $\int_A f \mathrm{d}\mu \le \int_B f \mathrm{d}\mu$.

3 The main results

In this section, let (X,Σ,μ) be a fuzzy measure space. For a given $f \in \mathscr{F}^\mu(X)$ and $A \in \Sigma$, we set

$$\Gamma = \left\{\alpha \mid \alpha \ge 0, \mu(A \cap F_\alpha) > \mu(A \cap F_\beta) \text{ for any } \beta > \alpha\right\}.$$

It is easy to see that

$$\int_A f \mathrm{d}\mu = \bigvee_{\alpha \in \Gamma} \left(\alpha \wedge \mu(A \cap F_\alpha)\right).$$

If $X = \mathbb{R}$ the set of real numbers, Σ is the Borel field and μ is the Lebesgue measure, it is easy to see that (X,Σ,μ) is a fuzzy measure space; but it should be noted that the Sugeno integral is not an extension of the Lebesgue integral.

The concept of (γ,η)-convexity is introduced as follows.

Definition 5. *Let I and J be two closed subintervals of $[0,+\infty]$. Let $\gamma_{x,y} : [0,1] \to I$ be a geodesic arc joining the points $x,y \in I$ and $\eta_{u,v} : [0,1] \to J$ be a geodesic arc joining the points $u,v \in J$. A real valued function $f : I \to J$ is said to be (γ,η)-convex if*

$$f\left(\gamma_{x,y}(t)\right) \le \eta_{f(x),f(y)}(t)$$

for all $x,y \in I$ and $t \in [0,1]$.

Remark. How do we distinguish different cases of (γ,η)-convex functions from each other? The answer is using the following inequality:
For a (γ,η)-convex function $f : [a,b] \to [c,d]$, we have

$$f(x) = f\left(\gamma_{a,b}(\gamma_{a,b}^{-1}(x))\right) \le \eta_{f(a),f(b)}(\gamma_{a,b}^{-1}(x)) \qquad (2)$$

for all $x \in [a,b]$. So, it is easy to show that for all $x \in [a,b]$, the inequality is sharp.

In the following theorem, some generalizations of Hadamard inequality for different geodesic convex functions are given.

Theorem 2. *Let $I,J \subseteq \mathbb{R}$, $a,b \in I^o$ with $a<b$ and $c,d \in J^o$ with $c<d$. For the particular geodesic arcs $\gamma:[0,1]\to I$ and $\eta:[0,1]\to J$ defined by $\gamma_{x,y}(t)=(1-t)x+ty$ and $\eta_{u,v}(t)=u^{1-t}v^t$, the following inequalities hold:*

1. *If $f:I\to I$ is a (γ,γ)-convex function, then*

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \frac{f(a)+f(b)}{2}.$$

2. *If $f:I\to J\subseteq(0,\infty)$ is a (γ,η)-convex function with $f(a)\neq f(b)$, then*

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \frac{f(a)}{\ln\left(\frac{f(b)}{f(a)}\right)}\left(\frac{f(b)}{f(a)}-1\right).$$

3. *If $f:J\subseteq(0,\infty)\to I$ is a (η,γ)-convex function, then*

$$\frac{1}{c-d}\int_c^d f(x)\mathrm{d}x \le f(c)+\frac{f(d)-f(c)}{\ln\left(\frac{d}{c}\right)}\left(\frac{d\ln\left(\frac{d}{c}\right)}{d-c}-1\right).$$

4. *If $f:J\subseteq(0,\infty)\to J\subseteq(0,\infty)$ is a (η,η)-convex function with $f(c)\neq f(d)$, then*

$$\frac{1}{c-d}\int_c^d f(x)\mathrm{d}x \le \frac{cf(c)}{d-c}\left(\frac{\left(\frac{d}{c}\right)^{\left(\log_{\frac{d}{c}}\left(\frac{f(d)}{f(c)}\right)+1\right)}-1}{\log_{\frac{d}{c}}\left(\frac{f(d)}{f(c)}\right)+1}\right).$$

Proof. The first inequality is the well-known classical Hadamard inequality. By using the inequality (2) for the assumed particular geodesic arcs, we have the following inequalities:

– The function $f:[a,b]\to[a,b]$ is (γ,γ)-convex iff

$$f(x)\le f(a)+\frac{x-a}{b-a}(f(b)-f(a))$$

for all $x\in[a,b]$.

– The function $f:[a,b]\to[c,d]$ is (γ,η)-convex iff

$$f(x)\le f(a)\left(\frac{f(b)}{f(a)}\right)^{\frac{x-a}{b-a}}$$

for all $x\in[a,b]$.

– The function $f:[c,d]\to[a,b]$ is (η,γ)-convex iff

$$f(x)\le f(c)+\log_{\frac{d}{c}}\frac{x}{f(c)}(f(d)-f(c))$$

for all $x\in[c,d]$.

– The function $f:[c,d]\to[c,d]$ is (η,η)-convex iff

$$f(x)\le f(c)\left(\frac{f(d)}{f(c)}\right)^{\log_{\frac{d}{c}}\frac{x}{f(c)}}$$

for all $x\in[c,d]$.

It is enough to integrate from the both sides of the above four inequalities over $[a,b]$ or $[c,d]$ to obtain the assertion of theorem.

Theorem 3. *Consider the fuzzy measure space $(\mathbb{R},\Sigma,\mu)$. Let $\gamma:[0,1]\to[a,b]$ and $\eta:[0,1]\to[c,d]$ be two invertible geodesic arcs. If $f:[a,b]\to[c,d]$ is a (γ,η)-convex function, then*

$$\int_a^b f\mathrm{d}\mu \le \begin{cases} \bigvee_{\alpha\in\left[f(a),f(b)\right)}\left(\alpha\wedge\mu\left(\left[\gamma_{a,b}\left(\eta^{-1}_{f(a),f(b)}(\alpha)\right),b\right]\right)\right), \\ \quad \gamma,\eta \text{ are comonotone,} \\ \bigvee_{\alpha\in\left[f(b),f(a)\right)}\left(\alpha\wedge\mu\left(\left[a,\gamma_{a,b}\left(\eta^{-1}_{f(a),f(b)}(\alpha)\right)\right]\right)\right), \\ \quad \gamma,\eta \text{ are countermonotone.}\end{cases}$$

Proof. By the (γ,η)-convexity of f and the property (F_3) of fuzzy measures, we have

$$\int_a^b f(x)\mathrm{d}\mu = \int_a^b f\left(\gamma_{a,b}(\gamma^{-1}_{a,b}(x))\right)\mathrm{d}\mu \tag{3}$$

$$\le \int_a^b \eta_{f(a),f(b)}(\gamma^{-1}_{a,b}(x))\mathrm{d}\mu. \tag{4}$$

If γ and η are comonotone, then $\eta\circ\gamma^{-1}$ is an increasing function. So, by Definition 4 we have

$$\begin{aligned}&\int_a^b \eta_{f(a),f(b)}(\gamma^{-1}_{a,b}(x))\mathrm{d}\mu\\ &=\bigvee_{\alpha\ge0}\left(\alpha\wedge\mu\left([a,b]\cap\eta_{f(a),f(b)}(\gamma^{-1}_{a,b}(x))\ge\alpha\right)\right)\\ &=\bigvee_{\alpha\ge0}\left(\alpha\wedge\mu\left(\left\{x\ge\gamma_{a,b}\left(\eta^{-1}_{f(a),f(b)}(\alpha)\right)\right\}\right)\right)\\ &=\bigvee_{\alpha\ge0}\left(\alpha\wedge\mu\left(\left[\gamma_{a,b}\left(\eta^{-1}_{f(a),f(b)}(\alpha)\right),b\right]\right)\right).\end{aligned} \tag{5}$$

Since $\eta\circ\gamma^{-1}$ is increasing, we have

$$\begin{aligned}&a\le\gamma_{a,b}\left(\eta^{-1}_{f(a),f(b)}(\alpha)\right)<b\\ &\Rightarrow\eta_{f(a),f(b)}(\gamma^{-1}_{a,b}(a))\le\alpha<\eta_{f(a),f(b)}(\gamma^{-1}_{a,b}(b))\\ &\Rightarrow\eta_{f(a),f(b)}(0)\le\alpha<\eta_{f(a),f(b)}(1)\\ &\Rightarrow f(a)\le\alpha<f(b).\end{aligned} \tag{6}$$

Thus, $\Gamma=\left[f(a),f(b)\right)$ and we only need to consider $\alpha\in\left[f(a),f(b)\right)$. It follows from (3), (5) and (6) that

$$\begin{aligned}&\int_a^b \eta_{f(a),f(b)}(\gamma^{-1}_{a,b}(x))\mathrm{d}\mu\\ &\quad\le\bigvee_{\alpha\in\left[f(a),f(b)\right)}\left(\alpha\wedge\mu\left(\left[\gamma_{a,b}\left(\eta^{-1}_{f(a),f(b)}(\alpha)\right),b\right]\right)\right).\end{aligned}$$

If γ and η are countermonotone, then $\eta\circ\gamma^{-1}$ is a decreasing function. So, by Definition 4 we have

$$\begin{aligned}&\int_a^b \eta_{f(a),f(b)}(\gamma_{a,b}^{-1}(x))\mathrm{d}\mu\\&=\bigvee_{\alpha\geq 0}\left(\alpha\wedge\mu\left([a,b]\cap\eta_{f(a),f(b)}(\gamma_{a,b}^{-1}(x))\geq\alpha\right)\right)\\&=\bigvee_{\alpha\geq 0}\left(\alpha\wedge\mu\left(\left\{x\leq\gamma_{a,b}\left(\eta_{f(a),f(b)}^{-1}(\alpha)\right)\right\}\right)\right)\\&=\bigvee_{\alpha\geq 0}\left(\alpha\wedge\mu\left(\left[a,\gamma_{a,b}\left(\eta_{f(a),f(b)}^{-1}(\alpha)\right)\right]\right)\right).\end{aligned}\tag{7}$$

Since $\eta\circ\gamma^{-1}$ is decreasing, we have

$$\begin{aligned}&a\leq\gamma_{a,b}\left(\eta_{f(a),f(b)}^{-1}(\alpha)\right)<b\\&\Rightarrow\eta_{f(a),f(b)}(\gamma_{a,b}^{-1}(b))\leq\alpha<\eta_{f(a),f(b)}(\gamma_{a,b}^{-1}(a))\\&\Rightarrow\eta_{f(a),f(b)}(0)\leq\alpha<\eta_{f(a),f(b)}(1)\\&\Rightarrow f(b)\leq\alpha<f(a).\end{aligned}\tag{8}$$

Thus, $\Gamma=\big[f(b),f(a)\big)$ and we only need to consider $\alpha\in\big[f(b),f(a)\big)$. It follows from (3), (7) and (8) that

$$\begin{aligned}&\int_a^b \eta_{f(a),f(b)}(\gamma_{a,b}^{-1}(x))\mathrm{d}\mu\\&\quad\leq\bigvee_{\alpha\in[f(b),f(a))}\left(\alpha\wedge\mu\left(\left[a,\gamma_{a,b}\left(\eta_{f(a),f(b)}^{-1}(\alpha)\right)\right]\right)\right).\end{aligned}$$

Remark. Let $f:[a,b]\to[c,d]$ be a (γ,η)-convex function, Σ be the Borel field and μ be the Lebesgue measure on $\mathbb{R}$. Then

$$\int_a^b f\mathrm{d}\mu\leq\begin{cases}\bigvee_{\alpha\in[f(a),f(b))}\left(\alpha\wedge\left(b-\gamma_{a,b}\left(\eta_{f(a),f(b)}^{-1}(\alpha)\right)\right)\right),\\ \quad\gamma,\eta\text{ are comonotone,}\\ \bigvee_{\alpha\in[f(b),f(a))}\left(\alpha\wedge\left(\gamma_{a,b}\left(\eta_{f(a),f(b)}^{-1}(\alpha)\right)-a\right)\right),\\ \quad\gamma,\eta\text{ are countermonotone.}\end{cases}$$

In particular, we investigate the results of Theorem 3 for the geodesic arcs $\gamma:[0,1]\to I$ and $\eta:[0,1]\to J$ defined by $\gamma_{x,y}(t)=(1-t)x+ty$ and $\eta_{u,v}(t)=u^{1-t}v^t$.

Corollary 1. *Consider the fuzzy measure space* $(\mathbb{R},\Sigma,\mu)$. *Let* $f:[a,b]\to[a,b]$ *be a* (γ,γ)*-convex function. Then*

$$\begin{aligned}&\int_a^b f\mathrm{d}\mu\\&\leq\begin{cases}\bigvee_{\alpha\in[f(a),f(b))}\left(\alpha\wedge\mu\left(\left[a+(b-a)\frac{\alpha-f(a)}{f(b)-f(a)},b\right]\right)\right),\\ \quad f(a)<f(b),\\ f(a)\wedge\mu([a,b]),\quad f(a)=f(b),\\ \bigvee_{\alpha\in[f(b),f(a))}\left(\alpha\wedge\mu\left(\left[a,a+(b-a)\frac{\alpha-f(a)}{f(b)-f(a)}\right]\right)\right),\\ \quad f(a)>f(b).\end{cases}\end{aligned}$$

Proof. It is easy to see that

$$\gamma_{a,b}\left(\gamma_{f(a),f(b)}^{-1}(\alpha)\right)=a+(b-a)\frac{\alpha-f(a)}{f(b)-f(a)}$$

and the assertion of the corollary comes from the assertion of Theorem 3 with particular geodesic arcs.

Remark. The case of Corollary 1 is ordinary convexity. If we assume that Σ is the Borel field and μ is the Lebesgue measure on $\mathbb{R}$, then

$$\int_a^b f\mathrm{d}\mu\leq\begin{cases}\frac{(b-a)f(b)}{b-a+f(b)-f(a)},\ f(a)<f(b),\\ f(a)\wedge(b-a),\ f(a)=f(b),\\ \frac{(b-a)f(a)}{b-a+f(a)-f(b)},\ f(a)>f(b).\end{cases}$$

This particular case has been investigated in Theorem 3 of [17].

Corollary 2. *Consider the fuzzy measure space* $(\mathbb{R},\Sigma,\mu)$. *Let* $f:[a,b]\to[c,d]$ *be a* (γ,η)*-convex function. Then*

$$\begin{aligned}&\int_a^b f\mathrm{d}\mu\\&\leq\begin{cases}\bigvee_{\alpha\in[f(a),f(b))}\left(\alpha\wedge\mu\left(\left[a+(b-a)\log_{\frac{f(b)}{f(a)}}\frac{\alpha}{f(a)},b\right]\right)\right),\\ \quad f(a)<f(b),\\ f(a)\wedge\mu([a,b]),\quad f(a)=f(b),\\ \bigvee_{\alpha\in[f(b),f(a))}\left(\alpha\wedge\mu\left(\left[a,a+(b-a)\log_{\frac{f(b)}{f(a)}}\frac{\alpha}{f(a)}\right]\right)\right),\\ \quad f(a)>f(b).\end{cases}\end{aligned}$$

Proof. One can easily see that

$$\gamma_{a,b}\left(\eta_{f(a),f(b)}^{-1}(\alpha)\right)=a+(b-a)\log_{\frac{f(b)}{f(a)}}\frac{\alpha}{f(a)}$$

and the assertion of the corollary comes from the assertion of Theorem 3 with particular geodesic arcs.

Remark. The case of Corollary 2 is log-convexity. If we assume that Σ is the Borel field and μ is the Lebesgue measure on $\mathbb{R}$, then

$$\int_a^b f\mathrm{d}\mu\leq\begin{cases}\bigvee_{\alpha\in[f(a),f(b))}\left(\alpha\wedge(b-a)\log_{\frac{f(b)}{f(a)}}\frac{f(b)}{\alpha}\right),\\ \quad f(b)>f(a),\\ f(a)\wedge(b-a),\quad f(a)=f(b),\\ \bigvee_{\alpha\in[f(b),f(a))}\left(\alpha\wedge(b-a)\log_{\frac{f(a)}{f(b)}}\frac{f(a)}{\alpha}\right),\\ \quad f(b)<f(a).\end{cases}$$

This particular case has been investigated in Theorem 3.8 of [10].

Corollary 3. *Consider the fuzzy measure space $(\mathbb{R},\Sigma,\mu)$. Let $f:[c,d]\to[a,b]$ be an (η,γ)-convex function. Then*

$$\int_a^b f\,\mathrm{d}\mu \le \begin{cases} \bigvee_{\alpha\in[f(a),f(b))}\left(\alpha\wedge\mu\left(\left[a\left(\frac{b}{a}\right)^{\frac{\alpha-f(a)}{f(b)-f(a)}},b\right]\right)\right), \\ \quad f(a)<f(b), \\ f(a)\wedge\mu([a,b]), \quad f(a)=f(b), \\ \bigvee_{\alpha\in[f(b),f(a))}\left(\alpha\wedge\mu\left(\left[a,a\left(\frac{b}{a}\right)^{\frac{\alpha-f(a)}{f(b)-f(a)}}\right]\right)\right), \\ \quad f(a)>f(b). \end{cases}$$

Proof. Obviously,

$$\eta_{a,b}\left(\gamma^{-1}_{f(a),f(b)}(\alpha)\right)=a\left(\frac{b}{a}\right)^{\frac{\alpha-f(a)}{f(b)-f(a)}}$$

and the assertion of the corollary concluded by the assertion of Theorem 3 with particular geodesic arcs.

Corollary 4. *Consider the fuzzy measure space $(\mathbb{R},\Sigma,\mu)$. Let $f:[c,d]\to[c,d]$ be an (η,η)-convex function. Then*

$$\int_a^b f\,\mathrm{d}\mu \le \begin{cases} \bigvee_{\alpha\in[f(a),f(b))}\left(\alpha\wedge\mu\left(\left[a\left(\frac{b}{a}\right)^{\log_{\frac{f(b)}{f(a)}}\frac{\alpha}{f(a)}},b\right]\right)\right), \\ \quad f(a)<f(b), \\ f(a)\wedge\mu([a,b]), \quad f(a)=f(b), \\ \bigvee_{\alpha\in[f(b),f(a))}\left(\alpha\wedge\mu\left(\left[a,a\left(\frac{b}{a}\right)^{\log_{\frac{f(b)}{f(a)}}\frac{\alpha}{f(a)}}\right]\right)\right), \\ \quad f(a)>f(b). \end{cases}$$

Proof. Clearly,

$$\eta_{a,b}\left(\eta^{-1}_{f(a),f(b)}(\alpha)\right)=a\left(\frac{b}{a}\right)^{\log_{\frac{f(b)}{f(a)}}\frac{\alpha}{f(a)}}$$

and the assertion of the corollary concluded by the assertion of Theorem 3 with particular geodesic arcs.

Example 1. It is assumed that the geodesic arcs $\gamma:[0,1]\to I$ and $\eta:[0,1]\to J$ are defined by $\gamma_{x,y}(t)=(1-t)x+ty$ and $\eta_{u,v}(t)=u^{1-t}v^t$. If we denote the right hand side functions of the inequality (2) by $g(x)$, then for the assumed particular geodesic arcs, we give the following examples:

– The function $f:[1,3]\to[0,+\infty]$ defined by $f(x)=\frac{1}{\ln^2(x^2+1)}$ is (γ,γ)-convex and satisfies the assertion of Corollary 1, see Figure 1 up.
– The function $f:[1,2]\to[0,+\infty]$ defined by $f(x)=x^x$ is (γ,η)-convex and satisfies the assertion of Corollary 2, see Figure 1 down.
– The function $f:\left[\frac{\pi}{4},\frac{\pi}{2}\right]\to[0,+\infty]$ defined by $f(x)=x^{\sin^2(x)}$ is (η,γ)-convex and satisfies the assertion of Corollary 3, see Figure 2 up.
– The function $f:[1,2]\to[0,+\infty]$ defined by $f(x)=\sqrt[4]{\cosh(2x)}$ is (η,η)-convex and satisfies the assertion of Corollary 4, see Figure 2 down.

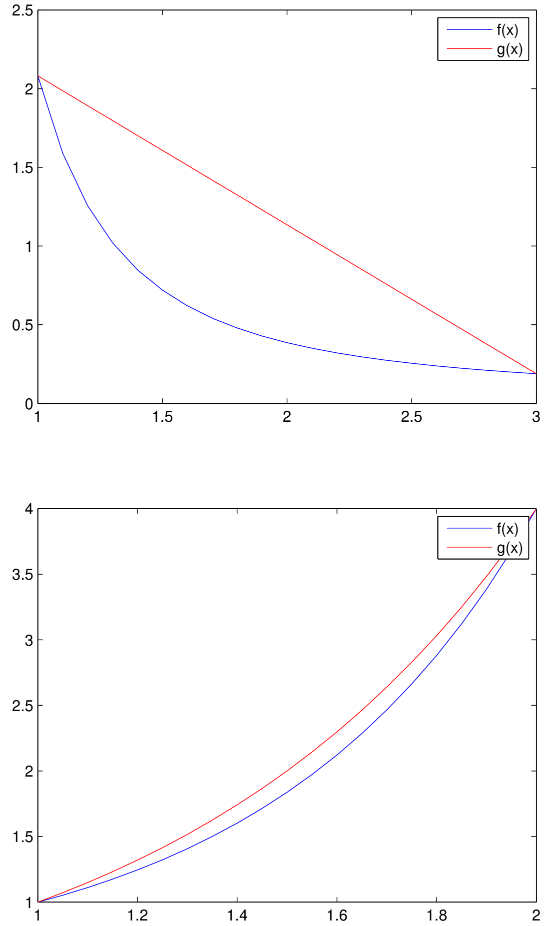


Fig. 1: A (γ,γ)-convex function (up) and a (γ,η)-convex function (down) dominated by their corresponding $g(x)$.

4 Applications

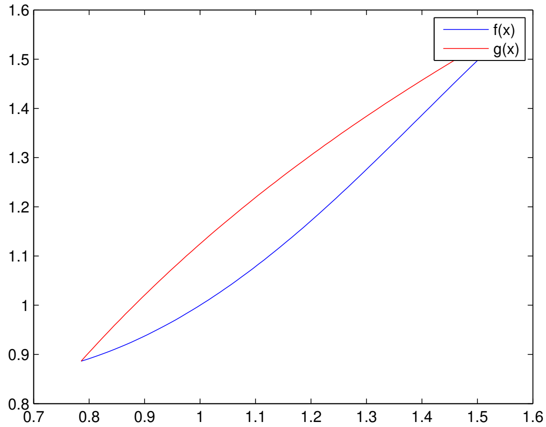
In this section, we are going to investigate some applications of the previous results for new definitions of geodesic convex functions.

Case 1. For the particular geodesic arcs $\gamma:[0,1]\to I$ and $\eta:[0,1]\to J$ defined by $\gamma_{x,y}(t)=x+(y-x)t$ and $\eta_{u,v}(t)=u\left(\frac{v}{u}\right)^{\sin\frac{\pi}{2}t}$, the following class of geodesic convex functions $f:I\to J$ is introduced:

$$f(a+(b-a)t)\le\left(\frac{f(a)}{f(b)}\right)^{\sin\frac{\pi}{2}t},$$

where $a,b\in I^o$ with $a<b$.

Consider the fuzzy measure space $(\mathbb{R},\Sigma,\mu)$. If $f:I\to J$ is a (γ,η)-convex function, then Theorem 3 gives us the



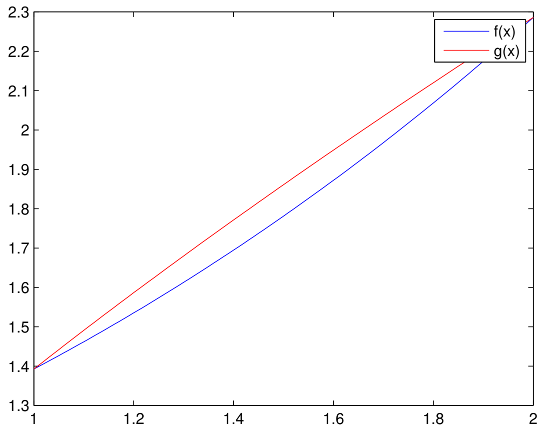


Fig. 2: A (η,γ)-convex function (up) and a (η,η)-convex function (down) dominated by their corresponding $g(x)$.

following result:

$$\int_a^b f\mathrm{d}\mu \le \begin{cases} \bigvee_\alpha \left(\alpha \wedge \mu\left(\left[a+\dfrac{2(b-a)}{\pi}\arcsin\left(\log_{\frac{f(b)}{f(a)}}\dfrac{\alpha}{f(a)}\right), b\right]\right)\right), \\ \alpha \in [f(a), f(b)),\ \ f(b) > f(a), \\ \bigvee_\alpha \left(\alpha \wedge \mu\left(\left[a, a+\dfrac{2(b-a)}{\pi}\arcsin\left(\log_{\frac{f(b)}{f(a)}}\dfrac{\alpha}{f(a)}\right)\right]\right)\right), \\ \alpha \in [f(b), f(a)),\ \ f(a) > f(b). \end{cases}$$

Case 2. For the particular geodesic arcs $\gamma : [0,1] \to I$ and $\eta : [0,1] \to J$ defined by $\gamma_{x,y}(t) = x\left(\frac{y}{x}\right)^{\sin\frac{\pi}{2}t}$ and $\eta_{u,v}(t) = u + (v-u)t$, the following class of geodesic convex functions $f : I \to J$ is introduced:

$$f\left(a\left(\frac{b}{a}\right)^{\sin\frac{\pi}{2}t}\right) \le f(a)\big(f(b) - f(a)\big)t,$$

where $a, b \in I^o$ with $a < b$.

Consider the fuzzy measure space $(\mathbb{R}, \Sigma, \mu)$. If $f : I \to J$ is a (γ,η)-convex function, then Theorem 3 gives us the following result:

$$\int_a^b f\mathrm{d}\mu \le \begin{cases} \bigvee_\alpha \left(\alpha \wedge \mu\left(\left[a\left(\dfrac{b}{a}\right)^{\sin\left(\frac{\alpha - f(a)}{f(b)-f(a)}\frac{\pi}{2}\right)}, b\right]\right)\right), \\ \alpha \in [f(a), f(b)),\ \ f(b) > f(a), \\ \bigvee_\alpha \left(\alpha \wedge \mu\left(\left[a, a\left(\dfrac{b}{a}\right)^{\sin\left(\frac{\alpha - f(a)}{f(b)-f(a)}\frac{\pi}{2}\right)}\right]\right)\right), \\ \alpha \in [f(b), f(a)),\ \ f(a) > f(b). \end{cases}$$

Case 3. For the particular geodesic arcs $\gamma : [0,1] \to I$ and $\eta : [0,1] \to J$ defined by $\gamma_{x,y}(t) = x\left(\frac{y}{x}\right)^t$ and $\eta_{u,v}(t) = u\left(\frac{v}{u}\right)^{\sin\frac{\pi}{2}t}$, the following class of geodesic convex functions $f : I \to J$ is introduced:

$$f\left(a\left(\frac{b}{a}\right)t\right) \le f(a)\left(\frac{f(b)}{f(a)}\right)^{\sin\frac{\pi}{2}t},$$

where $a, b \in I^o$ with $a < b$.

Considering the fuzzy measure space $(\mathbb{R}, \Sigma, \mu)$, for (γ,η)-convex function $f : I \to J$ according to Theorem 3 we have the following result:

$$\int_a^b f\mathrm{d}\mu \le \begin{cases} \bigvee_\alpha \left(\alpha \wedge \mu\left(\left[a\left(\dfrac{b}{a}\right)^{\frac{2}{\pi}\arcsin\left(\log_{\frac{f(b)}{f(a)}}\frac{\alpha}{f(a)}\right)}, b\right]\right)\right), \\ \alpha \in [f(a), f(b)),\ \ f(b) > f(a), \\ \bigvee_\alpha \left(\alpha \wedge \mu\left(\left[a, a\left(\dfrac{b}{a}\right)^{\frac{2}{\pi}\arcsin\left(\log_{\frac{f(b)}{f(a)}}\frac{\alpha}{f(a)}\right)}\right]\right)\right), \\ \alpha \in [f(b), f(a)),\ \ f(a) > f(b). \end{cases}$$

Case 4. For the particular geodesic arcs $\gamma : [0,1] \to I$ and $\eta : [0,1] \to J$ defined by $\gamma_{x,y}(t) = x\left(\frac{y}{x}\right)^{\sin\frac{\pi}{2}t}$ and $\eta_{u,v}(t) = u\left(\frac{v}{u}\right)^t$, the following class of geodesic convex functions $f : I \to J$ is introduced:

$$f\left(a\left(\frac{b}{a}\right)^{\sin\frac{\pi}{2}t}\right) \le f(a)u\left(\frac{f(b)}{f(a)}\right)t,$$

where $a,b \in I^o$ with $a < b$.

Considering the fuzzy measure space $(\mathbb{R},\Sigma,\mu)$, for (γ,η)-convex function $f : I \to J$ according to Theorem 3 we have the following result:

$$\int_a^b f \mathrm{d}\mu \leq \begin{cases} \bigvee_\alpha \left(\alpha \wedge \mu \left(\left[a\left(\frac{b}{a}\right)^{\sin\left(\frac{\pi}{2}\log_{\frac{f(b)}{f(a)}} \frac{\alpha}{f(a)}\right)}, b \right] \right) \right), \\ \alpha \in [f(a), f(b)), \ f(b) > f(a), \\ \bigvee_\alpha \left(\alpha \wedge \mu \left(\left[a, a\left(\frac{b}{a}\right)^{\sin\left(\frac{\pi}{2}\log_{\frac{f(b)}{f(a)}} \frac{\alpha}{f(a)}\right)} \right] \right) \right), \\ \alpha \in [f(b), f(a)), \ f(a) > f(b). \end{cases}$$

5 Conclusion

The concept of geodesic convexity with respect to the Riemannian metrics plays an important role in nonlinear optimization, e.g., in necessary and sufficient optimality criteria and in the connectedness of the solution set in linear and nonlinear complementarity systems. In this paper, a refinement of Hadamard inequality for geodesic convex functions is considered, and was generalized to the definition of Sugeno integral. This is the first paper of this kind which deals with Hadamard integral inequality for geodesic convexity, and extend this notion to nonlinear integrals. In addition, after introducing the concept of (γ,η)-convexity, an upper bound is found for each (γ,η)-convex, which is not necessarily linear.

Conflict of Interest

The authors declare that they have no conflict of interest.

References

[1] T. Rapcsák, *Geodesic convexity on* $\mathbb{R}^n$, in Generalized convexity, Springer Berlin Heidelberg, 91–103, (1994).

[2] T. Rapcsák, *Smooth Nonlinear Optimization in* $\mathbb{R}^n$, Vol. 19, Springer Science & Business Media, (2013).

[3] D.G. Luenberger, The gradient projection methods along geodesics, *Manag. Sci.*, **18**, 620–631 (1972).

[4] A. Iqbal, I. Ahmad and S. Ali, Some properties of geodesic semi-E-convex functions, *Nonlinear Anal. Theory, Methods Appl.*, **74**, 6805–6813 (2011).

[5] A. Kiliçman and W. Saleh, On geodesic semi strongly E-convex functions, *J. Interdiscipl. Math.*, **19**, 1039–1055 (2016).

[6] M. Sugeno, *Theory of Fuzzy Integrals and its Applications*, in Ph.D. Dissertation, Tokyo Institute of Technology, (1974).

[7] S. Abbaszadeh and A. Ebadian, Nonlinear integrals and Hadamard-type inequalities, *Soft Comput.*, **22**, 2843–2849 (2018).

[8] S. Abbaszadeh and M. Eshaghi, A Hadamard type inequality for fuzzy integrals based on r-convex functions, *Soft Comput.*, **20**, 3117–3124 (2016).

[9] S. Abbaszadeh, M.E. Gordji, E. Pap and A. Szakál, Jensen-type inequalities for Sugeno integral, *Inf. Sci.*, **376**, 148–157 (2017).

[10] S. Abbaszadeh, M. Eshaghi and M. de la Sen, The Sugeno fuzzy integral of log-convex functions, *J. Inequal. Appl.*, **2015**, 362 (2015).

[11] E. Pap and M. Štrboja, *Generalizations of Integral Inequalities for Integrals Based on Nonadditive Measures*, in Topics in Intelligent Engineering and Informatics 3, Intelligent Systems: Models and Applications (Ed. E. Pap), Springer, 3–22 (2013).

[12] A. Szakál, E. Pap, S. Abbaszadeh and M. Eshaghi Gordji, *Jensen Inequality with Subdifferential for Sugeno Integral*, In Mexican International Conference on Artificial Intelligence, pp. 193–199. Springer, Cham, (2016).

[13] C. Udriste, *Convex Functions and Optimization Methods on Riemannian Manifolds*, Kluwer Academic, (1994).

[14] D. Ralescu and G. Adams, The fuzzy integral, *J. Math. Anal. Appl.*, **75**, 562–570 (1980).

[15] E. Pap, *Null-additive Set Functions*, Kluwer, Dordrecht, (1995).

[16] Z. Wang and G. Klir, *Fuzzy Measure Theory*, Plenum, New York, (1992).

[17] J. Caballero and K. Sadarangani, Hermite-Hadamard inequality for fuzzy integrals, *Appl. Math. Comput.*, **215**, 2134–2138 (2009).

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