

# Direct Solution of the 7 parameters transformation problem

Amir Hashemi<sup>1,\*</sup>, Mahzad Kalantari<sup>2</sup> and Michel Kasser<sup>3</sup>

<sup>1</sup>Department of Mathematical Sciences, Isfahan University of Technology, Isfahan, 84156-83111, Iran

<sup>2</sup>ESIEE Paris, Cité DESCARTES, 93162 Noisy le Grand Cedex, France

<sup>3</sup>HEIG-VD, CH 1401 Yverdon-les-bains, Switzerland

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**Abstract:** In this paper, we propose a new algorithm to solve the 7 parameters transformation problem. Using Gröbner bases techniques and solving polynomial systems, we find, in a direct manner and without the need of a linearization, the values of the translation, rotation and a scale factor. Then, with the aid of simulated data, we will confront the algorithm to all sorts of configurations. Finally, we show the application of this algorithm in photogrammetry.

**Keywords:** Direct resolution, Photogrammetry, 7 parameters transformation, Gröbner bases, Solving polynomial systems.

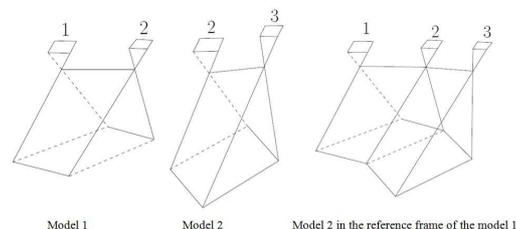
## 1. Introduction

In this paper, we consider the acquisition of some images, already oriented and localized by couple, and look for how one can assemble many couples together. For this purpose, we perform an approached setting up, in order to perform a bundle adjustment that relies entirely on the co-linearity equations. However these equations are not linear and therefore require initial values, to be linearized. More precisely, it permits the setting in geometry of a set of  $N$  images in the same referential.

One can say that today the external orientation is a problem (the pose problem) whose resolution is quite classical. In theory it requires only 3 known points, but in practice at least an extra fourth or even a fifth point are required in order to find a unique solution, to say nothing even of the unreliability of a calculation based on the minimal number of points in case of an outlier. The aim of this paper is to present a new alternative.

For a large set of images, the external orientation allows the setting in reference of a couple of image in relation to a 3D model of points, and the goal is to put progressively all the successive 3D models obtained from neighbouring couples in the same reference frame. To be more precise, let us suppose that we have 3D points from the couple of the images 1 and 2 (the reference of the system is the optical center of the image 1). If the relative

orientation is obtained between the images 2 and 3, we will also have the 3D points that will be in the reference of the image 2. In order to be able to put the image 3 in the first reference, we will put the 3D points taken from the couple 2 and 3 in the reference frame of the model taken from the couple of images 1 and 2. To illustrate this point, these steps are presented in Figure 1.



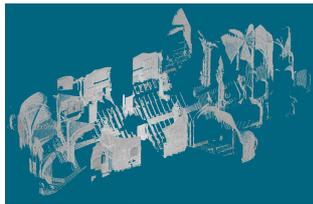
**Figure 1:** Setting in reference a model in relation to another

This problem is the equivalent of the passage from a reference frame to another one; it is therefore a similitude in the 3D space. The equations of the 3D similitude are those used for the absolute orientation. Otherwise it is possible to perform a compensation by model, instead of a bundle compensation, under the technical denomination

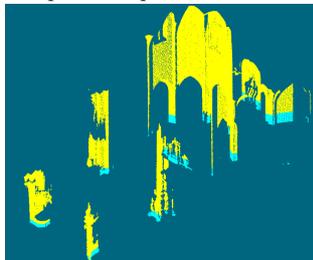
\* Corresponding author e-mail: [amir.hashemi@cc.iut.ac.ir](mailto:amir.hashemi@cc.iut.ac.ir)

of photogrammetric aerotriangulation : bundle compensation by independent models.

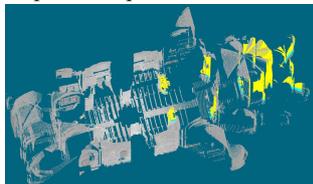
The 7 parameters transformation is very classical in geodesy [2], because it allows the passage of a reference frame to another. It transforms a set of points from a system to another one using a rotation, a translation and a scale factor. In geodesy [7], the angles of rotation being very small, an approximation is made for the rotation matrix. It leads to what is called a Helmert transformation. In a context of photogrammetry, unfortunately we cannot make this simplification for the rotation, because these angles may be large, and we use this transformation to connect the different models obtained for each couple. The result will be a set of images all in the same reference. It is also the case in the laser scanner surveys, in order to put in the same reference the clouds of points taken from different stations. In that case, the scale factor is in general very close to 1, see Figure 2.



(a) Cloud of points acquired from the first station



(b) Cloud of points acquired from the second station



(c) Setting in reference of the second station (yellow) in relation to the first one

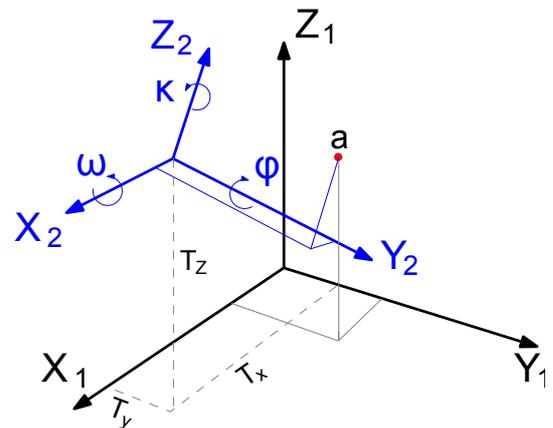
**Figure 2:** Example of setting in reference of the different laser clouds presented above, while using the algorithm presented below (Le Monastier). Credit: ENSG

In this paper, a new algorithm of resolution of the 7 parameters transformation is described. With the help of Gröbner bases and polynomial resolution [4], we find, in a direct manner and without the need of a linearization

(and thus of approached values), the values of the translation, rotation and a scale factor. In the continuation, with simulated data, we will test the algorithm in many different configurations. To conclude, we will show the application of this algorithm in photogrammetry.

## 2. The method of the 7 parameters: Direct calculation of the 3D similitude

**Algebraic modeling.** The 7 parameters transformation contains, as its name indicates, 7 unknowns; i.e. the scale factor  $\lambda$ , the rotation  $R$  and the translation  $T_x, T_y, T_z$ . As one can see in Figure 3, once these 7 parameters are obtained, we are able to link the whole system 2 to the system 1.



**Figure 3:** Illustration of a 7 parameters Helmert transform

$$\begin{bmatrix} X_a \\ Y_a \\ Z_a \end{bmatrix}_1 = \lambda R(\omega, \phi, \kappa) \begin{bmatrix} X_a \\ Y_a \\ Z_a \end{bmatrix}_2 + \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} \quad (1)$$

where  $R(\omega, \phi, \kappa)$  is the rotation matrix (see Figure 3). If one uses the Cayley representation for the rotation matrix, we will have the following system in which  $A$  is an antisymmetric matrix [1].

$$\begin{bmatrix} X_a \\ Y_a \\ Z_a \end{bmatrix}_1 = \lambda (I - A)^{-1} (I + A) \begin{bmatrix} X_a \\ Y_a \\ Z_a \end{bmatrix}_2 + \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} \quad (2)$$

By multiplying both sides of this equation by  $(I - A)$ , we get:

$$(I - A) \begin{bmatrix} X_a \\ Y_a \\ Z_a \end{bmatrix}_1 = \lambda (I + A) \begin{bmatrix} X_a \\ Y_a \\ Z_a \end{bmatrix}_2 + (I - A) \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} \quad (3)$$

In order to solve the 7 parameters, it is necessary to have at least 3 homologous points in both systems. In photogrammetry, topography and geodesy, three categories of control points may be defined: the points whose coordinates in  $X, Y$  and  $Z$  are known, the points whose only planimetric coordinates in  $X$  and  $Y$  are known, and finally the altimetric points whose only vertical coordinates (in  $Z$ ) are known. As one can see it in the equation 1, every couple of control points, which is in  $X, Y$  and  $Z$ , gives three equations. In order to construct a system of polynomials, seven equations are necessary, and therefore we must have at least one couple of points known in  $X, Y$  and  $Z$ , as well as one point in  $Z$ . In practice, one has almost always points known in  $X, Y$  and  $Z$ . For three points  $K, M$  and  $N$  expressed in the 2 systems we obtain the following system of 7 equations in 7 unknowns:

$$\begin{aligned}
 f_1 &= -Xk2 - cYk2 + bZk2 + \lambda Xk1 - \lambda cYk1 + \\
 &\quad \lambda bZk1 + T_x - cT_y + bT_z, \\
 f_2 &= cXk2 - Yk2 - aZk2 + \lambda cXk1 + \lambda Yk1 - \\
 &\quad \lambda aZk1 + cT_x + T_y - aT_z, \\
 f_3 &= -bXk2 + aYk2 - Zk2 - \lambda bXk1 + \lambda aYk1 + \\
 &\quad \lambda Zk1 - bT_x + aT_y + T_z, \\
 f_4 &= -Xm2 - cYm2 + bZm2 + \lambda Xm1 - \lambda cYm1 + \\
 &\quad \lambda bZm1 + T_x - cT_y + bT_z, \\
 f_5 &= cXm2 - Ym2 - aZm2 + \lambda cXm1 + \lambda Ym1 - \\
 &\quad \lambda aZm1 + cT_x + T_y - aT_z, \\
 f_6 &= -bXm2 + aYm2 - Zm2 - \lambda bXm1 + \lambda aYm1 + \\
 &\quad \lambda Zm1 - bT_x + aT_y + T_z, \\
 f_7 &= -bXn2 + aYn2 - Zn2 - \lambda bXn1 + \lambda aYn1 + \\
 &\quad \lambda Zn1 - bT_x + aT_y + T_z.
 \end{aligned}$$

The unknowns of this system are therefore  $\lambda, a, b, c, T_x, T_y$  and  $T_z$ . However, using some simple algebraic operations, we can modify these equations, to obtain a more convenient system with a smaller number of equations and unknowns. Indeed, with some linear combinations of the equations, the translation parameters can be eliminated. We will consider further the new equations

$$\begin{aligned}
 g_1 &= f_1 - f_4, \\
 g_2 &= f_2 - f_5, \\
 g_3 &= f_3 - f_7, \\
 g_4 &= f_6 - f_7.
 \end{aligned}$$

where the unknowns are  $\lambda, a, b$  and  $c$ . Once we calculate the values of these unknowns, we can compute easily the parameters of the translation.

**Resolution with the aid of Gröbner bases.** It is known that Gröbner bases approach can be useful to solve systems of algebraic equations with a finite number of solutions. In this direction, we recall first some basic definitions and notations related to Gröbner bases, and then we show how we can use them to solve a system of polynomial equations. For more details on Gröbner bases and its applications, we refer to [4].

The notion of Gröbner basis was introduced by Buchberger, who gave also the first algorithm to compute it (see [3]). This algorithm has been implemented in most general computer algebra systems like MAPLE, MATHEMATICA, SINGULAR, MACAULAY2, COCOA and SALSA software. To recall the precise definition of Gröbner bases, let  $R = K[x_1, \dots, x_n]$  be a polynomial ring where  $K$  is an arbitrary field. Let  $f_1, \dots, f_k$  be a sequence of  $k$  polynomials in  $R$ . The basic algebraic object associated with the system  $f_1 = 0, \dots, f_k = 0$  is the ideal generated by the polynomials  $f_1, \dots, f_k$  that we denote it by  $I = \langle f_1, \dots, f_k \rangle$ . Recall that this ideal is the set of all polynomials  $f_1 \cdot h_1 + \dots + f_k \cdot h_k$  where the  $h_i$ 's are arbitrary polynomials in  $R$ . Furthermore, we need a monomial ordering  $\prec$  on  $R$ , i.e. a well-ordering order on the set of monomials s.t. multiplication with a monomial respects the ordering. Note that a monomial is a power product of variables  $x_1, \dots, x_n$ . As a classical example of a monomial ordering, we recall the definition of the lexicographical ordering (LEX), denoted by  $\prec_{lex}$ . This ordering is a special monomial ordering having some interesting applied properties. Let  $\deg_i(m)$  be the degree in  $x_i$  of a monomial  $m$ . If  $m$  and  $m'$  are two monomials in  $R$ , then we write  $m \prec_{lex} m'$  if the most left non-zero entry in the sequence  $(\deg_1(m') - \deg_1(m), \dots, \deg_n(m') - \deg_n(m))$  is positive, see [4] for more details. Now, let  $\prec$  be a monomial ordering on  $R$ . The leading monomial of a polynomial  $f$  in  $R$  is the greatest monomial (with respect to  $\prec$ ) which appears in  $f$ , and we denote it by  $LM(f)$ . The leading coefficient of  $f$ , written  $LC(f)$ , is the coefficient of  $LM(f)$  in  $f$ . The leading term of  $f$  is  $LT(f) = LC(f)LM(f)$ . If  $G$  is a set of polynomials, we denote by  $LT(G)$  the monomial ideal generated by  $LT(g)$  for  $g$  in  $G$ . The leading term ideal of  $I$  is defined to be  $LT(I) = \langle LT(f) \mid f \in I \rangle$ .

**Definition 1.** (Gröbner basis) A finite subset  $G$  of  $I$  is called a Gröbner basis of  $I$  w.r.t.  $\prec$  if  $LT(G) = LT(I)$ .

**Proposition 1.** Every ideal  $I$  of the ring  $R$  has a Gröbner basis  $G$  w.r.t. a given monomial ordering. Moreover,  $G$  generates the same ideal  $I$ .

**Theorem 1.** Let  $I$  be an ideal of  $R$ ,  $\prec_{lex}$  the lexicographical ordering on  $R$  and  $G$  a Gröbner basis of  $I$  w.r.t.  $\prec_{lex}$ . Then, for each  $i$ , the intersection of  $G$  with  $K[x_i, \dots, x_n]$  is a Gröbner basis for the intersection of  $I$  with this ring.

This elimination property implies that to solve a polynomial system  $f_1 = 0, \dots, f_k = 0$ , we consider the

ideal  $I = \langle f_1, \dots, f_k \rangle$  and we compute a Gröbner basis  $G$  of  $I$  w.r.t.  $\prec_{lex}$ . Then, we can find a polynomial  $g_n$  in  $G$  in only  $x_n$ . Furthermore, by replacing the roots of  $g_n$  in  $G$ , we obtain another polynomial involving only  $x_{n-1}$ . This lifting continues all the way up until we calculate the values of all the variables.

*Example 1.* Let us consider the system  $f = xy - 2y = 0$  and  $g = x^2 - 2y^2 = 0$ . Assume that  $I$  is the ideal generated by  $f$  and  $g$  in  $\mathbb{R}[x, y]$  and  $y \prec_{lex} x$ . A Gröbner basis of  $I$  w.r.t.  $\prec_{lex}$  is the set  $\{f, g, h\}$  where  $h = y^3 - 2y$ . It can be shown easily that solving the new system  $f = g = h = 0$  is equivalent to the system  $f = g = 0$ . However, solving the new system is simpler than the original one. The roots of  $h$  (the values of  $y$ ) are  $0, \sqrt{2}, -\sqrt{2}$ . Finally, replacing each of these roots in  $g$ , we can calculate the corresponding value for  $x$ , and thus the solutions of the system are  $(0, 0), (2, \sqrt{2}), (2, -\sqrt{2})$ .

**Macaulay Matrix.** Our approach for computing Gröbner bases is based on constructing Macaulay matrix and followed by a Gaussian elimination procedure (see [6]). This matrices were first used for solving polynomial system by Macaulay in [8] where he generalized Sylvester's matrix to multivariate polynomials. The idea is to construct a matrix whose lines contain the multiples of the polynomials in the original system, the columns representing a basis of monomials up to a given degree. It was observed by Lazard [6] that for a high enough degree, ordering the columns according to a monomial ordering and performing a row reduction without column pivoting on the matrix gives directly a Gröbner basis for the input system. It seems that this solver was not previously used in photogrammetry community. Moreover, our experiments show that our proposed solver is correct, numerically stable, robust and reasonably fast.

Our approach consists in finding the matrix representation of the useful multiples of the input polynomials. In addition, this construction does not depend on the value of input points. Therefore, for any given value of input points, we can calculate directly this matrix, and then a Gröbner basis of the corresponding ideal is obtained via only a single Gaussian elimination. Once we have a Gröbner basis, we can compute the solutions (see [4]). However, the advantage of this method is that it avoids all the useless computations.

We recall now the definition of a Macaulay matrix and we explain how we could use it to compute a Gröbner basis of an ideal. Let  $I = \langle f_1, \dots, f_k \rangle$  be an ideal of  $R = K[x_1, \dots, x_n]$  and  $\prec$  a monomial ordering on  $R$ . Let  $G = \{g_1, \dots, g_s\}$  be a Gröbner basis of  $I$  w.r.t.  $\prec$ . Since, for each  $i$ ,  $g_i$  belongs to  $I$ , then we can write  $g_i = f_1.h_{i1} + \dots + f_k.h_{ik}$  for some polynomials  $h_{ij}$  in  $R$ . Suppose that we know the monomials appearing in the  $h_{ij}$ 's. For each monomial  $m$  in  $h_{ij}$ , we have  $mf_i$  is a useful multiple. For instance, for the above example, we observe that  $h = (1/2x + 1)f - 1/2yg$ . Thus, the set of all useful multiples for constructing a Gröbner basis of  $\langle f, g \rangle$  is  $\{f, g, xf, yg\}$ . Indeed, to find this set, we must perform

such a computation for each element of the Gröbner basis. We are interested in computing such a set for calculating a Gröbner basis of  $I$ . A natural question that may arise is: *How can we compute the set of all useful multiples for a given ideal?* Using the MAPLE package Groebner, we can compute it. For example, let  $F$  be the set of given polynomials. Then, the function `Basis(F, T, output = extended)` outputs  $G$ ; a Gröbner basis of the ideal generated by  $F$  w.r.t. the monomial ordering  $T$ , and also  $C$ ; a transformation matrix expressing  $G$  in terms of  $F$ . For example, if we consider the polynomials of the above example, and set  $F = [f, g]$  and  $T = plex(x, y)$ , i.e.  $y \prec_{lex} x$ , then the above function returns  $G = [h, f, g]$  and the corresponding transformation matrix  $\begin{bmatrix} 1/2x + 1 & -1/2y \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ . Thus,  $h = (1/2x + 1)f - 1/2yg$ . It should be pointed out that, for the polynomial system describing in the previous subsection, we replace the coordinates of 3 generic chosen homologous points in the equations  $g_1, \dots, g_4$  and compute then its Gröbner basis for an appropriate monomial ordering. From this basis, we can compute the set of all useful multiples. Note that this set would be also the set of useful multiples appearing in the representation of the elements of a Gröbner basis of  $\langle g_1, \dots, g_4 \rangle$  if we replace in the  $g_i$ 's the coordinates of the points by those of another 3 homologous points. So, we can suppose that we have already computed the set of all useful multiples for the ideal  $I$ , and we call it  $S$ .

Let  $d$  be the maximum degree of monomials appearing in  $S$ . We can now build the Macaulay matrix  $M(f_1, \dots, f_k)$  (to simplify we denote it by  $M$ ), as follows: Write down horizontally all the monomials of degree at most  $d$ , ordered following the given monomial ordering  $\prec$  (the first one being the largest one). Hence, each column of the matrix is indexed by a monomial of degree at most  $d$ . For each  $mf_i$  in  $S$ , we write down the coefficients of  $mf_i$  under their corresponding monomials.

$$\begin{array}{c}
 \vdots \\
 mf_i \\
 \vdots
 \end{array}
 \left(
 \begin{array}{c}
 \text{monomials of degree at most } d \\
 \dots \\
 \dots
 \end{array}
 \right)$$

For any row in the matrix, consider the monomial indexing the first non-zero column of this row. It is called the leading monomial of the row, and is the leading monomial of the corresponding polynomial. Gaussian elimination applied on this matrix leads to a Gröbner basis of  $I$  (see [6]). Let  $M'$  be the Gaussian elimination form of  $M$ . As a direct corollary of the main result of [6] we have:

**Corollary 1.** *The set of polynomials corresponding to the rows of  $M'$  forms a Gröbner basis for  $I$  w.r.t.  $\prec$ .*

*Example 2.* Let us consider again the polynomials  $f = xy - 2y$  and  $g = x^2 - 2y^2$ . Let also  $I = \langle f, g \rangle$  be an

ideal in  $\mathbb{R}[x,y]$  and  $y \prec_{lex} x$ . As we have already shown that the useful multiples of  $f$  and  $g$  that we need to compute a Gröbner basis of  $I$  is  $f, xf, g$  and  $yg$ . We construct now the matrix  $M = M(f, g)$  as follows:

$$M = \begin{matrix} & x^3 & x^2y & x^2 & y^2x & xy & x & y^3 & y^2 & y & 1 \\ \begin{matrix} f \\ xf \\ g \\ yg \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \end{pmatrix} \end{matrix}$$

After applying Gaussian elimination on  $M$ , the following matrix  $M'$  is obtained:

$$M' = \begin{pmatrix} x^3 & x^2y & x^2 & y^2x & xy & x & y^3 & y^2 & y & 1 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 \end{pmatrix}$$

Now, if we transform the rows of  $M'$  to polynomials, we get the Gröbner basis  $\{x^2y - 2xy, x^2 - 2y^2, 2xy - 2y^3, y^3 - 2y\}$  for  $I$  w.r.t.  $\prec_{lex}$ . Remark that, this basis seems to be different from that presented in the above example. However, both the sets are Gröbner bases for the same ideal (because the leading term ideal of both sets generate  $LT(I)$ , see the definition of Gröbner bases). Indeed, we can define the concept of reduced Gröbner basis which is unique. Since this is not the main subject of this article, we do not elaborate it here.

It is worth noting that, for the polynomial system of the previous subsection, once we compute the useful multiples, when the value of the input points changes, only the coefficients of polynomials change. Thus, using Lazard's approach, we build a Macaulay matrix (and we may compute it directly when the value of the input points changes), and a Gaussian elimination on this matrix gives a Gröbner basis of the ideal. For the polynomials  $g_1, \dots, g_4$ , the matrix  $M$  has a dimension  $24 \times 28$ . Here is the list of the 24 useful multiples of polynomials that we have computed:

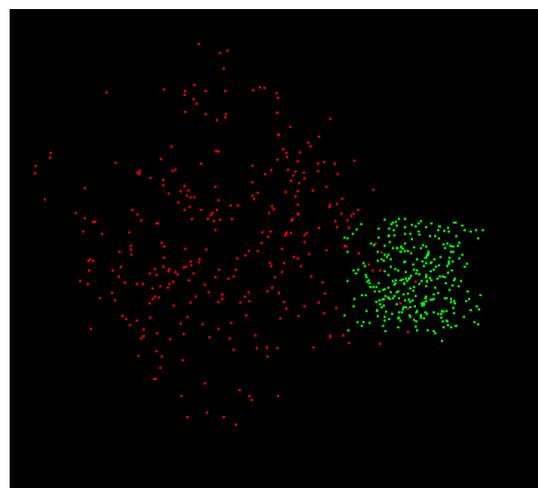
$$\begin{aligned} &g_4, \lambda g_4, \lambda^2 g_4, \lambda^3 g_4, c g_4, \\ &\lambda c g_4, \lambda^2 c g_4, \lambda^3 c g_4, g_3 \\ &\lambda g_3, \lambda^2 g_3, \lambda^3 g_3, c g_3, \lambda c g_3 \\ &\lambda^2 c g_3, \lambda^3 c g_3, g_2, \lambda g_2 \\ &\lambda^2 g_2, \lambda^3 g_2, c g_2, \lambda c g_2 \\ &\lambda^2 c g_2, \lambda^3 c g_2, g_1, \lambda, g_1 \\ &\lambda^2 g_1, \lambda^3 g_1, c g_1, \lambda c, g_1 \\ &\lambda^2 c g_1, \lambda^3 c g_1. \end{aligned}$$

### 3. Evaluation of the 7 parameters method

Now, we evaluate with synthetic data the precision of calculation using the present method. It is necessary to make vary many parameters, i. e. the rotation on the three axes, the translation in  $X, Y$  and  $Z$ , and the scale factor. In addition to these parameters, we have to add noise on the input values, with realistic standard deviations. It will allow us to value the behaviour of the algorithm in presence of noise.

**Configuration of the simulation.** As it would be without interest, as impossible to allow for an interpretation, to vary all parameters at the same time, we have proceeded in several steps. In any case the values of the coordinates in input of the two systems have received a noise with standard deviations ranging from 10cm to 50cm, with a step of 10cm. To simplify the writings, the units of length (meters) are not indicated, since they are without impact on the results. Then several values on the rotation, translation and scale factor, are used. Each time, one only of these parameters varies. The values selected for angles were  $20^\circ, 40^\circ, 60^\circ$  and  $80^\circ$ , for scale factor, 0.5, 1, 1.5 and 2, and for translations, 2000, 4000 and 6000m.

**Calculation of the points for the simulation.** In a cube whose side is 2000 (m), 1000 points are randomly sampled. These 1000 points will constitute the input points for the first system. In each configuration the rotation, translation and scale factor are applied to these 1000 points, in order to get a set of coordinates of the second system. In each configuration 500 random samples are got. As an example, for a relative translation of (5000, 1500, 500), a scale factor of 1.5 and a rotation with an angle of around the  $X$  axis of 42 degrees, the  $Y$  axis of 36 degrees, the  $Z$  axis of 24 degrees, we get the Figure 4:



**Figure 4:** In red, the points in the first system. In green, the points obtained with the given configuration

Once the two coordinates sets in the two systems are obtained, we add noise on the  $X, Y$  and  $Z$  values of all points, with r. m. s. varying from 10cm to 50cm.

Here we illustrate with some results of the simulation. A complete version of this one can be found in [5]. They have been performed for the rotations around  $X, Y$ , and  $Z$ , for the translations in  $X, Y$  and  $Z$ , and for the scale values already described.

**Variation of the rotation.** In this configuration the angle of rotation on each of the axes varies in turn. The translation is equal to  $(2000, 1000, 500)$ . The scale factor is equal to 1.5. We made vary the angle of rotation around the  $X, Y$ , and  $Z$  axes, from 0 to 80 degrees, with a step of 20 degrees. In every case only one of the components of the rotation varies, and the two others are fixed to zero. We illustrate the results for a rotation of 20 and 80 degrees around the  $X$  axis:

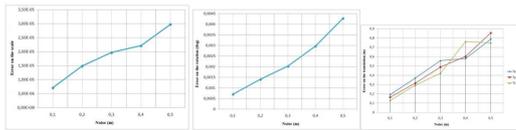


Figure 5: Rotation  $(20^\circ, 0^\circ, 0^\circ)$

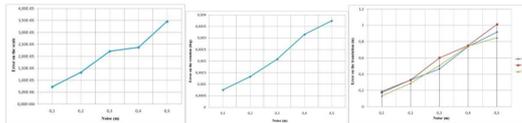


Figure 6: Rotation  $(80^\circ, 0^\circ, 0^\circ)$

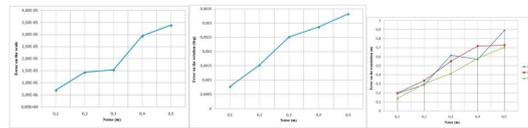


Figure 7: Rotation  $(0^\circ, 20^\circ, 0^\circ)$

**Variation of the translation.** In this configuration the three angles of rotation are fixed and are equal to  $40^\circ, 20^\circ$  and  $30^\circ$ . The scale factor is equal to 2. The translation varies on every axis from 0 to 6000, with steps of 2000. The case of translation 0 is the one of a panoramic configuration. Here we illustrate the results for a translation of 0 m (panoramic pictures) and a translation of 6000m.

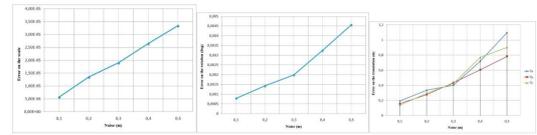


Figure 8: Rotation  $(0^\circ, 80^\circ, 0^\circ)$

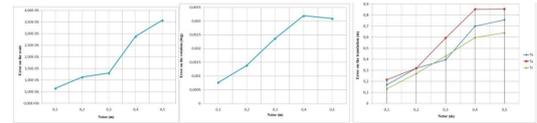


Figure 9: Rotation  $(0^\circ, 0^\circ, 20^\circ)$

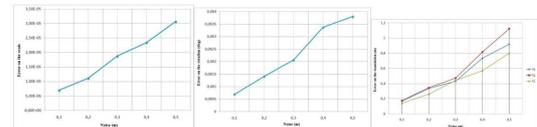


Figure 10: Rotation  $(0^\circ, 0^\circ, 80^\circ)$

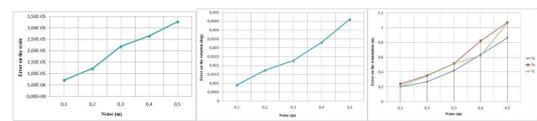


Figure 11: Translation  $(0, 0, 0)$  Case of panoramic images

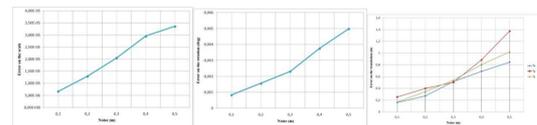


Figure 12: Translation  $(6000, 0, 0)$

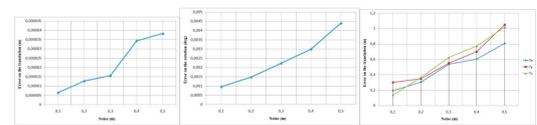


Figure 13: Translation  $(0, 6000, 0)$

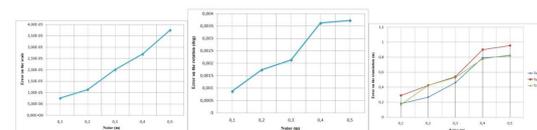


Figure 14: Translation  $(6000, 0, 0)$

**Variation of the scale factor.** In this configuration only the scale varies. The 4 values which are taken in account are 0.5, 1, 1.5 and 2. The values of the rotation are equal to 40 degrees, 20 degrees and 30 degrees. The translation is equal to (2000, 1000, 500). Here we illustrate the results for scales of 0.5 and 2.

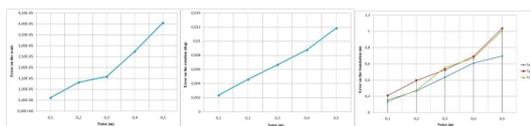


Figure 15: Scale = 0.5

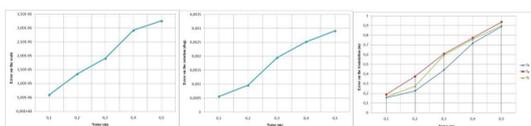


Figure 16: Scale = 2

For all of these tests, we have observed that the errors were rising with the r.m.s. of the noise, and that in nearly all simulations, the error on the scale factor was lower than  $4 \times 10^{-5}$ , the errors on the rotations were lower than  $0.005^\circ$ , and the errors on the translations were lower than 1m. These extremely good values prove the efficiency of the method in any geometric configuration.

## 4. Conclusions

The present algorithm of resolution of the 3D similitude presented here has been tested in different configurations. This method calculates the elements in a direct manner using Gröbner bases technique, therefore without need of approached values. In all these simulations, some very different configurations have been tested in presence of noise, and we have seen that the algorithm behaves very well in all these configurations. Without representing strictly speaking a definitive proof, these are strong indications on the reliability of this method. The method of the 7 parameters presented here can be used as an alternative to the classic bearing. The application of this method ranges widely beyond photogrammetry and computer vision, as for example in the domain of geodesy and topography.

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**Amir Hashemi** received his PhD degree in computer science at Université Paris 6 in France. His research interests are in the areas of computer algebra including Gröbner bases (computation and applications), comprehensive Gröbner bases (computation

and applications), solving (parametric) polynomial systems, involutive bases, triangular decomposition, differential (computer) algebra. He has published research articles in reputed international journals of Mathematics and computer science. He is now an assistant professor of Mathematics at Isfahan university of technology in Iran.



**Mahzad Kalantari** is born in 1982 in Ispahan (Iran). She got her engineer’s diploma and master degree in electronics at ESIEE PARIS. She has got in 2009 a PhD in Informatics, for a research devoted to photogrammetry and computer vision. She is now electronic engineer at

SNCF (French National Railways).



**Michel Kasser** is Professor in Geodesy and Advanced Geomatics at HEIG-VD (Switzerland). As University Professor in France, he has been formerly Director of the ESGT (engineering school for surveyors), then Director of the LAREG (Laboratory of

Geodesy) at IGN-France, and Director of the ENSG, the main French engineering school in Geomatics. Formerly Director of the GRGS (Groupe de Recherches en Geodesie Spatiale), he has been researcher in geodesy for geophysics in many active areas of the world with the Institut de Physique du Globe de Paris, and researcher in laser instrumentation with the Laboratoire d'Optique Applique where he developed the FTLRS, the smallest Satellite Laser Ranging station ever built. He has been President of the Association Francaise de Topographie (main professional body in France for Geomatics). He has got 3 patents in different laser applications, and published many scientific papers, mainly in laser instrumentation, geodesy, geophysics and photogrammetry. He gives lectures in Geomatics, Geodesy and Photogrammetry.