Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/050406

Finite Difference Approximation Method for Two-Dimensional Space-Time Fractional Diffusion Equation Using Nonsingular Fractional Derivative

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Received: 2 Mar. 2019, Revised: 2 Jun. 2019, Accepted: 1 Jul. 2019
Published online: 1 Oct. 2019

Abstract: In this work, we utilized the nonsingular kernel fractional derivative, known as Caputo-Fabrizio fractional derivative, to solve for the numerical solution of two-dimensional space-time fractional diffusion equation using finite difference approximation. Analysis for unconditional stability and convergence have been presented. Interestingly, by using the nonsingular kernel fractional derivative, it is found that the convergence generates a second order accuracy weighted by the memory kernel of the fractional derivative. In addition, fractional order dependency of the convergence have been discussed and compared to some previous works. Moreover, the obtained finite difference approximation method was employed to solve for a given example. Numerical test verified the analysis of this study.

Keywords: Finite difference, Caputo-Fabrizio fractional derivative, two-dimensional space-time fractional diffusion equation, convergence, stability.

1 Introduction, motivation and preliminaries

Recent developments of fractional calculus have attracted many researchers due to their potential applications in modelling real world phenomenon [1,2,3,4]. Interests in describing physical and chemical processes involving derivatives and integrals of fractional orders grows significantly [5,6,7].One of the powerful tools in modelling physical process is by means of fractional differential equation, especially when the model has memory that plays a fundamental role. Perhaps, due to the presence of the fractional order, mathematical design plays more accurate and important role in real-world problems which involve a convolution of a function and its memory [8,9] that can describe different scales. Fractional derivative is essentially one of the most important tools that are widely investigated over the history of calculus. In fact, the significance of its applications in pure mathematics, physics [10] and engineering [11] has become more evident over the past decades. However, the older version of fractional derivative contains a singularity [12] that gives a hard situation due to the risk of singularity and could possibly lead to non-full memory of the description of physical problem. Nonetheless, several studies have been reported utilizing the fractional derivative with singularity and many numerical approximations have been investigated to avoid the point of singularity while increasing the accuracy.

In the work of Zhuang et. al [13], implicit difference for two dimensional time fractional diffusion equation on a finite domain has been studied using the Caputo fractional derivative and they showed that the accuracy is dependent to the power of fractional order. Contribution of this work is essentially the extension of their work since we consider the recently introduced fractional derivative, without risking for singularity. Another study that deals with fractional derivative with singularity is the work of Zhang et.al [14], where they increased the accuracy of finite element method on nonuniform mesh and new refinement technique for temporal accuracy of fractional time diffusion equation was introduced. Anh, et al [15] presented the spectral representation of the solution for fractional diffusion equation with random initial condition. Moreover, Langlands, et al [5] presented their result on the problem they proposed using an implicit numerical schemes. Liu et.al [16] solved the fractional diffusion equation by transforming the partial differential to a system of ordinary differential equation. Higher order approximation on such fractional derivative with singularity have been reported such as the spectral method [17,18,19], but it is rather difficult to obtain a higher order accuracy due

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to the nonsingularity.

With these encountered problems and inconvenience, new fractional derivative was introduced by Caputo and Fabrizio [20] to avoid the singularity of the kernel. This new fractional derivative contains an exponential kernel, which is one of its interesting features that can construct and describe structures at different scales. Further properties of this new fractional derivative without singular kernel was investigated by Losada and Nieto [21]. With this new fractional deriative operator, additional merits are obtained in some published journals for the scrutiny of groundwater pollution equation [22], nonlinear Fisher's reaction-diffusion equation [23] and Magnetohydrodynamics free convection flow [24]. It should be noted that Akman, et.al discretized the Caputo-Fabrizio derivative [25].

Enticed by the above inferences, the targets of this paper are to solve for the numerical approximation using the nonsingular kernel fractional derivative with the consideration of an exponential memory and obtain a more accurate numerical solution to explain physical real-world phenomenon and its application to fractional time diffusion equation and investigate its stability and convergence using the following definitions of Caputo – Fabrizio fractional derivative.

Definition 1. *Let $f \in H^1(a,b), b > a, \alpha \in (0,1)$ then the Caputo-Fabrizio fractional derivative is defined as*

$$ {}^{CF}D_t^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha}\int_0^t f'(s)\exp\left\{-\alpha\frac{t-s}{1-\alpha}\right\}ds, \tag{1}$$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$. It can also be applied to the function that does not belong to $H^1(a,b)$, such as the Caputo-Fabrizio (CF) fractional derivative can be reformulated as

$$ {}^{CF}D_t^{\alpha} f(t) = \frac{\alpha M(\alpha)}{1-\alpha}\int_0^t (f(t)-f(s))\exp\left\{-\alpha\frac{t-s}{1-\alpha}\right\}ds. \tag{2}$$

In view of definition 1, the integral term has no singular behavior, which makes it as one of its interesting feature compare to the previous version of the fractional derivative, which is the Caputo derivative.
This paper is organized as follows: in section 2, we focus on the main numerical analysis of this study where the main results of this work have been discussed, such as the numerical approximation, unconditional stability, convergence, order of accuracy, and etc. in section 3, test function were solved and the obtained finite difference approximation method was employed in order or verify the numerical analysis. Lastly, section 4 concluded this paper.

2 Finite difference approximation for 2D fractional diffusion equation

Throughout the paper, we will be considering the two dimensional fractional diffusion equation of the form

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = N(x,y,t)\frac{\partial^2 u}{\partial x^2} + M(x,y,t)\frac{\partial^2 u}{\partial y^2} + F(x,y,t), \tag{3}$$

where

$$u = u(x,y,t)$$

N, M are space-time dependent coefficient. Following the given initial and boundary condition as

$$u(x,y,0) = \phi(x,y), \quad (x,y) \in \Gamma, \tag{4}$$

$$u(x,y,t)|_{\partial\Gamma} = 0, \tag{5}$$

such that $\Gamma = \{(x,y)|x \in [0,X], y \in [0,Y]\}$ and $N, M > 0$. If the time fractional derivative follows the definition of Caputo-Fabrizio fractional derivative (1) of order α as

$$ {}^{CF}D_t^{\alpha} u(x,y,t) = \frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{M(\alpha)}{1-\alpha}\int_0^t \frac{\partial u(x,y,s)}{\partial s}\exp\left\{-\alpha\frac{t-s}{1-\alpha}\right\}ds. \tag{6}$$

Defining a uniform mesh along spatial and temporal coordinates, we can subdivide the intervals as follows:

$$\begin{aligned}
[0,T] &\implies 0 = t_o < t_1 < t_2 < \cdots < t_A = T, \\
[0,X] &\implies 0 = t_o < x_1 < x_2 < \cdots < x_B = X, \\
[0,Y] &\implies 0 = t_o < y_1 < y_2 < \cdots < y_C = Y,
\end{aligned}$$

where the spatial and temporal iteration can be defined as $t_n = n\tau, x_l = lh_x, y_m = mh_y$ and the increments as $\tau = T/A, h_x = X/B, h_y = Y/C$. Now, let the numerical approximation be $u^n_{l,m}$ of $y(x_l, y_m, t_n)$, $f^n_{l,m}$ of $f(x_l, y_m, t_n)$ and $\phi_{l,m}$ of $\phi(x_l, y_m)$. Using the fundamental theorem for calculus, we can write the approximated numerical scheme of equation (6) as

$$\begin{aligned}
\frac{\partial^\alpha u(x_l, y_m, t_{n+1})}{\partial t^\alpha} &= \frac{1}{1-\alpha}\int_0^{t_{n+1}} \frac{\partial u(x_l, y_m, s)}{\partial s} \exp\left[-\frac{\alpha}{1-\alpha}(t_{n+1}-s)\right] ds \\
&\approx \frac{1}{1-\alpha}\sum_{j=0}^{n}\int_{j\tau}^{(j+1)\tau} \frac{\partial u(x_l, y_m, t_j)}{\partial s} \exp\left[-\frac{\alpha}{1-\alpha}(t_{n+1}-s)\right] ds \\
&\approx \frac{1}{1-\alpha}\sum_{j=0}^{n} \frac{u(x_l, y_m, t_{j+1}) - u(x_l, y_m, t_j)}{\tau}\int_{j\tau}^{(j+1)\tau} \exp\left[-\frac{\alpha}{1-\alpha}(t_{n+1}-s)\right] ds \\
&\approx \frac{1}{1-\alpha}\sum_{j=0}^{n} \frac{u(x_l, y_m, t_{j+1}) - u(x_l, y_m, t_j)}{\tau}\int_{(n-j)\tau}^{(n-j+1)\tau} \exp\left[-\frac{\alpha}{1-\alpha}(\mathscr{T})\right] d\mathscr{T} \\
&\approx \frac{1}{1-\alpha}\sum_{j=0}^{n} \frac{u(x_l, y_m, t_{n-j+1}) - u(x_l, y_m, t_{n-j})}{\tau}\int_{j\tau}^{(j+1)\tau} \exp\left[-\frac{\alpha}{1-\alpha}(\mathscr{T})\right] d\mathscr{T} \\
&\approx \frac{\exp\left(-\frac{\alpha\tau}{1-\alpha}\right) - 1}{\alpha\tau}\left[u(x_l, y_m, t_{n+1}) - u(x_l, y_m, t_n)\right] \\
&+ \frac{1}{\alpha\tau}\sum_{j=1}^{n}\left[u(x_l, y_m, t_{n-j+1}) - u(x_l, y_m, t_{n-j}),\right] D_j
\end{aligned} \tag{7}$$

where

$$D_j = \exp\left(-\frac{\alpha\tau}{1-\alpha}(j+1)\right) - \exp\left(-\frac{\alpha\tau}{1-\alpha}j\right).$$

Theorem 1. *Let $u^n_{l,m}$ be the numerical approximation for a function $u(x,y,t)$ on the domain $[0,X]\times[0,Y]\times[0,T]$. Then the time-fractional derivative of u in Caputo-Fabrizio fractional derivative sense of order $0<\alpha<1$ can be written as*

$$\frac{\partial^\alpha u(x_l, y_m, t_n)}{\partial t^\alpha} = \frac{1}{\alpha}\sum_{j=0}^{n}\left[\frac{u(x_l, y_m, t_{j+1}) - u(x_l, y_m, t_j)}{\tau}\right] D_j + \mathscr{O}_\alpha(\tau). \tag{8}$$

Proof. From the fundamental theorem of calculus, the following differential equation holds

$$\frac{\partial u(x_l, y_m, s)}{\partial s} = \frac{\partial u(x_l, y_m, t_j)}{\partial s} + \mathscr{O}(\tau), \quad t_j \le s \le t_{j+1} \tag{9}$$

$$\frac{\partial u(x_l, y_m, t_j)}{\partial s} = \frac{\partial u(x_l, y_m, t_{j+1}) - u(x_l, y_m, t_j)}{\tau} + \mathscr{O}(\tau) \tag{10}$$

then following the definition of Caputo-Fabrizio fractional derivative and doing the same method as in equation (7) we can obtain

$$\frac{\partial^\alpha u(x_l, y_m, t_n)}{\partial t^\alpha} = \frac{1}{\alpha}\sum_{j=0}^{n}\left[\frac{u(x_l, y_m, t_{j+1}) - u(x_l, y_m, t_j)}{\tau}\right] D_j + \frac{1}{\alpha}\sum_{j=0}^{n} D_j \mathscr{O}(\tau) \tag{11}$$

and notice that

$$\sum_{j=0}^{n}\left[\exp\left(-\frac{\alpha\tau}{1-\alpha}(j+1)\right) - \exp\left(-\frac{\alpha\tau}{1-\alpha}(j)\right)\right] = \exp\left(-\frac{\alpha\tau}{1-\alpha}\right),$$

then equation (11) becomes

$$\frac{\partial^\alpha u(x_l, y_m, t_n)}{\partial t^\alpha} = \frac{1}{\alpha}\sum_{j=0}^{n}\left[\frac{u(x_l, y_m, t_{j+1}) - u(x_l, y_m, t_j)}{\tau}\right] D_j + \frac{1}{\alpha}\exp\left(-\frac{\alpha\tau}{1-\alpha}\right)\mathscr{O}(\tau) \tag{12}$$

we can immediately write the error estimate for temporal fractional derivate as

$$\mathscr{O}_\alpha(\tau) = \frac{1}{\alpha}\exp\left(-\frac{\alpha\tau}{1-\alpha}\right)\mathscr{O}(\tau)$$

which satisfies the claimed estimate for the time-fractional derivative only at level $t = t_n$.

The complete truncation term for space-time fractional derivative will be discussed later.

Lemma 1.*Suppose that*

$$\mathscr{R}^{\alpha}_{\tau}u(x_l,y_m,t_n)=\frac{1}{\alpha}\sum_{j=1}^{n}\left[\frac{u(x_l,y_m,t_{j+1})-u(x_l,y_m,t_j)}{\tau}\right]D_j.$$

Then the following inequalities hold

$$\left|\frac{\partial^{\alpha}u(x_l,y_m,t_n)}{\partial t^{\alpha}}-\mathscr{R}^{\alpha}_{\tau}u(x_l,y_m,t_n)\right|\leq\mathscr{M}\tau\int_{(j-1)\tau}^{j\tau}\exp\left[-\frac{\alpha}{1-\alpha}(t_n-s)\right]ds\leq\mathscr{M}_1\tau, \tag{13}$$

where $\mathscr{M},\mathscr{M}_1$ are constants.

*Proof.*The proof of this lemma is straightforward, using the theorem 2.1 and the fact that

$$n\mathscr{O}(\tau)<\int_{(j-1)\tau}^{j\tau}\exp\left[-\frac{\alpha}{1-\alpha}(t_j-s)\right]ds$$

(by direct manipulation of the integral and the error estimate found in theorem 2.1) and this finishes the proof.

Descritizing 3 and adopting a second-order space derivative starting from temporal level at $t=t_{n+1}$ where $1\leq n\leq A$ for a given temporal grid size A we have

$$\begin{aligned}u^{n+1}_{l,m}-u^{n}_{l,m}+\sum_{j=1}^{n}\left(u^{n-j+1}_{l,m}-u^{n-j}_{l,m}\right)d_j=\sigma_1\left(N^{n+1}_{l,m}(u^{n+1}_{l+1,m}-2u^{n+1}_{l,m}-u^{n+1}_{l-1,m})\right)\\+\sigma_2\left(M^{n+1}_{l,m}(u^{n+1}_{l,m+1}-2u^{n+1}_{l,m}-u^{n+1}_{l,m-1})\right)\\+\frac{\alpha\tau}{\left[\exp\left(-\frac{\alpha}{1-\alpha}\tau\right)-1\right]}f^{n+1}_{l,m},\end{aligned} \tag{14}$$

where

$$\sigma_1=\frac{\alpha\tau}{\left[\exp\left(-\frac{\alpha}{1-\alpha}\tau\right)-1\right]h_x^2},\quad\sigma_2=\frac{\alpha\tau}{\left[\exp\left(-\frac{\alpha}{1-\alpha}\tau\right)-1\right]h_y^2},\quad d_j=\frac{D_j}{\exp\left(-\frac{\alpha}{1-\alpha}\tau\right)-1}.$$

Equation (14) is in implicit form for numerical solution of equation (3). Now, rearranging we have

$$\begin{aligned}u^{n+1}_{l,m}(1+2\lambda_1+2\lambda_2)-\lambda_1(u^{n+1}_{l+1,m}-u^{n+1}_{l-1,m})-\lambda_2(u^{n+1}_{l,m+1}-u^{n+1}_{l,m-1})\\=u^{n}_{l,m}-\sum_{j=1}^{n}\left(u^{n-j+1}_{l,m}-u^{n-j}_{l,m}\right)d_j+\frac{\alpha\tau}{\left[\exp\left(-\frac{\alpha}{1-\alpha}\tau\right)-1\right]}f^{n+1}_{l,m},\end{aligned} \tag{15}$$

where

$$\lambda_1=\sigma_1N^{n+1}_{l,m},\quad\lambda_2=\sigma_2M^{n+1}_{l,m}.$$

Thus, to solve for the solution, we can transform equation (15) into matrix form:
For $n=0$:

$$\begin{aligned}u^{1}_{l,m}(1+2\lambda_1+2\lambda_2)-\lambda_1(u^{1}_{l+1,m}-u^{1}_{l-1,m})-\lambda_2(u^{1}_{l,m+1}-u^{1}_{l,m-1})\\=u^{0}_{l,m}+\alpha\tau\exp\left(\frac{\alpha}{1-\alpha}\tau\right)f^{1}_{l,m}.\end{aligned} \tag{16}$$

For $n>0$:

$$\begin{aligned}u^{n+1}_{l,m}(1+2\lambda_1+2\lambda_2)-\lambda_1(u^{n+1}_{l+1,m}-u^{n+1}_{l-1,m})-\lambda_2(u^{n+1}_{l,m+1}-u^{n+1}_{l,m-1})\\=u^{n}_{l,m}-\sum_{j=1}^{n}\left(u^{n-j+1}_{l,m}-u^{n-j}_{l,m}\right)d_j+\frac{\alpha\tau}{\left[\exp\left(-\frac{\alpha}{1-\alpha}\tau\right)-1\right]}f^{n+1}_{l,m}\\=\left(\frac{2-\left(\exp\left[\frac{\alpha}{1-\alpha}\tau\right]+\exp\left[-\frac{\alpha}{1-\alpha}\tau\right]\right)}{1-\exp\left[\frac{\alpha}{1-\alpha}\tau\right]}\right)u^{n}_{l,m}+u^{0}_{l,m}d_j\\-\sum_{j=1}^{n-1}u^{n-j}_{l,m}(d_j-d_{j-1})+\frac{\alpha\tau}{\left[\exp\left(-\frac{\alpha}{1-\alpha}\tau\right)-1\right]}f^{n+1}_{l,m}.\end{aligned} \tag{17}$$

One can easily write

$$u^1 = \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \\ \vdots \\ u_{B-1}^1 \end{bmatrix}, \quad f^1 = \begin{bmatrix} f_1^1 \\ f_2^1 \\ f_3^1 \\ \vdots \\ f_{B-1}^1 \end{bmatrix}, \quad u^0 = \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{B-1} \end{bmatrix} \tag{18}$$

and

$$u_l^n = \begin{bmatrix} u_{l,1}^n \\ u_{l,2}^n \\ u_{l,3}^n \\ \vdots \\ u_{l,C-1}^n \end{bmatrix}, \quad f_l^n = \begin{bmatrix} f_{l,1}^n \\ f_{l,2}^n \\ f_{l,3}^n \\ \vdots \\ f_{l,C-1}^n \end{bmatrix}, \quad u_l^0 = \phi_l = \begin{bmatrix} \phi_{l,1} \\ \phi_{l,2} \\ \phi_{l,3} \\ \vdots \\ \phi_{l,C-1} \end{bmatrix}. \tag{19}$$

Hence, we can write the matrix form as

$$\begin{cases} \kappa \mathbf{u}^1 &= \mathbf{u}^0 + \alpha\tau \exp\left(\frac{\alpha}{1-\alpha}\tau\right)\mathbf{f}^1 \\ \kappa \mathbf{u}^{n+1} &= \sum_{j=0}^{n-1} u^{n-j}(d_j - d_{j+1}) + d_j \mathbf{u}^0 + \frac{\alpha\tau}{\exp\left(-\frac{\alpha}{1-\alpha}\tau\right)-1}\mathbf{f}^{n+1} \\ \mathbf{u}^0 &= \phi \end{cases}, \tag{20}$$

where κ is the matrix coefficients.

Lemma 2.*The approximate solution of two-dimensional fractional diffusion equation in difference approximation using nonsingular kernel fractional derivative is unconditionally stable.*

*Proof.*The proof of this lemma is the consequence of Theorem 2.4.

Theorem 2.*For all $n \geq 0$ and $r \geq n$ such that $\|\omega^r\|_{n\to\infty} \leq \|\omega^0\|_{n\to\infty}$ holds.*

*Proof.*Suppose $u_{l,m}^{\hat{n}}$ be the approximate solution of (15) and let the error estimate be

$$\omega_{l,m}^n = u_{l,m}^n - u_{l,m}^{\hat{n}} \tag{21}$$

which satisfies

$$\omega_{l,m}^1(1+2\lambda_1+2\lambda_2) - \lambda_1(\omega_{l+1,m}^1 - \omega_{l-1,m}^1) - \lambda_2(\omega_{l,m+1}^1 - \omega_{l,m-1}^1) = d_0\omega_{l,m}^0, \tag{22}$$

$$\begin{aligned} \omega_{l,m}^{n+1}(1+2\lambda_1+2\lambda_2) - \lambda_1(\omega_{l+1,m}^{n+1} - \omega_{l-1,m}^{n+1}) - \lambda_2(\omega_{l,m+1}^{n+1} - \omega_{l,m-1}^{n+1}) = d_j\omega_{l,m}^0 \\ + \left(\frac{2 - \left(\exp\left[\frac{\alpha}{1-\alpha}\tau\right] + \exp\left[-\frac{\alpha}{1-\alpha}\tau\right]\right)}{1-\exp\left[\frac{\alpha}{1-\alpha}\tau\right]}\right)\omega_{l,m}^n - \sum_{j=1}^{n-1}\omega_{l,m}^{n-j}(d_j - d_{j-1}), \end{aligned} \tag{23}$$

or in matrix form

$$\begin{cases} \kappa\omega^1 = \omega^0 \\ \kappa\omega^{n+1} = (d_0-d_1)\omega^n + (d_1-d_2)\omega^{n-1} + \cdots + (d_{n-1}-d_n)\omega^1 + d_n\omega^0 \\ \omega^0 \end{cases}$$

such that

$$\omega^n = \begin{bmatrix} \omega_1^n \\ \omega_2^n \\ \omega_3^n \\ \vdots \\ \omega_{B-1}^n \end{bmatrix}, \quad \omega_l^n = \begin{bmatrix} \omega_{l,1}^n \\ \omega_{l,2}^n \\ \omega_{l,3}^n \\ \vdots \\ \omega_{l,C-1}^n. \end{bmatrix} \tag{24}$$

We define

$$|\omega_{\mu,\nu}^r| = \max_{1\leq l\leq B-1; 1\leq m\leq C-1} |\omega_{l,m}^r|.$$

So that, the inequality

$$|\omega^0_{\mu,\nu}| \le \|\omega^0\|_{n\to\infty}$$

is satisfied. Now,

$$\begin{aligned}
|\omega^1_{\mu,\nu}| &\le |\omega^1_{\mu,\nu}|(1+2\lambda_1+2\lambda_2) - \lambda_1(|\omega^1_{\mu+1,\nu}| - |\omega^1_{\mu-1,\nu}|) - \lambda_2(|\omega^1_{\mu,\nu+1}| - |\omega^1_{\mu,\nu-1}|) \\
&\le \left|\omega^1_{\mu,\nu}(1+2\lambda_1+2\lambda_2) - \lambda_1(\omega^1_{\mu+1,\nu} - \omega^1_{\mu-1,\nu}) - \lambda_2(\omega^1_{\mu,\nu+1} - \omega^1_{\mu,\nu-1})\right| \\
&\le |\omega^0_{\mu,\nu}| \\
&\le \|\omega^0\|_{n\to\infty}. \qquad (25)
\end{aligned}$$

This also means that

$$\|\omega^1\|_{n\to\infty} \le \|\omega^0\|_{n\to\infty}. \qquad (26)$$

One can easily check that for any $r \ge n \ge 0$, we have

$$\begin{aligned}
|\omega^{n+1}_{\mu,\nu}| &= |\omega^{n+1}_{\mu,\nu}|(1+2\lambda_1+2\lambda_2) - \lambda_1(|\omega^{n+1}_{\mu,\nu}| - |\omega^{n+1}_{\mu,\nu}|) - \lambda_2(|\omega^{n+1}_{\mu,\nu}| - |\omega^{n+1}_{\mu,\nu}|) \\
&\le |\omega^{n+1}_{\mu,\nu}|(1+2\lambda_1+2\lambda_2) - \lambda_1(|\omega^{n+1}_{\mu+1,\nu}| - |\omega^{n+1}_{\mu-1,\nu}|) - \lambda_2(|\omega^{n+1}_{\mu,\nu+1}| - |\omega^{n+1}_{\mu,\nu-1}|) \\
&\le |\omega^{n+1}_{\mu,\nu}(1+2\lambda_1+2\lambda_2) - \lambda_1(\omega^{n+1}_{\mu+1,\nu} - \omega^{n+1}_{\mu-1,\nu}) - \lambda_2(\omega^{n+1}_{\mu,\nu+1} - \omega^{n+1}_{\mu,\nu-1}|) \\
&= (d_0 - d_1)|\omega^n_{\mu,\nu}| + \sum_{j=1}^{n-1}(d_j - d_{j+1})|\omega^{n-j}_{\mu,\nu}| + d_j|\omega^0_{\mu,nu}| \\
&\le (d_0 - d_1)\|\omega^n\| + \sum_{j=1}^{n-1}(d_j - d_{j+1})\|\omega^{n-j}\| + d_j\|\omega^0\| \\
&\le \|\omega^0\|_{n\to\infty.} \qquad (27)
\end{aligned}$$

Therefore,

$$\|\omega^r\|_{n\to\infty} \le \|\omega^0\|_{n\to\infty}. \qquad (28)$$

Interestingly, with the use of the nonsingular kernel fractional derivative, more simplified form of unconditionally stable numerical approximation can be obtained from the expressions shown above.

Theorem 3(Convergence). *Suppose $u(x_l, y_m, t_n)$ be the exact solution for the fractional partial differential equation (3) and let*

$$U^n_{l,m} = u(x_l, y_m, t_n) - u^n_{l,m}$$

such that

$$\mathbf{U}^n = \begin{bmatrix} \mathbf{U}^n_1 \\ \mathbf{U}^n_2 \\ \mathbf{U}^n_3 \\ \vdots \\ \mathbf{U}^n_{B-1} \end{bmatrix}, \quad \mathbf{U}^n_l = \begin{bmatrix} \mathbf{U}^n_{l,1} \\ \mathbf{U}^n_{l,2} \\ \mathbf{U}^n_{l,3} \\ \vdots \\ \mathbf{U}^n_{l,C-1} \end{bmatrix}.$$

Then,

$$|\mathscr{O}^{n+1}_{l,m}| \le \mathscr{N}\left(\tau^2 \exp\left[-\frac{\alpha}{1-\alpha}\tau\right] + \alpha\tau h_x^2 + \alpha\tau h_y^2\right)$$

holds for some constant $\mathscr{N}$.

*Proof.*Using equations (16) and (17), we have

$$U^1_{l,m}(1+2\lambda_1+2\lambda_2) - \lambda_1(U^1_{l+1,m} - U^1_{l-1,m}) - \lambda_2(U^1_{l,m+1} - U^1_{l,m-1}) = \mathscr{O}^0_{l,m} \qquad (29)$$

$$\begin{aligned}
U^{n+1}_{l,m}(1+2\lambda_1+2\lambda_2) &- \lambda_1(U^{n+1}_{l+1,m} - U^{n+1}_{l-1,m}) - \lambda_2(U^{n+1}_{l,m+1} - U^{n+1}_{l,m-1}) \\
&= \left(\frac{2 - \left(\exp\left[\frac{\alpha}{1-\alpha}\tau\right] + \exp\left[-\frac{\alpha}{1-\alpha}\tau\right]\right)}{1 - \exp\left[\frac{\alpha}{1-\alpha}\tau\right]}\right)U^n_{l,m} + U^0_{l,m}d_j \\
&\quad - \sum_{j=1}^{n-1} U^{n-j}_{l,m}(d_j - d_{j-1}) + \mathscr{O}^{n+1}_{l,m}. \qquad (30)
\end{aligned}$$

If the temporal level $t = t_{n+1}$ is adopted, we can immediately identify that

$$\begin{aligned}\mathscr{O}_{l,m}^{n+1} &= \sum_{j=0}^{n} D_j \left(u(x_l,y,m,t_{n-j+1}) - u(l,m,t_{n-j}\right) \\ &- \lambda_1 \left(u(x_{l+1},y_m,t_{n+1}) - 2u(x_l,y_m,t_{n+1}) + u(x_{l-1},y_m,t_{n+1})\right) \\ &- \lambda_1 \left(u(x_l,y_{m-1},t_{n+1}) - 2u(x_l,y_m,t_{n+1}) + u(x_l,y_{m-1},t_{n+1})\right) \\ &- \frac{\alpha\tau}{\exp\left(-\frac{\alpha}{1-\alpha}\tau\right) - 1} f_{l,m}^{n+1}. \end{aligned} \tag{31}$$

Now, from *Theorem 2.1* and Taylor expansion for second order partial differential equation for space variable, we can get the following equalities

$$\frac{\partial^\alpha u(x_l,y,m,t_{n+1})}{\partial t^\alpha} = \frac{1}{\alpha\tau} \sum_{j=0}^{n} \left(u(x_l,y,m,t_{n-j+1}) - u(l,m,t_{n-j}\right) D_j + \mathscr{O}_\alpha(\tau) \tag{32}$$

$$\frac{\partial^2 u(x_l,y,m,t_{n+1})}{\partial x^2} = \left[\frac{u(x_{l+1},y_m,t_{n+1}) - 2u(x_l,y_m,t_{n+1}) + u(x_{l-1},y_m,t_{n+1})}{h_x^2}\right] + \mathscr{O}_x(h_x^2) \tag{33}$$

$$\frac{\partial^2 u(x_l,y,m,t_{n+1})}{\partial y^2} = \left[\frac{u(x_l,y_{m+1},t_{n+1}) - 2u(x_l,y_m,t_{n+1}) + u(x_l,y_{m-1},t_{n+1})}{h_y^2}\right] + \mathscr{O}_y(h_y^2). \tag{34}$$

Thus,

$$\mathscr{O}_{l,m}^{n+1} = \mathscr{O}\left(\tau^2 \exp\left[-\frac{\alpha}{1-\alpha}\tau\right] + \alpha\tau h_x^2 + \alpha\tau h_y^2\right). \tag{35}$$

Therefore, for constant $\mathscr{N}$ it holds that

$$|\mathscr{O}_{l,m}^{n+1}| \le \mathscr{N}\left(\tau^2 \exp\left[-\frac{\alpha}{1-\alpha}\tau\right] + \alpha\tau h_x^2 + \alpha\tau h_y^2\right). \tag{36}$$

This theorem shows that the truncation term of numerical approximation for temporal mesh points in Caputo-Fabrizio fractional derivative sense is in second order of time increment weighted by the fractional derivative exponential memory kernel. In comparison to some previous literature [13,14], it has been shown that the accuracy of the numerical solution was in fractional order, in here we can see that the second order accuracy can be generated, in accordance with the classical methods, but weighted by the memory kernel of the nonsingular fractional derivative. We present the next lemma to verify the convergency.

Lemma 3.*For any $n \ge 0$ and $r \ge n$. Then the norm*

$$\|\xi^{\mathbf{r}}\|_{n\to\infty} \le \mathscr{N}\,[d_n]^{-1}\left(\tau^2 \exp\left[-\frac{\alpha}{1-\alpha}\tau\right] + \alpha\tau h_x^2 + \alpha\tau h_y^2\right)$$

holds for some constant $\mathscr{N}$.

*Proof.*Suppose $\left\|\xi^{\mathbf{1}}\right\| = \max_{1\le l\le B-1, 1\le m\le C-1} |\xi_{l,m}^1| = |\xi_{\mu,v}^1|$ and

$$\|\xi^{\mathbf{n}}\|_{n\to\infty} \le \mathscr{N}\,[d_{n-1}]^{-1}\left(\tau^2 \exp\left[-\frac{\alpha}{1-\alpha}\tau\right] + \alpha\tau h_x^2 + \alpha\tau h_y^2\right)$$

such that

$$\begin{aligned}|\xi_{\mu,v}^1| &\le |\xi_{\mu,v}^1|(1+2\lambda_1+2\lambda_2) - \lambda_1(|\xi_{\mu,v}^1| - |\xi_{\mu,v}^1|) - \lambda_2(|_{\mu,v}^1| - |\xi_{\mu,v}^1|) \\ &\le \left|\xi_{\mu,v}^1(1+2\lambda_1+2\lambda_2) - \lambda_1(\xi_{\mu+1,v}^1 - \xi_{\mu-1,v}^1) - \lambda_2(\xi_{\mu,v+1}^1 - \xi_{\mu,v-1}^1)\right| \\ &= |\mathscr{O}_{\mu,v}^1| \le \mathscr{N}\,[d_0]^{-1}\left(\tau^2 \exp\left[-\frac{\alpha}{1-\alpha}\tau\right] + \alpha\tau h_x^2 + \alpha\tau h_y^2\right). \end{aligned} \tag{37}$$

Now, for $n > 0$ and $r \geq n$, let $|\xi_{\mu,\nu}^{n+1}| = \max_{1\leq l\leq B-1, 1\leq m\leq C-1} |\xi_{l,m}^{n+1}|$ we have

$$\begin{aligned}
|\xi_{\mu,\nu}^{n+1}| &\leq \left|\xi_{\mu,\nu}^{1}(1+2\lambda_1+2\lambda_2) - \lambda_1(\xi_{\mu+1,\nu}^{1} - \xi_{\mu-1,\nu}^{1}) - \lambda_2(\xi_{\mu,\nu+1}^{1} - \xi_{\mu,\nu-1}^{1})\right| \\
&= |\sum_{j=0}^{n-1}(d_j - d_{j+1})\xi_{\mu,\nu}^{n-j} + \mathscr{O}_{l,m}^{n+1}| \\
&\leq \sum_{j=0}^{n-1}(d_j - d_{j+1})|\xi_{\mu,\nu}^{n-j}| + |\mathscr{O}_{l,m}^{n+1}| \\
&\leq \sum_{j=0}^{n-1}(d_j - d_{j+1})|\xi_{\mu,\nu}^{n-j}| + \mathscr{N}\left(\tau^2 \exp\left[-\frac{\alpha}{1-\alpha}\tau\right] + \alpha\tau h_x^2 + \alpha\tau h_y^2\right) \\
&\leq \sum_{j=0}^{n-1}(d_j - d_{j+1})\|\xi_{\mu,\nu}^{\mathbf{n-j}}\|_{n\to\infty} + \mathscr{N}\left(\tau^2 \exp\left[-\frac{\alpha}{1-\alpha}\tau\right] + \alpha\tau h_x^2 + \alpha\tau h_y^2\right) \\
&\leq \left(\sum_{j=0}^{n-1}(d_j - d_{j+1}) + d_n\right)(d_n)^{-1}\mathscr{N}\left(\tau^2 \exp\left[-\frac{\alpha}{1-\alpha}\tau\right] + \alpha\tau h_x^2 + \alpha\tau h_y^2\right) \\
&= (d_n)^{-1}\mathscr{N}\left(\tau^2 \exp\left[-\frac{\alpha}{1-\alpha}\tau\right] + \alpha\tau h_x^2 + \alpha\tau h_y^2\right).
\end{aligned} \tag{38}$$

Now, if the constant $\mathscr{N}$ exists and is α-dependent, then we must have

$$\begin{aligned}
\lim_{n\to\infty} \frac{(d_n)^{-1}}{\exp\left[\frac{\alpha}{1-\alpha}\tau\right]} &= \lim_{n\to\infty} \frac{1}{\exp\left[\frac{\alpha}{1-\alpha}\tau\right]} \left[\frac{\exp\left(-\frac{\alpha\tau}{1-\alpha}(n+1)\right) - \exp\left(-\frac{\alpha\tau}{1-\alpha}n\right)}{\exp\left[\frac{-\alpha}{1-\alpha}\tau\right] - 1}\right]^{-1} \\
&= \lim_{n\to\infty} \frac{1}{\exp\left[\frac{\alpha}{1-\alpha}\tau\right]} \left[\frac{\exp\left(-\frac{\alpha\tau}{1-\alpha}\right)}{\exp\left[\frac{-\alpha}{1-\alpha}\tau\right] - 1}\right]^{-1} \\
&= \exp\left(-\frac{\alpha\tau}{1-\alpha}\right) - 1 \\
&= \exp\left(-\frac{\alpha}{1-\alpha}\frac{T}{A}\right) - 1.
\end{aligned} \tag{39}$$

Since $\tau = \frac{T}{A}$ vanishes, the norm of error vanishes as $n \to A \to \infty$. Hence, the approximation converges if $n\tau \leq T$ is finite. Moreover, this finishes the proof.

The stability and convergence of the numerical approximation for the solution of two-dimensional space-time fractional diffusion equation has been demonstrated. Unconditional stability were analyzed as well as convergency. It was shown that the numerical approximation, in terms of temporal variable, using nonsingular fractional derivative, generates second order convergency rate weighted by the memory kernel of the fractional derivative. Interestingly, singularity of the numerical approximation was avoided.

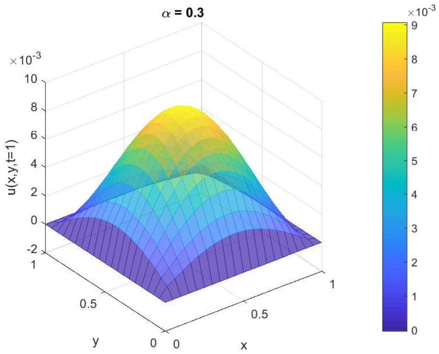
3 Numerical test

To test the validity of the presented algorithm. Consider the two-dimensional diffusion equation without the forcing term f, with constant coefficients, and an exact solution

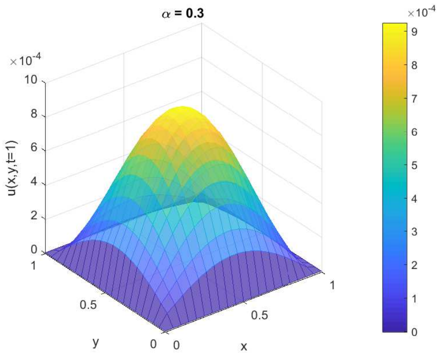
$$u(x,y,t) = \sin(\pi x)\sin(\pi y)(t^2+1)$$

and initial and boundary conditions

$$u(x,y,0) = \sin(\pi x)\sin(\pi y), \quad u(x,y,t) = 0$$
$$\Gamma = \{(x,y,t) : 0 \leq x \leq 1; 0 \leq y \leq 1, 0 \leq t \leq 1\}.$$

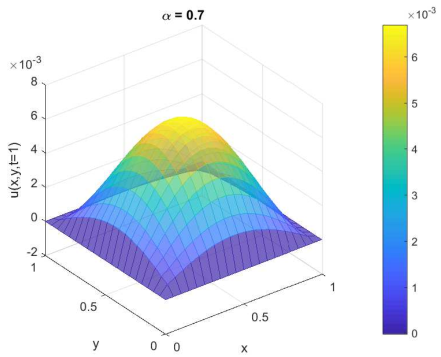


(a)

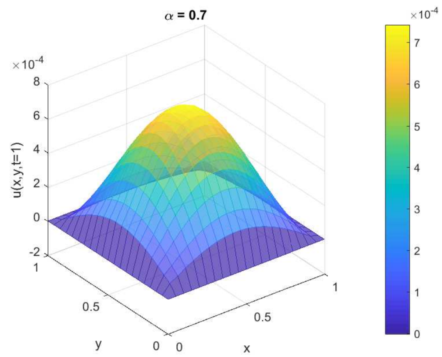


(b)

Fig. 1: Mesh plot for numerical solution of example 1 with $\alpha = 0.3$, $h_x = h_y = 0.05$, (left) $\tau = 0.1$, and (right) $\tau = 0.01$.



(a)



(b)

Fig. 2: Mesh plot for numerical solution of example 1 with $\alpha = 0.7$, $h_x = h_y = 0.05$, (left) $\tau = 0.1$, and (right) $\tau = 0.01$.

The maximum error estimate and convergency rate of the presented numerical solution is defined as

$$\omega_\infty = \max_{1\le l\le B-1, 1\le m\le C-1} |u(x_l, y_m, t_n) - u^n_{l,m}|, \quad \mathbf{e} = \log_2\left(\frac{\omega(B,C,A/2)}{\omega(B,C,A)}\right).$$

Table 1 presents the different maximum errors at different fractional order and convergence rate. The numerical solution is in excellent agreement to the presented analytical solution of the problem. In addition the convergence

Table 1: Maximum error at t=1.0 with different values of fractional order, time steps, and space steps.

α	τ	$h_x = h_y$	ω_∞	e
0.3	0.1	$\frac{1}{20}$	9.1132e-3	2.1308
		$\frac{1}{40}$	9.1211e-3	2.1020
		$\frac{1}{100}$	9.1242e-3	2.1015
	0.01	$\frac{1}{20}$	9.2461e-4	3.1272
		$\frac{1}{40}$	9.2776e-4	3.1257
		$\frac{1}{100}$	9.2843e-4	3.1254
	0.001	$\frac{1}{20}$	9.2531e-5	4.1302
		$\frac{1}{40}$	9.2938e-5	4.1283
		$\frac{1}{100}$	9.3025e-5	4.1279
0.7	0.1	$\frac{1}{20}$	6.7086e-3	2.2360
		$\frac{1}{40}$	6.7113e-3	2.2341
		$\frac{1}{100}$	6.7134e-3	2.2335
	0.01	$\frac{1}{20}$	7.4208e-4	3.2227
		$\frac{1}{40}$	7.4473e-4	3.2212
		$\frac{1}{100}$	7.4553e-4	3.2208
	0.001	$\frac{1}{20}$	7.4920e-5	4.2219
		$\frac{1}{40}$	7.5246e-5	4.2201
		$\frac{1}{100}$	7.5320e-5	4.2196

presented in Theorem 2.5 is satisfied with an improved accuracy compare to the previous method of fractional derivative with singularities. Furthermore, the maximum error generates a second order accuracy weighted by an exponential kernel of the fractional derivative with α-dependency. On the other, simulation of numerical solution is shown to demonstrate the continuous dependence of the numerical solution to fractional order α.

4 Conclusion

Finite difference approximation for the two-dimensional space-time fractional diffusion equation, with bounded domain, using the Caputo-Fabrizio fractional derivative have been presented in this work. Analysis for the stability and convergence have been established as well, which shows that the numerical solution generates a convergence second order accuracy weighted by the memory kernel of the used fractional derivative. In addition, using the nonsingular kernel fractional derivative, singularity of the numerical solution can be avoided.

Acknowledgment

The authors would like to express their gratitude to the Department of Physics, Mindanao State University for the unconditional support and motivation that lead to the success of this study. The Department of Science and Technology-Science Education Institute is highly acknowledge for the financial support throughout the years. The authors would also like to thank the anonymous reviewers for the noble and valuable comments and suggestions that improved this manuscript.

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