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A decomposition Algorithm for Solving Stochastic Multi-Level Large Scale Quadratic Programming Problem

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Abstract: We present a decomposition algorithm to solve a multi-level large scale quadratic programming problem with stochastic parameters in the objective functions. In the first phase of the solution algorithm and to avoid the complexity of this problem, the stochastic nature of the problem is converted into the equivalent crisp problem. In the second phase, Taylor series is combined with a decomposition algorithm to obtain the optimal solution for this problem. An illustrative example is discussed to demonstrate the correctness of the proposed solution method.

Keywords: Large scale problems; stochastic programming; quadratic programming; multilevel programming;
MSC 2000: 90C06; 90C05; 90C99.

1 Introduction

Decision problems of chance-constrained or stochastic optimization arise when certain coefficients of an optimization model are not fixed or known but instead, to some extent, are of probabilistic quantities. In most real life problems in mathematical programming, the parameters are considered as random variables [1,2].

Multi-level programming techniques were developed to solve decentralized problems with multiple decision-makers in hierarchical organization, where each unit or department independently seek its own interest, but is affected by the actions of other units through externalities [3,4,5,6,7].

In large scale programming which closely describe and represent real world decision situations, various factors of the real world system should be reflected in the description of the objective functions and constraints. These objective functions and constraints involve many parameters and experts may assign them different values [8,9].

Notable studies have been carried out in the area of stochastic multi-level programming problems. In [10] Kumar and Baran presented a fuzzy goal programming (FGP) procedure for solving multilevel programming problems (MLPPs) having chance constraints in hierarchical decision organizations. The proposed approach converted the chance constraints of a problem into their respective deterministic equivalent in the decision making context. Then, the objective functions of decision makers (DMs) located at different hierarchical levels are converted into fuzzy goals by introducing an imprecise aspiration level to each of them to make decision in an uncertain environment.

In [6], Pramanik et al. used the fuzzy goal programming approach to solve chance constrained quadratic bi-level programming problem. Chance constraints were converted into equivalent deterministic constraints by the prescribed distribution functions. The quadratic membership functions were formulated by using the individual best solution of the quadratic objective functions subject to the equivalent deterministic constraints.

After the publication of the Dantzig and Wolfe decomposition method [11], numerous subsequent works on multi-objective large scale and multi-level large scale programming problems were carried out [12,13].

Osman et al. [13] presented a method for solving a special class of large scale fuzzy multi-objective integer problems depending on the decomposition algorithm. Furthermore, Abo-Sinna and Abou-Elenin extended the technique for order preference by similarity ideal solution (TOPSIS) to resolve large scale multiple objective programming problems involving fuzzy parameters [8].

Benzi et al. [9] developed and compared multilevel algorithms for solving large scale bound constrained

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nonlinear problems via interior point methods. It shows how a multilevel continuation strategy can be used to obtain good initial guesses for each nonlinear iteration. A minimal surface problem is used to illustrate the various approaches.

Sultan et al. [14] presented an algorithm to solve a three-level large scale linear programming problem in which the objective functions at every level are to be maximized. An algorithm for solving a three-planner model and a solution method for treating this problem are suggested. It attempted to optimize the problem separately at each level as a large scale programming problem using the Dantzig and Wolfe decomposition method. Therefore, it handled the optimization process through a series of sub problems that can be solved independently.

Currently, the challenging task for academic research is to address large-scale complex optimization problems under various uncertainties. Therefore, investigations on the development of chance-constrained multi-level large scale programming problem are required.

In this paper, an attempt to solve a multi-level large scale quadratic programming problem with stochastic parameters in the objective functions based on a decomposition algorithm is considered.

This paper is organized as follows: in Section 2 the model of a multilevel large scale quadratic programming problem with stochastic parameters in the objective functions is formulated. In Section 3, the decomposition method of large scale three-level linear programming problem is presented. An algorithm for solving a three-level large scale quadratic programming problem (TLLSQPP) with stochastic parameters in objective functions is suggested in Section 4. In addition, a numerical example is provided in Section 5 to clarify the results and the solution algorithm. Finally, conclusion and future works are reported in Section 6.

2 Problem Formulation and Solution Concept

The three-level large scale quadratic programming problem (TLLSQPP) with stochastic parameters in the objective functions may be formulated as follows:

[First Level]

$$\underset{x_1,x_2}{\mathrm{Max}}\, F_1 = A_1 x + \frac{1}{2} x^T L_1 x, \tag{1}$$

where $x_3,\ldots,x_m$ solves;

[Second Level]

$$\underset{x_3,x_4}{\mathrm{Max}}\, F_2 = A_2 x + \frac{1}{2} x^T L_2 x, \tag{2}$$

where $x_5,\ldots,x_m$ solves;

[Third Level]

$$\underset{x_5,x_6}{\mathrm{Max}}\, F_3 = A_3 x + \frac{1}{2} x^T L_3 x, \tag{3}$$

where $x_7,\ldots,x_m$ solves.

Subject to

$$x \in G, \tag{4}$$

where

$$\begin{aligned} G = \{ a_{01}x_1 + a_{02}x_2 + a_{0m}x_m &\le b_0, \\ d_1 x_1 \qquad\qquad\qquad &\le b_1, \\ d_2 x_2 \qquad\quad &\le b_2, \\ d_m x_m &\le b_m, \\ x_1,\ldots,x_m &\ge 0. \} \end{aligned}$$

$F_i : R^m \to R, (i = 1,2,3)$ are the first level objective function, the second level objective function, and the third level objective function, respectively, (L_1, L_2, L_3) are $m \times m$ real matrices contain random stochastic coefficient and (A_1, A_2, A_3) are $1 \times m$ matrices.

In the above problem (1)–(4), x is $m \times 1$ are real vector variables, G is the large scale linear constraint set where, $b = (b_0,\ldots,b_m)^T$ is $(m+1)$ vector, and $a_{01},\ldots, a_{0m}, d_1,\ldots,d_m$ are constants.

Therefore, the first level decision maker (FLDM) has x_1, x_2 indicating the first decision level choice, the second level decision maker (SLDM) and the third level decision maker (TLDM) have x_3, x_4 and x_5, x_6 indicating the second decision level choice and the third decision level choice, respectively.

Definition 1. For any $(x_1, x_2 \in G_1 = \{x_1, x_2 | (x_1,\ldots,x_m) \in G\})$ given by the FLDM and $(x_3, x_4 \in G_2 = \{x_3, x_4 | (x_1,\ldots,x_m) \in G\})$ given by the SLDM, if the decision-making variable $(x_5, x_6 \in G_3 = \{x_5, x_6 | (x_1, \ldots, x_m) \in G\})$ is the Pareto optimal solution of the TLDM, then $(x_1,\ldots,x_m)$ is a feasible solution for the TLLSQPP with stochastic parameters in objective functions.

Definition 2. If $x^* \in R^m$ is a feasible solution of the TLLSQPP; no other feasible solution $x \in G$ exist, such that $F_1(x^*) \le F_1(x)$; so x^* is the Pareto optimal solution for the TLLSQPP with stochastic parameters in objective functions.

The basic idea in treating the TLLSQPP with stochastic parameters in objective functions is to convert the probabilistic nature of this problem into an equivalent deterministic. In this case, the set of objective functions can be rewritten in the deterministic form as [2]:

$$\begin{aligned} f_r(x) = \sum_{j=1}^{n} a_{rj} x_j + k_1^r \sum_{j=1}^{n} x_j^2 E(L_j^r) \\ + k_2^r \sqrt{\sum_{j=1}^{n} \frac{1}{2} x_j^2 \sigma^2 (L_j^r) x}, \quad r = 1,2,\ldots,k, \end{aligned} \tag{5}$$

where $E(L_j^r)$ = mean of L_j^r and $\sigma^2(L_j^r)$ = variance of L_j^r, and k_1^r, k_2^r, are non-negative constants whose values indicate the relative importance of the mean and the standard deviation of the variable L_j^r for maximization. If $k_1^r = k_2^r = 1$, it is an indication that equal importance is given to the maximization of the mean as well as the standard deviation of L_j^r.

3 Decomposition Algorithm for the Three-Level Large Scale Linear Programming Problem

The TLLSQPP with stochastic parameters in objective functions can be understood as the following deterministic TLLSQPP.

[First Level]

$$\underset{x_1,x_2}{\text{Max}}\ f_1(x) = A_1 x + \frac{1}{2} x^T L_1' x, \tag{6}$$

where $x_3,\ldots,x_m$ solves;

[Second Level]

$$\underset{x_3,x_4}{\text{Max}}\ f_2(x) = A_2 x + \frac{1}{2} x^T L_2' x, \tag{7}$$

where $x_5,\ldots,x_m$ solves;

[Third Level]

$$\underset{x_5,x_6}{\text{Max}}\ f_3(x) = A_3 x + \frac{1}{2} x^T L_3' x, \tag{8}$$

where $x_7,\ldots,x_m$ solves.

Subject to

$$x \in G. \tag{9}$$

To solve a three-level large scale quadratic programming problem using decomposition algorithms is a complex problem. Taylor series can overcome this complexity by obtaining polynomial objective functions equivalent to quadratic objective functions.

$$H_i(x) \cong \hat{f}_i(x) = f_i(x_i^*) + \sum_{j=1}^{n} (x_j - x_{ij}^*) \frac{\partial f_i(x_i^*)}{dx_j}, (i = 1,2,3). \tag{10}$$

So the equivalent TLLSPP can be written as:

[First Level]

$$\underset{x_1,x_2}{\text{Max}}\ H_1(x), \tag{11}$$

where $x_3,\ldots,x_m$ solves;

[Second Level]

$$\underset{x_3,x_4}{\text{Max}}\ H_2(x), \tag{12}$$

where $x_5,\ldots,x_m$ solves;

[Third Level]

$$\underset{x_5,x_6}{\text{Max}}\ H_3(x), \tag{13}$$

where $x_7,\ldots,x_m$ solves.

Subject to

$$x \in G. \tag{14}$$

The three-level large scale linear programming problem is solved by adopting the leader-follower Stakelberg strategy combined with Dantzig and Wolf decomposition method [8,11]. First, the optimal solution that is acceptable to the FLDM is obtained using the decomposition method to break the large scale problem into n-sub problems that can be solved directly.

The decomposition principle is based on representing the TLLSLPP in terms of the extreme points of the sets $d_j x_j \le b_j, x_j \ge 0, j = 1,2,\ldots,m$. To do so, the solution space described by each $d_j x_j \le b_j, x_j \ge 0, j = 1,2,\ldots,m$ must be bounded and closed.

Then by inserting the FLDM decision variable to the SLDM in order to seek the optimal solution using Dantzig and Wolf decomposition method [11], then the decomposition method break the large scale problem into n-sub problems that can be solved directly. Finally the TLDM repeat the same action till the optimal solution is obtained for his/her problem which is the optimal solution to the TLLSLPP.

Theorem 1. The decomposition algorithm terminates in a finite number of iterations, yielding a solution of the large scale problem.

To prove theorem 1 above, the reader is referred to [11].

4 An Algorithm

A solution algorithm to solve a three-level large-scale quadratic programming problem (TLLSQPP) with stochastic parameters in objective functions is described in a series of steps. The suggested algorithm can be summarized in the following manner:

Step 1. Determine $E\{L_j^i\}$ and $Var\{L_j^i\}, (i = 1,2,\ldots,m)$.

Step 2. Calculate

$$f_r(x) = \sum_{j=1}^{n} a_{rj} x_j + k_1^r \sum_{j=1}^{n} x_j^2 E(L_j^r) + k_2^r \sqrt{\sum_{j=1}^{n} \frac{1}{2} x_j^2 \sigma^2(L_j^r) x}, \quad r = 1,2,\ldots,k.$$

Step 3. Formulate the equivalent (TLLSQPP).

Step 4. Convert problem (TLLSQPP) into (TLLSLPP) using Taylor series approach the transformation for the FLDM, SLDM, and TLDM.

$$H_i(x) \cong \hat{f}_i(x) = f_i(x_i^*) + \sum_{j=1}^{n} (x_j - x_{ij}^*) \frac{\partial f_i(x_i^*)}{dx_j}, (j = 1,2,3).$$

Step 5. Formulate the FLDM problem, go to Step 6.

Step 6. Convert the master problem in terms of the extreme points of sets $d_j x_j \le b_j, x_j \ge 0, j = 1,2,3.$

Step 7. Determine the extreme points $x_j = \sum_{k=1}^{k_j} \beta_{jk} \hat{x}_{jk}, j = 1,2,3$ using Balinski's algorithm [15].

Step 8. Set $k = 1$.

Step 9. Compute $z_{jk} - c_{jk} = C_B B^{-1} P_{jk} - c_{jk}$, go to Step 10.

Step 10. If $z_{jk}^* - c_{jk}^* \le 0$, then go to Step 11; otherwise, the optimal solution has been reached, go to Step 16.

Step 11. Determine $\hat{X}_{jk}$ associated with min $\{z_{jk}^* - c_{jk}^*\}$.

Step 12. B_{jk} associated with extreme point $\hat{X}_{jk}$ must enter the solution.

Step 13. Determine leaves variable.

Step 14. The new basis is determined by replacing the vector associated with leaving variable with the vector B_{jk}, go to Step 15.

Step 15. Set $k = k+1$, go to Step 9.

Step 16. If the SLDM obtain the optimal solution go to Step 20, otherwise go to Step 17.

Step 17. Set $(x_1, x_2) = (x_1^F, x_2^F)$ to the SLDM constraints, go to Step 18.

Step 18. Formulate the SLDM problem, go to Step 8.

Step 19. If the TLDM obtain the optimal solution go to Step 22, otherwise go to Step 20.

Step 20. Set $(x_1, x_2, x_3, x_4) = (x_1^F, x_2^F, x_3^S, x_4^S)$ to the TLDM constraints, go to Step 21.

Step 21. Formulate the TLDM problem, go to Step 8.

Step 22. $(x_1^F, x_2^F, x_3^S, x_4^S, x_5^T, x_6^T, \ldots, x_m^T)$ is as an optimal solution for the three-level large scale linear programming problem, then stop.

5 Numerical Example

To demonstrate the solution for the (STLLSQPP) the following problem is considered:

[First Level]

$$\underset{x_1,x_2}{\text{Max}}\, F_1(x,\theta^1) = \underset{x_1,x_2}{\text{Max}}\, (1+\theta_1^1)x_1^2 + \theta_2^1 x_2^2 + x_5 + x_6,$$

where x_3, x_4, x_5, x_6 solves;

[Second Level]

$$\underset{x_3,x_4}{\text{Max}}\, F_2(x,\theta^2) = \underset{x_3,x_4}{\text{Max}}\, x_1 + (2+\theta_3^2)x_3^2 + \theta_4^2 x_4^2 + 2x_5,$$

where x_5, x_6 solves;

[Third Level]

$$\underset{x_5,x_6}{\text{Max}}\, F_3(x,\theta^3) = \underset{x_5,x_6}{\text{Max}}\, x_1 + 2x_2 + \theta_5^3 x_5^2 + (\theta_6^3+1)x_6^2.$$

Subject to

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \le 40,$$
$$3x_1 + x_2 \le 36,$$
$$4x_3 + 2x_4 \le 16,$$
$$x_5 + 4x_6 \le 20,$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \ge 0,$$

where θ_j^i are independent normal distribution with the following means and variances:

Random variable	θ_1^1	θ_2^1	θ_3^2	θ_4^2	θ_5^3	θ_6^3
Mean	1	2	3	4	5	1
Variance	4	9	16	25	16	25

Now the (TLLSQPP) with stochastic parameters in the objective functions can be understood as the following (TLLSQPP):

[First Level]

$$\underset{x_1,x_2}{\text{Max}}\, f_1(x,\theta^1) = \underset{x_1,x_2}{\text{Max}}\, 4x_1^2 + 5x_2^2 + x_5 + x_6,$$

where x_3, x_4, x_5, x_6 solves;

[Second Level]

$$\underset{x_3,x_4}{\text{Max}}\, f_2(x,\theta^2) = \underset{x_3,x_4}{\text{Max}}\, x_1 + 9x_3^2 + 9x_4^2 + 2x_5,$$

where x_5, x_6 solves;

[Third Level]

$$\underset{x_5,x_6}{\text{Max}}\, f_3(x,\theta^3) = \underset{x_5,x_6}{\text{Max}}\, x_1 + 2x_2 + 9x_5^2 + 7x_6^2.$$

Subject to

$$x_1+x_2+x_3+x_4+x_5+x_6 \le 40,$$
$$3x_1+x_2 \le 36,$$
$$4x_3+2x_4 \le 16,$$
$$x_5+4x_6 \le 20,$$
$$x_1,x_2,x_3,x_4,x_5,x_6 \ge 0.$$

The 1st order Taylor polynomial series is used to convert the quadratic function to linear function. Therefore, the (TLLSQPP) is written as:

[First Level]

$$\underset{x_1,x_2}{\text{Max}}\, H_1(x_1,x_2) = \underset{x_1,x_2}{\text{Max}}\, 8x_1+20x_2+x_5+x_6-24,$$

where x_3,x_4,x_5,x_6 solves;

[Second Level]

$$\underset{x_3,x_4}{\text{Max}}\, H_2(x_3,x_4) = \underset{x_3,x_4}{\text{Max}}\, x_1+18x_3+18x_4+2x_5-26,$$

where x_5,x_6 solves;

[Third Level]

$$\underset{x_5,x_6}{\text{Max}}\, H_3(x_5,x_6) = \underset{x_5,x_6}{\text{Max}}\, x_1+2x_2+18x_5+14x_6-16.$$

Subject to

$$x_1+x_2+x_3+x_4+x_5+x_6 \le 40,$$
$$3x_1+x_2 \le 36,$$
$$4x_3+2x_4 \le 16,$$
$$x_5+4x_6 \le 20,$$
$$x_1,x_2,x_3,x_4,x_5,x_6 \ge 0.$$

The FLDM's problem is formulated as follows:

$$\text{Max } H_1(x) = \underset{x_1,x_2}{\text{Max}}\, 8x_1+20x_2+x_5+x_6-24.$$

Subject to

$$x \in G.$$

After 5 iterations the FLDM's optimal solution is obtained

$$(x_1^F,x_2^F,x_3^F,x_4^F,x_5^F,x_6^F) = (0,36,0,0,4,0).$$

Now set $(x_1,x_2) = (0,36)$ to the SLDM constraints.

Secondly, the SLDM's problem can be solved as follows:

$$\text{Max } H_2 = \underset{x_3,x_4}{\text{Max}}\, 18x_3+18x_4+2x_5-26.$$

Subject to

$$x_3+x_4+x_5+x_6 \le 4,$$
$$4x_3+2x_4 \le 16,$$
$$x_5+4x_6 \le 20,$$
$$x_3,x_4,x_5,x_6 \ge 0.$$

The SLDM will repeat the same action as the FLDM till the attainment of optimal solution $(x_3^S,x_4^S,x_5^S,x_6^S) = (0,4,0,0)$, now set $(x_3,x_4) = (0,4)$ to the TLDM constraints.

Finally, the TLDM's problem can be solved as follows:

$$\text{Max } H_3 = \underset{x_5,x_6}{\text{Max}}\, 18x_5+14x_6+56.$$

Subject to

$$x_5+x_6 \le 0,$$
$$x_5+4x_6 \le 20,$$
$$x_5,x_6 \ge 0.$$

The TLDM will repeat the same action as the FLDM and the SLDM till the attainment of optimal solution $(x_5^T,x_6^T) = (0,0)$ So $(x_1^F,x_2^F,x_3^S,x_4^S,x_5^T,x_6^T) = (0,36,0,4,0,0)$ is the optimal solution for three-level large scale linear programming problem, where $H_1 = 696$, $H_2 = 46$, and $H_3 = 56$.

6 Summary and Concluding Remarks

This paper presented a solution for multi-level large scale quadratic programming problem with stochastic parameters in the objective functions based on a decomposition algorithm. In the first phase of the solution algorithm and to avoid the complexity of this problem, the stochastic nature of the problem was converted into the equivalent crisp problem. In the second phase, Taylor series was combined with a decomposition algorithm to obtain the optimal solution of this problem. Finally, a numerical example was given to clarify the results of this paper.

However, there are several points open for future discussion, which should be explored and studied in the area of stochastic multi-level large scale optimization such as:

1. A decomposition algorithm for solving stochastic multi-level large scale integer quadratic programming problem.

2. A decomposition algorithm for solving stochastic multi-level large scale fractional programming problem.

3. A decomposition algorithm for solving stochastic multi-level large scale mixed integer quadratic programming problem.

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