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Hadamard Exponential Hankel Matrix, Its Eigenvalues and Some Norms

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Abstract: In this paper, we study the $n \times n$ Hadamard exponential Hankel matrix of the form $H_n = \left[e^{i+j}\right]_{i,j=0}^{n-1}$. We found ℓ_p norms, two upper bounds for spectral norm and eigenvalues of this matrix. Finally, we give an application related Hadamard inverse, Hadamard product and eigenvalues of this matrix.

Keywords: Hankel matrix; Hadamard exponential; Hadamard product; norm; spectral norm; eigenvalue.

1. Introduction

In [1], Reams proved that $e^{\circ A}$ is positive semidefinite where $A \in \mathrm{R}^{n \times n}$ be symmetric and positive semidefinite. $A \in \mathrm{R}^{n \times n}$ be almost positive (semi) definite then $e^{\circ A}$ is positive (semi) definite. Moreover, Reams gave some proofs related Hadamard inverse and Hadamard square root of symetric matrices.

In [2], Solak and Bozkurt found an upper and lower bound of Cauchy-Hankel matrix in the form

$$H_n = \left[1/(a+(i+j)b)\right]_{i,j=1}^{n}$$

where $b \neq 0$, a and b are any numbers and a/b is not integer.

In [3], Solak and Bozkurt determined bounds for the spectral and ℓ_p norm of Cauchy-Hankel matrices of the form

$$H_n = \left[1/(g+h(i+j))\right]_{i,j=1}^{n} \equiv \left[1/(g+kh)\right]_{i,j=1}^{n},\ k = 0, 1, \ldots, n-1,$$

where k is defined by $i + j = k \ (\mathrm{mod}\ n)$.

In [4], Güngör found lower bounds for the spectral norm and Euclidean norm of Cauchy-Hankel matrix in the form

$$H_n = \left[1/(g+(i+j)h)\right]_{i,j=1}^{n}.$$

In [5], Türkmen and Bozkurt obtained an upper bounds for the spectral norm of the Cauchy-Hankel matrices of the form

$$H_n = \left[1/(g+(i+j)h)\right],$$

where $g = 1/k$ and $h = 1$.

In [6], Nallı studied the Hadamard exponential GCD matrices of the form

$$E = \left[e^{(i,j)}\right]_{i,j=1}^{n},$$

where (i, j) is the greatest common divisor of i and j. Nallı gave the structure theorem and calculated the determinant, trace, inverse and determined upper bound for determinant of this matrix.

A Hankel matrix is an $n\times n$ matrix

$$H_n=\left[h_{i,j}\right]_{i,\ j=0}^{n-1} \quad , \tag{1}$$

where $h_{i,\ j}=h_{i+j}$, i.e., a matrix of the form

$$H_n=\begin{bmatrix} h_0 & h_1 & h_2 & \cdots & h_{n-1} \\ h_1 & h_2 & h_3 & \cdots & h_n \\ h_2 & h_3 & h_4 & \cdots & h_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{n-1} & h_n & h_{n+1} & \cdots & h_{2n-2} \end{bmatrix}. \tag{2}$$

Let $A=\left(a_{ij}\right)$ is an $m\times n$ matrix, then Hadamard exponential and Hadamard inverse of the matrix A is defined by

$$e^{\circ A}=\left(e^{a_{ij}}\right)$$

and

$$A^{\circ(-1)}=\left(\frac{1}{a_{ij}}\right),$$

respectively [1].

Let $A=\left(a_{ij}\right)$ is an $m\times n$ matrix, then transpose of the matrix A is $n\times m$ matrix and defined by $A^T=(a_{ji})$.

Let $A=\left(a_{ij}\right)$ is an $m\times n$ complex matrix. The ℓ_p norm of A is defined by

$$\|A\|_p=\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_{ij}\right|^p\right)^{1/p}. \tag{3}$$

If $p=2$ then ℓ_2 norm is called Frobenius or Euclidean norm and showed by $\|A\|_F$.

Let A be $m\times n$ complex matrix. Then the spectral norm of the matrix A is defined by

$$\|A\|_2=\sqrt{\max_{1\le i\le n}|\lambda_i|}, \tag{4}$$

where λ_i numbers are eigenvalues of the matrix AA^H and the matrix A^H is conjugate transpose of the matrix A.

The inequality, between the Frobenius norm and the spectral norm

$$\frac{1}{\sqrt{n}}\|A\|_F\le\|A\|_2\le\|A\|_F \tag{5}$$

is valid [7].

The spectral radius is known the maximum of the absolute values of the eigenvalues of a matrix. That is, for $n\times n$ matrix A, the spectral radius of A is defined as $\rho\left(A\right)=\max_{1\le i\le n}\left|\lambda_i\right|$ where λ_i are eigenvalues of the matrix A.

Let $A=\left(a_{ij}\right)$ and $B=(b_{ij})$ is an $m\times n$ matrices. Then, the Hadamard product of A and B is entrywise product and defined by

$$A\circ B=(a_{ij}b_{ij}) \text{ [1].}$$

Define the maximum column length norm $c_1(\cdot)$ and maximum row length norm $r_1(\cdot)$ on $m\times n$ matrix $A=(a_{ij})$ by

$$c_1(A)\equiv\max_j\sqrt{\sum_i\left|a_{ij}\right|^2}=\max_j\left\|\left[a_{ij}\right]_{i=1}^m\right\|_F$$

and

$$r_1(A) \equiv \max_i \sqrt{\sum_j |a_{ij}|^2} = \max_i \left\| \left[a_{ij} \right]_{j=1}^{n} \right\|_F .$$

Let $A=(a_{ij})$, $B=(b_{ij})$ and $C=(c_{ij})$ is an $m\times n$ matrices. If $C=A\circ B$ then

$$\|C\|_2 \le r_1(A)c_1(B) \text{ [8].} \tag{6}$$

Let $A=(a_{ij})$ is an $n\times n$ matrix. The principal k - minors of matrix A are defined by

$$A\begin{pmatrix} i_1, i_2, \ldots, i_k \\ i_1, i_2, \ldots, i_k \end{pmatrix} = \det\begin{bmatrix} a_{i_1 i_1} & a_{i_1 i_2} & \cdots & a_{i_1 i_k} \\ a_{i_2 i_1} & a_{i_2 i_2} & \cdots & a_{i_1 i_k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_k i_1} & a_{i_k i_2} & \cdots & a_{i_k i_k} \end{bmatrix},$$

where $1 \le i_1 < i_2 < \cdots < i_k \le n \quad (1 \le k \le n)$.

Let $A=(a_{ij})$ is an $n\times n$ matrix. The equation

$$\det(\lambda I - A) = \lambda^n + a_1\cdot\lambda^{n-1} + a_2\cdot\lambda^{n-2} + \cdots + a_{n-1}\cdot\lambda + a_n = 0 \tag{7}$$

is called the characteristic equation of the matrix A. Characteristic polynomial of matrix A is a monic polynomial and coefficients of this polynomial can obtain using principal minors by

$$a_k = (-1)^k \cdot \sum_{k=1}^{n} A\begin{pmatrix} i_1, i_2, \ldots, i_k \\ i_1, i_2, \ldots, i_k \end{pmatrix}. \tag{8}$$

As a special, we can write

$$\begin{aligned} a_1 &= (-1)\cdot\left\{ A\begin{pmatrix} 1 \\ 1 \end{pmatrix} + A\begin{pmatrix} 2 \\ 2 \end{pmatrix} + \cdots + A\begin{pmatrix} n \\ n \end{pmatrix} \right\} \\ &= -\{a_{11} + a_{22} + \cdots + a_{nn}\} = -tr(A) \end{aligned}$$

and

$$\begin{aligned} a_n &= (-1)^n \cdot A\begin{pmatrix} i_1, i_2, \ldots, i_n \\ i_1, i_2, \ldots, i_n \end{pmatrix} \\ &= (-1)^n \cdot \det(A). \end{aligned}$$

Taking $h_{i+j} = i + j$ in (1) we get a Hankel matrix

$$H_n = \begin{bmatrix} 0 & 1 & 2 & \cdots & n-1 \\ 1 & 2 & 3 & \cdots & n \\ 2 & 3 & 4 & \cdots & n+1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n-1 & n & n+1 & \cdots & 2n-2 \end{bmatrix} \tag{9}$$

and Hadamard exponential of this matrix is

$$e^{\circ H_n} = \begin{bmatrix} 1 & e & e^2 & \cdots & e^{n-1} \\ e & e^2 & e^3 & \cdots & e^n \\ e^2 & e^3 & e^4 & \cdots & e^{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e^{n-1} & e^n & e^{n+1} & \cdots & e^{2n-2} \end{bmatrix}. \tag{10}$$

It is known that

$$\sum_{k=0}^{n-1} x^k = 1 + x + x^2 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}.$$

Using this equality, it can be write

$$\sum_{k=1}^{n-1} x^k = x + x^2 + \cdots + x^{n-1} = \frac{x^n - x}{x - 1}. \qquad (11)$$

If we take the derivative of both side of equality (11)

$$\frac{d}{dx}\left\{\sum_{k=1}^{n-1} x^k\right\} = \frac{d}{dx}\left\{\frac{x^n - 1}{x - 1}\right\}$$
$$= \sum_{k=1}^{n-1} k \cdot x^{k-1} = \frac{(n-1)\cdot x^n - n \cdot x^{n-1} + 1}{(x-1)^2}$$

thus we get

$$\sum_{k=1}^{n-1} k \cdot x^k = \frac{(n-1)\cdot x^{n+1} - n \cdot x^n + x}{(x-1)^2}. \qquad (12)$$

In this paper, we investigate ℓ_p norm, spectral norm and eigenvalues of $e^{\circ H_n}$ in (10). After, we give some results for determinant and spectral radius of this matrix. Finally, we give an application related Hadamard product and Hadamard inverse as a theorem.

2. Main Results

Theorem 2.1 *Let* $e^{\circ H_n}$ *as in* (10). *Then the* ℓ_p *norm of this matrix is*

$$\left\| e^{\circ H_n} \right\|_p = \frac{\left(e^{pn} - 1\right)^{2/p}}{\left(e^p - 1\right)^{2/p}}.$$

Proof. If we calculate p th power of ℓ_p norm of $e^{\circ H_n}$ we get

$$\left\| e^{\circ H_n} \right\|_p^p = \sum_{k=1}^{n}\left[k \cdot \left(e^{k-1}\right)^p\right] + \sum_{k=1}^{n-1}\left[(n-k)\cdot\left(e^{n+k-1}\right)^p\right]$$
$$= n \cdot e^{p(n-1)} + e^{-p} \cdot \sum_{k=1}^{n-1}\left[k \cdot \left(e^p\right)^k\right]$$
$$+ n \cdot e^{p(n-1)} \cdot \sum_{k=1}^{n-1}\left(e^p\right)^k - e^{p(n-1)} \cdot \sum_{k=1}^{n-1}\left[k \cdot \left(e^p\right)^k\right]$$
$$= n \cdot e^{p(n-1)} + \left(e^{-p} - e^{p(n-1)}\right)\cdot \sum_{k=1}^{n-1}\left[k \cdot \left(e^p\right)^k\right] + n \cdot e^{p(n-1)} \cdot \sum_{k=1}^{n-1}\left(e^p\right)^k.$$

Using (11) and (12) we get

$$\left\| e^{\circ H_n} \right\|_p^p = \frac{\left(e^{pn} - 1\right)^2}{\left(e^p - 1\right)^2}.$$

If we take $(1/p)$ th power of the both-hand side we get

$$\left\| e^{\circ H_n} \right\|_p = \frac{\left(e^{pn} - 1\right)^{2/p}}{\left(e^p - 1\right)^{2/p}}. \qquad (13)$$

Theorem 2.2 *Let* $e^{\circ H_n}$ *as in* (10). *Then, following inequalities for the spectral norm of* $e^{\circ H_n}$

$$\left\| e^{\circ H_n} \right\|_2 \le \frac{e^{2n} - 1}{e^2 - 1}$$

and

$$\left\|e^{\circ H_n}\right\|_2 \geq \frac{e^{2n}-1}{\sqrt{n}\left(e^2-1\right)}$$

are valid.

Proof. For $p=2$ in (13) we get the Frobenius norm of $e^{\circ T_n}$ by

$$\left\|e^{\circ H_n}\right\|_E = \frac{e^{2n}-1}{e^2-1}.$$

Using inequality (5) we get

$$\left\|e^{\circ H_n}\right\|_2 \leq \frac{e^{2n}-1}{e^2-1}$$

and

$$\left\|e^{\circ H_n}\right\|_2 \geq \frac{e^{2n}-1}{\sqrt{n}\left(e^2-1\right)}.$$

Theorem 2.3 *Le* $e^{\circ H_n}$ *as in* (10). *Then, second upper bound for the spectral norm of* $e^{\circ H_n}$ *is*

$$\left\|e^{\circ H_n}\right\|_2 \leq \frac{\sqrt{(e^{4n-4}-e^{2n-2}+e^2-1)(e^{4n-2}-e^{2n-2})}}{(e^2-1)}.$$

Proof. We can write

$$e^{\circ H_n} = \begin{bmatrix} 1 & e & e^2 & \cdots & 1 \\ e & e^2 & e^3 & \cdots & 1 \\ e^2 & e^3 & e^4 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e^{n-1} & e^n & e^{n+1} & \cdots & e^{2n-2} \end{bmatrix} \circ \begin{bmatrix} 1 & 1 & 1 & \cdots & e^{n-1} \\ 1 & 1 & 1 & \cdots & e^n \\ 1 & 1 & 1 & \cdots & e^{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} = A \circ B.$$

Then, using (11) we get

$$r_1(A) = \sqrt{e^{2n-2}\left(1+\sum_{k=1}^{n-1}(e^2)^k\right)} = \sqrt{e^{2n-2}\left(\frac{e^{2n}-1}{e^2-1}\right)}$$
$$= \sqrt{\frac{e^{4n-2}-e^{2n-2}}{e^2-1}}$$

and

$$c_1(B) = \sqrt{1+e^{2n-2}\left(1+\sum_{k=1}^{n-2}(e^2)^k\right)} = \sqrt{1+e^{2n-2}\left(\frac{e^{2n-2}-1}{e^2-1}\right)}$$
$$= \sqrt{\frac{e^{4n-4}-e^{2n-2}+e^2-1}{e^2-1}}.$$

From (6) an upper bound is found by

$$\left\|e^{\circ H_n}\right\|_2 \leq r_1(A)c_1(B) = \frac{\sqrt{(e^{4n-4}-e^{2n-2}+e^2-1)(e^{4n-2}-e^{2n-2})}}{(e^2-1)}.$$

Theorem 2.4 *Let* $e^{\circ H_n}$ *a Hankel matrix as in* (10) *and* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *are eigenvalues of* $e^{\circ H_n}$. *Then*

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = 0$$

and

$$\lambda_n = 1 + e^2 + e^4 + \cdots + e^{2n-2}.$$

Proof. Let characteristic equation of $e^{\circ H_n}$ is

$$\det\left(\lambda I - e^{\circ H_n}\right) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n = 0.$$

Now we can calculate the coefficients $a_1, a_2, \ldots, a_n$ using formula (8). It is seen that

$$a_1 = -tr(e^{\circ H_n}) = -(1 + e^2 + e^4 + \cdots + e^{2n-2}).$$

Because of k th row ($2 \le k \le n$) of $e^{\circ H_n}$ is e^{k-1} multiple of the first row, every $k \times k$ subdeterminants of $e^{\circ H_n}$ equal to 0 . Thus we can say easily that

$$a_2 = a_3 = \ldots = a_n = 0.$$

Then

$$\det\left(\lambda I - e^{\circ H_n}\right) = \lambda^n - (1 + e^2 + e^4 + \cdots + e^{2n-2}) \cdot \lambda^{n-1}$$
$$= \lambda^{n-1}(\lambda - (1 + e^2 + e^4 + \cdots + e^{2n-2})) = 0$$

and we can write

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = 0,\ \lambda_n = 1 + e^2 + e^4 + \cdots + e^{2n-2}.$$

Thus proof is completed.

Conclusion 2.5 *The spectral radius of* $e^{\circ H_n}$ *is*

$$1 + e^2 + e^4 + \cdots + e^{2n-2}$$

and

$$\det(e^{\circ H_n}) = 0.$$

Theorem 2.6 *Let* $e^{\circ H_n}$ *a Hankel matrix as in* (10). *Then the eigenvalues of the matrix* $\left(e^{\circ H_n}\right)^{\circ(-1)} \circ e^{\circ H_n}$ *are*

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = 0,\ \lambda_n = n.$$

Proof. If we write the Hadamard inverse of $e^{\circ H_n}$, it is easily seen that

$$\left(e^{\circ H_n}\right)^{\circ(-1)} = \begin{bmatrix} 1 & e^{-1} & e^{-2} & \cdots & e^{-(n-1)} \\ e^{-1} & e^{-2} & e^{-3} & \cdots & e^{-n} \\ e^{-2} & e^{-3} & e^{-4} & \cdots & e^{-(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e^{-(n-1)} & e^{-n} & e^{-(n+1)} & \cdots & e^{-(2n-2)} \end{bmatrix}.$$

If we write the Hadamard product of $\left(e^{\circ H_n}\right)^{\circ(-1)}$ and $e^{\circ H_n}$, we get

$$B = \left(e^{\circ H_n}\right)^{\circ(-1)} \circ e^{\circ H_n} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Let characteristic equation of B is

$$\det\left(\lambda I - B\right) = \lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} + \cdots + b_{n-1}\lambda + b_n = 0.$$

If we calculate $b_1, b_2, \ldots, b_n$ we see that immediately

$$b_1 = -n = -tr(B)$$

and

$$b_2 = b_3 = \cdots = b_n = 0$$

from formula (8). Then, we get

$$\det(\lambda I - B) = \lambda^n - n \cdot \lambda^{n-1}$$
$$= \lambda^{n-1}(\lambda - n) = 0.$$

Thus

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = 0,\ \lambda_n = n$$

and proof is completed.

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