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RBF-PS method and Fourier Pseudospectral method for solving stiff nonlinear partial differential equations

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Abstract: Radial basis function-Pseudospectral method and Fourier Pseudospectral (FPS) method are extended for stiff nonlinear partial differential equations with a particular emphasis on the comparison of the two methods. Fourth-order Runge-Kutta scheme is applied for temporal discretization. The numerical results indicate that RBF-PS method can be more accurate than standard Fourier pseudospectral method for many nonlinear wave equations.

Keywords: RBF-PS method, FPS method, Stiff PDEs.

1 Introduction

Radial basis function-Pseudospectral method (see the references [2], [3]), is extended for stiff nonlinear differential equations. We compare the RBF meshless method with well documented standard Fourier Pseudospectral method [4, 11] for the following nonlinear wave equations.

$$u_t + \alpha u u_x + \beta u_{xx} = 0 \quad \text{(Burgers equation)}, \tag{1}$$

$$u_t + \alpha u u_x + \sigma u_{xxx} = 0 \quad \text{(Kortewege-de Vries equation)}, \tag{2}$$

$$u_t + \alpha u u_x + \beta u_{xx} + \sigma u_{xxx} + \gamma u_{xxxx} = 0 \quad \text{(Kuramoto-Sivashinsky equation)}. \tag{3}$$

Burgers equation is an important fluid dynamic model. Many researchers studied this model for conceptual understanding of physical flows and for testing of various numerical methods [6]. The Korteweg-de Vries equation was derived by Korteweg and de Vries to model water in shallow canal [8]. The Kuramoto-Sivashinsky equation is the simplest equation that exhibit chaotic behavior [7, 10]. This equation contains both the second and fourth order derivative in its linear part. The second term acts as an energy source and the nonlinear term transfers energy from low to high wave numbers. While the fourth order term counter balance the destabilizing effect of the second term. All of the above equations are time-dependent stiff PDEs which are combination of low order nonlinear terms and higher order linear terms. In both RBF-PS and FPS techniques once we descretize the spatial part of the PDE we get a system of ODEs which can be solved by an appropriate ODE solver.

2 RBF meshless method

For a given set of centers $\{x_1, x_2, x_3, ..., x_N\} \subseteq R^d$ an RBF approximation of the solution $u(x,t)$ of a PDE given by

$$u(x,t) = \sum_{j=1}^{N} \lambda_j(t) \psi(\|x - x_j\|), \tag{4}$$

Where, $\psi(\|x-x_j\|)$ is a function known as radial basis function, $\|\cdot\|$ is the Euclidian norm and $\lambda=\{\lambda_1,\lambda_2,...,\lambda_N\}$ is the expansion coefficients vector, which may be obtained at the nodes. The nodes may or may not coincide at the nodes. However in this study we take the nodes $\{x_1,x_2,...,x_N\}$ to be coincide with the centers. We can use infinitely smooth radial basis functions with free parameter c such as multiquadrics $\psi(r)=\sqrt{r^2+c^2}$, inverse multiquadrics $\psi(r)=1/\sqrt{r^2+c^2}$ or piecewise smooth functions like cubic $\psi(r)=r^3$, thin plate splines $r^2\log(r)$ and Wendland's compactly supported functions like $\psi(r)=(1-cr)_+^6(35(cr)^2+18cr+3)$. In RBF meshless approach for each collocation points $x_i, i=1,2,...,N$, equation (4) may be written in the matrix-vector form as

$$u=A\lambda, \tag{5}$$

The entries of the matrix A are $A_{ij}=\psi(\|x_i-x_j\|)$.

Using equation (1) the derivatives u_x may be obtain by differentiating the radial basis functions and then evaluate at each point $x_i, i=1,2,...,N$, we have in matrix-vector notation

$$u_x=A_x\lambda. \tag{6}$$

Where the entries of the matrix A_x are $\frac{d}{dx}\psi\left(\|x-x_j\|_{x=x_i}\right)$. The differentiation matrix can be obtained by solving equations (5)-(6) for the value of λ. Thus we have,

$$u_x=A_xA^{-1}u=D_xu, \tag{7}$$

where, $D_x=A_xA^{-1}$ is the differentiation matrix. It should be noted that the differentiation matrix depends on the invertibility of the matrix A. It is well known that the matrix A is always invertible for distinct set of collocation points. In a similar way, we can write

$$u_{xx}=A_{xx}A^{-1}u=D_{xx}u, \tag{8}$$

Where, $D_{xx}=A_{xx}A^{-1}$, and the entries of the matrix A_{xx} are $\frac{d^2}{dx^2}\psi\left(\|x-x_j\|_{x=x_i}\right)$. Similarly we can compute differentiation matrices of higher order. Using the above differentiation matrices, the RBF-PS schemes corresponding to equations (1)-(3) are given as

$$\frac{du}{dt}=-\alpha u*D_xu-\beta D_{xx}u \quad \text{(Burgers equation)}, \tag{9}$$

$$\frac{du}{dt}=-\alpha u*D_xu-\sigma D_{xxx}u \quad \text{(Kortewege-de Vries equation)}, \tag{10}$$

$$\frac{du}{dt}=-\alpha u*D_xu-\beta D_{xx}u-\sigma D_{xxx}u-\gamma D_{xxxx}u \quad \text{(Kuramoto-Sivashinsky equation)}. \tag{11}$$

The schemes (9)-(11) are of the form

$$\frac{du}{dt}=F(u) \tag{12}$$

To descretize (12) in time we can use any ODE solver. In our computations we are using fourth-order Runge-Kutta method given as

$$r_1=F(u^n), r_2=F(u^n+\frac{\delta t}{2}r_1), r_3=F(u^n+\frac{\delta t}{2}r_2), r_4=F(u^n+\delta t r_3),$$

$$u^{n+1}=u^n+\frac{\delta t}{6}(r_1+2r_2+2r_3+r_4). \tag{13}$$

It should be noted that in RBF meshless method the differentiation matrices D_x, D_{xx} and D_{xxx} are computed only once outside the time-stepping procedure. Inside the time-stepping we require only matrix-vector multiplications. So this approach is much faster than the approach used in [9], where the interpolation coefficients are computed at each time-step.

3 Fourier Pseudospectral method

In this section, we present Fourier pseudospectral method proposed by sanders et al. [11]. In general the Fourier pseudospectral method is implemented with periodic boundary conditions. Let $u(x,t)$ be periodic on a spatial grid x_j of period 2π with the spacing of grid points $h = 2\pi / N$. The interval of length other than 2π can easily be handled by a scale factor. The discrete Fourier transformed of a function $u(x,t)$ is defined

$$\hat{u}_k = h\sum_{j=1}^{N} u_j e^{-ikx_j}, k = -\frac{N}{2}+1,\dots,\frac{N}{2}, \tag{14}$$

and the corresponding inverse discrete Fourier transform is

$$u_j = \frac{1}{2\pi}\sum_{k=-N/2+1}^{N/2} \hat{u}_k e^{ikxj}, \; j = 1,2,\dots,N, \tag{15}$$

where, k is wave number. If u is differential function with Fourier transform $\hat{u}$ then Fourier transform of the pth derivative u^p is

$$(u^p\hat{)}(k) = (ik)^p \hat{u}_k \tag{16}$$

In this method a given PDE is multiplied by an integrating factor and make a change of variable. This allow us to solve the linear part exactly and then use a numerical scheme to solve the transformed nonlinear equation. This procedure has been used for PDEs by Trefethen [12], Fornberg and Driscoll [5], Cox and Mattews [1]. We write KS equation (3) in the form

$$u_t + \frac{\alpha}{2}(u^2)_x + \beta u_{xx} + \sigma u_{xxx} + \gamma u_{xxxx} = 0. \tag{17}$$

The Burgers equation (1) can be reduced from KS equation (3) by setting $\sigma = 0$, $\gamma = 0$, while the KdV equation (2) can be obtained by setting $\beta = 0$ and $\sigma = 0$ in equation (3). The Fourier transform of the KS equation (3) is given as

$$\hat{u}_t + (\frac{\alpha ik}{2})(u^2\hat{)} - \{\beta k^2 + \alpha ik^3 - \gamma k^4\}\hat{u} = 0. \tag{18}$$

Let $l = -(\beta k^2 + \alpha ik^3 - \gamma k^4)$ multiply the above equation by e^{lt} and let $\hat{U} = e^{lt}\hat{u}$, we have

$$\hat{U}_t = -(\frac{\alpha ik}{2})e^{lt}(u^2\hat{)}. \tag{19}$$

The linear term is eliminated and the problem is no longer stiff. Working in the Fourier space we have

$$\hat{U}_t = -(\frac{i\alpha k}{2})e^{lt}F((F^{-1}(e^{-lt}\hat{U}))^2). \tag{20}$$

Where, F is the discrete Fourier transform operator. The nonlinearity involved in the models (1)-(3) causes a little trouble for an explicit time-stepping method if we use FPS method without using the method of integrating factor. However it would compute solution successfully but would need a very small time step for stability. The method of integrating factors allows us to take a relatively larger time step.

3 Numerical experiments

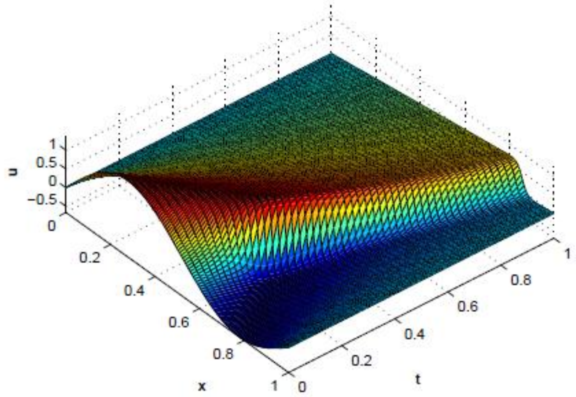
In this section, we employ three problems, the Burgers equation, the Korteweg-de Vries (KdV) equation and the Kuramoto-Sivashinsky (KS) equation to make a comparison between the Fourier Pseudospectral (FPS) method and RBF-PS methods.

Burgers equation: We solve the moving front problem given by the Burgers equation over the spatial interval $0 \le x \le 1$,with the initial solution and boundary conditions [13],

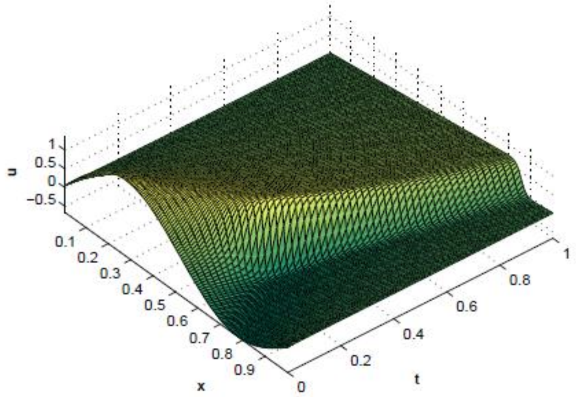
$$u(x,0) = \sin(2\pi x) + 0.5\sin(\pi x), \tag{21}$$

$$u(0,t) = 0, u(1,t) = 0. \tag{22}$$

The solution generates a steep front moving towards $x = 1$,which also decays with time due to the boundary conditions. The solution is advanced in time using fourth-order Runge-Kutta with step size $\delta t = 0.001$ for both RBF-PS method and FPS schemes and are shown in Figure 1. In the computations we used $\alpha = 1, \beta = -0.005, N = 150, t = 0.001$ and $c = 0.01$.



a- RBF-PS solution of Burgers equation.



b- FPS solution of Burgers equation.

Figure 1: RBF-PS and FPS solutions of Burgers equation, when, $\delta t = 0.001, \varepsilon = 0.005, N = 150,$ $c = 0.01, [a,b] = [0.1]$, corresponding to problem (21).

Korteweg-de Vries equation: We solve the KdV equation (3) using RBF-PS method and FPS methods over the spatial domain $0 \le x \le 16\pi$, with the single soliton initial solution and boundary conditions

$$u(x,0) = 2\sec h^2(x - x_0), \tag{23}$$

$$u(0,t) = 0, u(16\pi,t) = 0. \tag{24}$$

The L_∞ , L_2 error norms and two of the conservative quantities $I_1 = \int_a^b u du$, $I_2 = \int_a^b u^2 du$ are computed and compared. We define $E_i = \|I_i(t) - I_i(0)\| / I_i(0), i = 1,2$ as the relative errors in the conservative quantities. For time integration fourth-order Runge-Kutta method is used for both RBF-PS and FPS schemes. The results are shown in Table 1. We used $\alpha = 6, \sigma = 1, N = 25, x_0 = 10$ and MQ shape parameter $c = 1$. The exact solution of the KdV equation is given as

$$u(x,t) = 2\sec h^2(x - x_0 - 4t). \tag{25}$$

Again we solve the KdV equation (2) over the interval $-\pi \le x \le \pi$, but this time with the initial solution and boundary condition

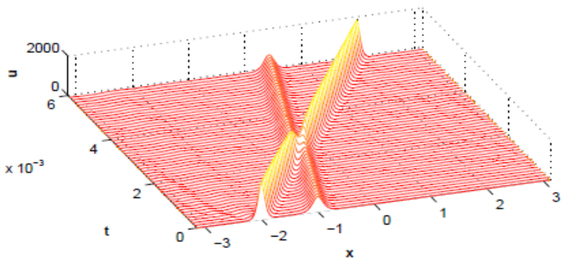
$$u(x.0) = 3A^2 \sec h^2(0.5A(x + 2)) + 3B^2 \sec h^2(0.5B(x + 1)), \tag{26}$$

$$u(-\pi,t)=0, u(\pi,t)=0, \tag{27}$$

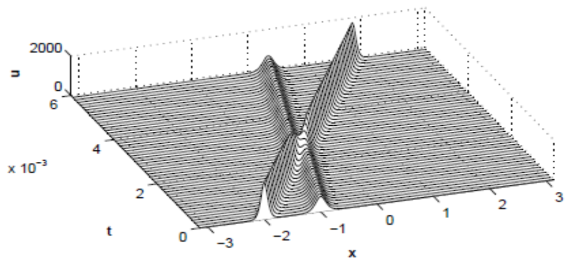
which is the interaction of two solitary waves centered at $x=-2$, and $x=-1$. Here the constants $3A^2$ and $3B^2$ are the amplitudes, which are proportional to the speeds of the two waves. So the wave of larger amplitude moves faster than the wave with smaller amplitude interacting it and passing through it cleanly. In our computations we used $A=25, B=16, c=0.0231, N=256, \delta t=10^{-6}, \alpha=1, \sigma=1$,
The results of both RBF meshless method and FPS methods are shown in Figure 2.

Table 1: Comparison of RBF-PS method and FPS method for the numerical solution KdV equation, when, $N=256, c=1, t=7, [a,b]=[0,16\pi]$, corresponding to problem (23).

	RBF-PS method				FPS method			
δt	L_∞	L_2	E_1	E_2	L_∞	L_2	E_1	E_2
1×10^{-3}	7.36×10^{-5}	4.41×10^{-4}	5.70×10^{-7}	3.65×10^{-8}	4.30×10^{-7}	1.31×10^{-6}	2.49×10^{-9}	6.23×10^{-6}
1×10^{-4}	7.14×10^{-5}	4.30×10^{-4}	2.13×10^{-8}	3.61×10^{-8}	3.85×10^{-9}	2.30×10^{-8}	2.49×10^{-9}	6.25×10^{-6}



a- RBF-PS method of KdV equation.



b- FPS method for KdV equation.

Figure 2: RBF-PS and FPS solutions of KdV equation, when $\delta t=10^{-6}, N=256, c=0.0231$
Corresponding to problem (26).

Kuramoto-Sivashinsky equation: Here we solved the KS equation (3) over the spatial domain $0\le x\le 16\pi$ with the initial solution and boundary conditions

$$u(x,0)=C+9-15[\tanh(D(x-x_0))+\tanh^2(D(x-x_0))-\tanh^3(D(x-x_0)) \tag{28}$$

$$u(0,t)=0, u(16\pi,t)=0 \tag{29}$$

The L_∞, L_2 error norms are computed. For time integration fourth-order Runge-Kutta method.
The results are shown in Table 2. In this computations we have used $\alpha=1, \beta=1, \sigma=4, \gamma=1, N=120,$ $C=6, D=1/2, c=1.5$ and the exact solution of the KS equation (3) is given by

$$u(x,t)=C+9-15[\tanh(D(x-x_0-Ct))+\tanh^2(D(x-x_0-Ct))-\tanh^3(D(x-x_0-Ct)) \tag{30}$$

Finally we consider the KS equation (3) with the initial solution and boundary conditions as

$$u(x,0)=\cos(\frac{x}{16})(1+\sin(\frac{x}{16})), \tag{31}$$

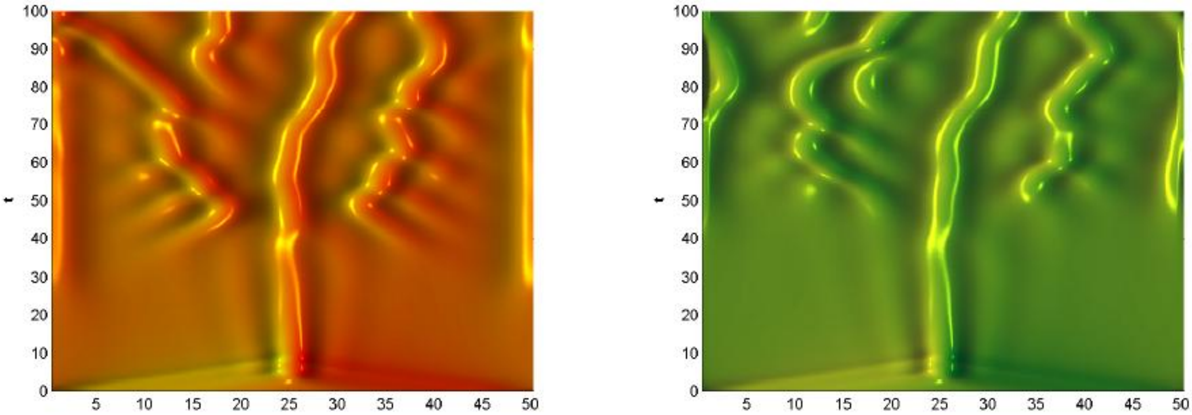
$$u(0,t)=0, u(16\pi,t)=0, \tag{32}$$

We solve this problem using both the RBF-PS method and FPS methods. For time integration we used fourth-order Runge-Kutta method. The solution is shown in Figure 3. Despite the sensitivity and chaotic behavior of the equation both the methods produced the correct solution over a very long run, which would be difficult to obtain by using some low order schemes. In this computation we used

$\alpha=2, \beta=1, \sigma=0, \gamma=1, N=120, \delta t=0.0001, c=1.5$, and the spatial domain, $0 \le x \le 16\pi$.

Table 2: Comparison of RBF-PS method and FPS method for the numerical solution of Kuramoto-Sivashinsky equation, when $c=1.5, N=120, t=1, [a,b]=[0.16\pi]$, corresponding to problem (28).

	RBF-PS method		FPS method	
δt	L_∞	L_2	L_∞	L_2
1×10^{-3}	4.78×10^{-4}	4.41×10^{-3}	1.70×10^{-3}	6.07×10^{-3}
1×10^{-4}	478×10^{-4}	4.41×10^{-3}	1.70×10^{-3}	6.07×10^{-3}



a- RBF-PS solution of KS equation. b- FPS solution of KS equation.

Figure 3: RBF-PS method and FPS method for the numerical solution of Kuramoto-Sivashinsky equation, when $\delta t=0.0001, N=120, c=1.5, [a,b]=[0.16\pi]$, up to time $t=100$, corresponding to (31).

3 Conclusions

Radial basis function-Pseudospectral (RBF-PS) and Fourier Pseudospectral (PS) methods are applied for some nonlinear wave equations. The results of the two methods for these stiff partial differential equation are compared. For Burgers equations both the method produced identical graphical solution. The FPS performed better than RBF meshless method in solving Burgers equation. However the RBF meshless method produced better results than FPS method for Kuramoto-Sivashinsky equation. Both the methods remained stable for time step size $\delta t \le 1.0\times10^{-3}$. However FPS method worked better with relatively larger time step δt . It is found that RBF meshless method is a reliable and robust approach of spectral level accuracy. It has great potential for application in engineering disciplines.

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