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Riemann-Liouville Fractional Derivative and Application to Model Chaotic Differential Equations

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Abstract: In this work, the stability analysis and numerical treatment of chaotic time-fractional differential equations are considered. The classical system of ordinary differential equations with initial conditions is generalized by replacing the first-order time derivative with the Riemann-Liouville fractional derivative of order α, for $0 < \alpha \leq 1$. In the numerical experiments, we observed that analysis of pattern formation in time-fractional coupled differential equations at some parameter value is almost similar to a classical process. A range of chaotic systems with current and recurrent interests which have many applications in biology, physics and engineering are taken to address any points and queries that may naturally occur.

Keywords: Chaotic systems, fractional calculus, numerical simulations, Riemann-Liouville derivative.

1 Introduction

Fractional calculus has a long history [1–3]. It is seen as the generalization of ordinary differentiation and integration to non-integer orders. Over the years, fractional calculus was found to play an important role in the modeling of a considerable number of real-life or physical phenomena; for instance, the predator-prey dynamic [4, 5], the modeling of memory-dependent and complex media such as porous media [6–9] and references therein. It has emerged as an essential tool for the study of dynamical systems where standard methods are not effective with strong limitations. A lot of research findings have revealed that majority of models that are based only on the integer (classical) order derivatives do not provide enough information to describe the complexity of such phenomena, on the basis of their mathematical and physical considerations. Some system of equations present delays which may be finite, infinite, or state-dependent. Others are subject to an impulsive effect. This paper aims to capture a wide reader of specialists such as biologist, economist, engineers, mathematicians as well as physicists on the application of fractional calculus to address various nonlinear problems.

A general time-fractional differential equation of order α is given by

$$\varphi_n \mathscr{D}_t^{\alpha_n} u(t) + \cdots + \varphi_1 \mathscr{D}_t^{\alpha_1} u(t) + \varphi_0 \mathscr{D}_t^{\alpha_0} u(t) = 0 \tag{1}$$

where $\varphi_\delta \in \mathbf{R}$, $\delta = 0, 1, 2, \ldots, n$ are the coefficients of the differential equations. Without loss of generality, we let $\alpha_n > \alpha_{n-1} > \alpha_{n-2} > \cdots > \alpha_0 \geq 0$. Following [3], we present the analytical solution of (1) in the general form

$$u(t) = \frac{1}{\varphi_n} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \sum_{\delta_0+\delta_1+\delta_2+\cdots+\delta_{n-2}=k} (k; \delta_0, \delta_1, \delta_2, \ldots, \delta_{n-2})$$
$$\times \prod_{i=0}^{n-2} \left(\frac{\varphi_i}{\varphi_n}\right)^{\delta_i} \chi_k \left(t, -\frac{\varphi_{n-1}}{\varphi_n}; \alpha_n - \alpha_{n-1}, \alpha_n + \sum_{j=0}^{n-2} (\alpha_{n-1} - \alpha_j)\delta_j + 1\right), \tag{2}$$

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where $(k;\delta_0,\delta_1,\delta_2,\ldots,\delta_{n-2})$ are multinomial coefficients and the term $\chi_\delta(t,\lambda;\xi,\tau)$ denotes the Mittag-Leffler function type, bear in mind that $\delta_0 \geq 0; \delta_1 \geq 0; \ldots, \delta_{n-2} \geq 0$, as suggested by Podlubny [3, 10]. This function is given by

$$\chi_\delta(t,\lambda;\xi,\tau) = t^{\xi\delta+\tau-1}E^{(\delta)}_{\xi,\tau}\left(\lambda t^{\xi}\right), \quad \text{for } \delta = 0,1,2,\ldots$$

where $E^{(\delta)}_{\xi,\tau}(v)$ is the $\delta-$th derivative of two parameters Mittag-Leffler function [11]

$$E^{(\delta)}_{\xi,\tau}(v) = \sum_{i=0}^{\infty}\frac{(i+\delta)!\,v^i}{\Gamma(\xi i+\xi_\delta+\tau)\cdot i!}, \quad \text{for } \delta = 0,1,2,\ldots.$$

The Laplace transform of function $\chi_\delta(t,\pm\lambda;\alpha,\eta)$ is defined by

$$\mathscr{L}\{\chi_\delta(t,\pm\lambda;\alpha,\eta)\} = \frac{\delta!\,y^{\alpha-\eta}}{(y^\alpha \pm \lambda)^{\delta+1}}. \tag{3}$$

for $y > |\lambda|^{1/\alpha}$. Readers are referred to [3, 12] for a list of some useful Laplace transforms and their inverse functions.

The non-local nature of fractional derivatives can be used to simulate accurately many natural phenomena containing long memory, for example, the groundwater and geo-hydrology models [6, 7, 13]. In recent years, there has been considerable interest in formulating different methods to numerically solve various types of differential and integral equations. One of the outstanding techniques is the use of spectral methods [14–21], due to their flexibility of implementation over finite and infinite intervals. A good survey of many appropriate numerical approximations can be found in [22, 23].

Precisely speaking, we consider the fractional ordinary differential system of the form

$$\mathscr{D}^{\alpha}_{0,t}\mathbf{u}(t) = f(t,u(t)), \quad t \in (0,T], \quad T > 0, \tag{4}$$

where $f \in C([0,T]\times\mathbf{R},\mathbf{R})$, $\alpha \in (0,1)$, and $n > 0$ is an integer not less than α. Here, we denote the term $\mathscr{D}^{\alpha}_{0,t}\mathbf{u}(t)$ is the Riemann-Liouville αth-order derivative of function $u(t)$ defined by

$${}^{RL}\mathscr{D}^{\alpha}_{0,t}\mathbf{u}(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\xi)^{n-\alpha-1}u^{(n)}d\xi, \quad n-1<\alpha<n, \quad n \in \mathbb{Z}^+. \tag{5}$$

The rest of this paper is structured as follows. There are several definitions of fractional derivatives, a quick tour based on the most useful and widely applied cases, such as the Riemann-Liouville, the Caputo, and the Grünwald-Letnikov types are briefly discussed in Section 2. Numerical method of approximation is given in Section 3. Modelling of three chaotic problems arising in biology, physics and financial economics is considered in Section 4. Finally, the main conclusion is outlined in Section 5.

2 Preliminary Background

In this section, we briefly introduced some of the useful preliminaries regarding fractional calculus theory [1, 3, 24–26].

The Riemann-Liouville fractional integral of order α for a function $u(t) \in C^1([0,b],\mathbb{R}^n)$; $b > 0$ is given by

$${}^{RL}\mathscr{I}^{\alpha}_{0}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1}u(\xi)d\xi, \quad 0<\alpha<\infty, \tag{6}$$

where $\Gamma(\cdot)$ denotes the Euler's gamma function.

The left Riemann-Liouville derivative and the right Riemann-Liouville derivative with order $\alpha > 0$ of the given function $u(t) \in C^1([0,b],\mathbb{R}^n)$ are respectively given as

$${}^{RL}\mathscr{D}^{\alpha}_{0,t}u(t) = \frac{d^n}{dt^n}\left[\mathscr{D}^{\alpha-n}_{a,t}u(t)\right] = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-\xi)^{n-\alpha-1}u(\xi)d\xi \tag{7}$$

and

$${}^{RL}\mathscr{D}^{\alpha}_{t,b}u(t) = (-1)^n\frac{d^n}{dt^n}\left[\mathscr{D}^{\alpha-n}_{t,b}u(t)\right] = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_t^b (\xi-t)^{n-\alpha-1}u(\xi)d\xi, \tag{8}$$

where n is an integer which satisfies $n-1 \leq \alpha < n$.

The Riemann-Liouville fractional derivative of order $\alpha \in (0,1]$ for a function $u(t) \in C^1([0,b],\mathbb{R}^n)$; $b > 0$ is given by

$$\begin{aligned} {}^{RL}\mathscr{D}_0^{\alpha} u(t) &= \frac{d^n}{dt^n}\mathscr{I}_0^{1-\alpha} u(t) \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\xi)^{n-\alpha-1} u(\xi)d\xi, \end{aligned} \tag{9}$$

for all $t \in [0,b]$ and $n-1 < \alpha < n$, where $n > 0$ is an integer.

The left Caputo derivative and the right Caputo derivative with order $\alpha > 0$ of the given function $u(t)$, for $t \in (0,b)$ are respectively given as

$${}^{C}\mathscr{D}_{0,t}^{\alpha} u(t) = \mathscr{D}_{0,t}^{\alpha-n}\left[u^{(n)}(t)\right] = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\xi)^{n-\alpha-1} u^{(n)}(\xi)d\xi, \tag{10}$$

and

$${}^{C}\mathscr{D}_{t,b}^{\alpha} u(t) = (-1)^n \mathscr{D}_{0,t}^{\alpha-n}\left[u^{(n)}(t)\right] = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^b (\xi-t)^{n-\alpha-1} u^{(n)}(\xi)d\xi \tag{11}$$

where n is an integer which satisfies $n-1 < \alpha < n$.

The Caputo fractional derivative of order $\alpha \in (0,1]$ for a function $u(t) \in C^1([0,b],\mathbb{R}^n)$; $b > 0$ is given by

$${}^{C}\mathscr{D}_{0,t}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\xi)^{n-\alpha-1} u^n(\xi)d\xi, \tag{12}$$

for all $t \in [0,b]$.

The Grunwald-Letnikov fractional derivative of order $\alpha > 0$ of a function $u(t)$ is given by

$${}^{GL}\mathscr{D}_{0,t}^{\alpha} u(t) = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{\left[\frac{t-\alpha}{h}\right]}(-1)^k\binom{\alpha}{k}u(t-kh), \tag{13}$$

where h is the time-step.

The one-parameter Mittag-Leffler function $E_{\alpha}(z)$ is defined by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{\omega^k}{\Gamma(\alpha k+1}. \tag{14}$$

This function arises in the solution of fractional integral equations and interpolates between the power law and exponential law (when $\alpha = 1$, we obtain $E_1(z) = e^z$) scenarios for phenomena modelled by classical and fractional ordinary equations [27–31].

3 Numerical Approximation and Stability Analysis of Time-Fractional Differential System with Riemann-Liouville Derivative

In this section, we first present the numerical approximation of the Riemann-Liouville time-fractional derivative with the power law kernel [32], and later discuss the stability analysis of time-fractional differential equations.

3.1 Riemann-Liouville Approximation

We shall present directly the numerical approximation, by definition, we have

$$\begin{aligned} &u(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\xi)^{-\alpha} f(\xi)d\xi, \\ &{}^{RL}\mathscr{D}_{0,t}^{\alpha}\{f(t)\} = \frac{d}{dt}u(t), \\ &\frac{d}{dt}u(t) = \frac{u(t_{j+1}-u(t_j))}{\Delta t}, \end{aligned} \tag{15}$$

where

$$u(t_{j+1})=\frac{1}{\Gamma(1-\alpha)}\int_0^{t_{j+1}}(t_{j+1}-\xi)^{-\alpha}f(\xi)d\xi$$

and

$$u(t_j)=\frac{1}{\Gamma(1-\alpha)}\int_0^{t_j}(t_j-\xi)^{-\alpha}f(\xi)d\xi$$

Our numerical approximation is presented as follows:

$$\begin{aligned}
u(t_{j+1}) &= \frac{1}{\Gamma(1-\alpha)}\int_0^{t_{j+1}}(t_{j+1}-\xi)^{-\alpha}f(\xi)d\xi,\\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{s=0}^{j}\int_{t_s}^{t_{s+1}}(t_{j+1}-\xi)^{-\alpha}\frac{f(t_{s+1})+f(t_s)}{2}d\xi,\\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{s=0}^{j}\frac{f(t_{s+1})+f(t_s)}{2}\int_{t_s}^{t_{s+1}}(t_{j+1}-\xi)^{-\alpha}d\xi,\\
&= \frac{1}{\Gamma(2-\alpha)}\sum_{s=0}^{j}\frac{f(t_{s+1})+f(t_s)}{2}\left[(t_{j+1}-t_{s+1})^{1-\alpha}-(t_{j+1}-t_s)^{1-\alpha}\right],
\end{aligned}\tag{16}$$

similarly,

$$\begin{aligned}
u(t_j) &= \frac{1}{\Gamma(1-\alpha)}\int_0^{t_j}(t_j-\xi)^{-\alpha}f(\xi)d\xi,\\
&= \frac{1}{\Gamma(2-\alpha)}\sum_{s=1}^{j}\frac{f(t_s)+f(t_{s-1})}{2}\left[(t_j-t_{s-1})^{1-\alpha}-(t_j-t_s)^{1-\alpha}\right]+\mathscr{O}(\Delta t).
\end{aligned}\tag{17}$$

Thus

$$\begin{aligned}
{}^{RL}\mathscr{D}_{0,t}^{\alpha}\{f(t)\} = \frac{1}{\Delta t\Gamma(2-\alpha)}&\left\{\sum_{s=0}^{j}\frac{f(t_{s+1})+f(t_s)}{2}\left[(t_{j+1}-t_{s+1})^{1-\alpha}-(t_{j+1}-t_s)^{1-\alpha}\right]\right.\\
&\left.-\sum_{s=1}^{j}\frac{f(t_s)+f(t_{s-1})}{2}\left[(t_j-t_{s-1})^{1-\alpha}-(t_j-t_s)^{1-\alpha}\right]\right\}\\
&+\mathbb{E}_{\alpha,j},
\end{aligned}\tag{18}$$

where

$$\mathbb{E}_{\alpha,j}=\frac{1}{\Delta t\Gamma(1-\alpha)}\left\{\sum_{s=0}^{j}\int_{t_s}^{t_{s+1}}\frac{f(u)-f(t_{s+1})}{(t_{j+1}-u)^{\alpha}}du-\sum_{s=0}^{j-1}\int_{t_s}^{t_{s+1}}\frac{f(u)-f(t_{s+1})}{(t_j-u)^{\alpha}}du\right\}.$$

Theorem 31(Atangana and Gómez-Aguilar [32]) Let f denotes a function not necessarily differentiable on $[a,T]$, then the fractional derivative of f of order α in the Riemann-Liouville sense is defined by

$$\begin{aligned}
{}^{RL}\mathscr{D}_{0,t}^{\alpha}\{u(t)\} = \frac{1}{\Delta t\Gamma(2-\alpha)}&\left\{\sum_{s=0}^{j}\frac{f(t_{s+1})+f(t_s)}{2}\left[(t_{j+1}-t_{s+1})^{1-\alpha}-(t_{j+1}-t_s)^{1-\alpha}\right]\right.\\
&\left.-\sum_{s=1}^{j}\frac{f(t_s)+f(t_{s-1})}{2}\left[(t_j-t_{s-1})^{1-\alpha}-(t_j-t_s)^{1-\alpha}\right]\right\}\\
&+\mathbb{E}_{\alpha,j},
\end{aligned}$$

where

$$|\mathbb{E}_{\alpha,j}|\leq C\left(t_{j+1}^{1-\alpha}-t_j^{1-\alpha}\right).$$

*Proof.*Readers are referred to Atangana and Gómez-Aguilar [32] for details.

3.2 Stability Analysis of Fractional Differential System in Riemann-Liouville Sense

Here, we shall consider the stability of fractional differential system in the sense of the Riemann-Liouville fractional derivative [24, 33]

$$ {}^{RL}\mathscr{D}_{0,t}^{\alpha}u(t) = Fu(t), \ \ 0 < \alpha < 1, \tag{19} $$

where $u(t) = [u_1(t), u_2(t), \ldots, u_n(t)]^{\mathrm{T}} \in \mathbb{R}^n$, $F = (f_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$, with initial condition taken the form

$$ {}^{RL}\mathscr{D}_{0,t}^{\alpha-1}u(t)|_{t=0} = u_0 = [u_{10}, u_{20}, \ldots, u_{n0}]^{\mathrm{T}}. \tag{20} $$

In what follows we analyze equation (19) for stability when F has zeros and non-zero eigenvalues.

Theorem 32 If all the eigenvalues of F satisfies

$$ |\arg(\lambda(F))| > \frac{\alpha\pi}{2}, \tag{21} $$

then the zero solution of fractional differential system (19) is asymptotically stable.

*Proof.*By taking the Laplace transform on both sides of equation (19), subject to the initial condition (20), we get

$$ u(s)s^{\alpha} - u_0 = FU(s), $$

which means that the solution of fractional differential system (19) can be written as

$$ u(t) = u_0 t^{\alpha-1} E_{\alpha,\alpha}(Ft^{\alpha}). $$

By following [10, 34], we assume that F is diagonalizable, such that

$$ \Lambda = A^{-1}FA = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n). $$

where A denotes an invertible matrix. Then,

$$ E_{\alpha,\alpha}(Ft^{\alpha}) = AE_{\alpha,\alpha}(\Lambda t^{\alpha})A^{-1} = A\,\mathrm{diag}[E_{\alpha,\alpha}(\lambda_1 t^{\alpha}), E_{\alpha,\alpha}(\lambda_2 t^{\alpha}), E_{\alpha,\alpha}(\lambda_3 t^{\alpha}), \ldots, E_{\alpha,\alpha}(\lambda_n t^{\alpha})]A^{-1}. $$

By using the two-parameter Mittag-Leffler function

$$ E_{\alpha,\alpha}(z) = -\sum_{s=2}^{k} \frac{z^{-s}}{\Gamma(\alpha-\alpha s)} + \mathscr{O}(|z|^{-1-k}), \tag{22} $$

and (21), we have

$$ E_{\alpha,\alpha}(\lambda_j t^{\alpha}) = -\sum_{s=2}^{k} \frac{(\lambda_j t^{\alpha})^{-s}}{\Gamma(\alpha-s\alpha)} + \mathscr{O}(|\lambda_j t^{\alpha}|^{-1-k}) \to 0,\ t\to+\infty,\ \ 1\le j\le n. $$

So that,

$$ \|E_{\alpha,\alpha}(\Lambda t^{\alpha})\| = \|\mathrm{diag}[E_{\alpha,\alpha}(\lambda_1 t^{\alpha}), E_{\alpha,\alpha}(\lambda_2 t^{\alpha}), E_{\alpha,\alpha}(\lambda_3 t^{\alpha}), \ldots, E_{\alpha,\alpha}(\lambda_n t^{\alpha})]\| \to 0. $$

Again, we recall that any matrix $F \in \mathbb{R}^{n\times n}$ can be written in its Jordan canonical form

$$ F = A\mathbf{J}A^{-1} \tag{23} $$

where $\mathbf{J}$ denotes the diagonal block matrix $\mathbf{J} = \mathrm{diag}(J_1, J_2, J_3, \ldots, J_M)$, and

$$ J_j = \begin{bmatrix} \lambda_j & 1 & & \\ & \lambda_j & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{bmatrix}_{n_j\times n_j}, \ \ j = 1, 2, \ldots, M. $$

Clearly,

$$E_{\alpha,\alpha}(Ft^{\alpha}) = A\,\mathrm{diag}[E_{\alpha,\alpha}(J_1t^{\alpha}), E_{\alpha,\alpha}(J_2t^{\alpha}), \ldots, E_{\alpha,\alpha}(J_Mt^{\alpha})]A^{-1},$$

and

$$\begin{aligned}
E_{\alpha,\alpha}(J_jt^{\alpha}) &= \sum_{s=0}^{\infty}\frac{(J_jt^{\alpha})^s}{\Gamma(\alpha s+\alpha)} = \sum_{s=0}^{\infty}\frac{(t^{\alpha})^s}{\Gamma(\alpha s+\alpha)}J_j^s \\
&= \sum_{s=0}^{\infty}\frac{(t^{\alpha})^s}{\Gamma(\alpha s+\alpha)}\begin{bmatrix} \lambda_j^s & K_s^1\lambda_j^{s-1} & \cdots & K_s^{n_j-1}\lambda_j^{s-n_j+1} \\ & \lambda_j^s & \ddots & \vdots \\ & & \ddots & K_s^1\lambda_j^{s-1} \\ & & & \lambda_j^s \end{bmatrix} \\
&= \begin{bmatrix} E_{\alpha,\alpha}(\lambda_jt^{\alpha}) & \frac{1}{1!}\frac{\partial}{\partial\lambda_j}E_{\alpha,\alpha}(\lambda_jt^{\alpha}) & \cdots & \frac{1}{(n_j-1)!}\left(\frac{\partial}{\partial\lambda_j}\right)^{n_j-1}E_{\alpha,\alpha}(\lambda_jt^{\alpha}) \\ & E_{\alpha,\alpha}(\lambda_jt^{\alpha}) & \ddots & \vdots \\ & & \ddots & \frac{1}{1!}\frac{\partial}{\partial\lambda_j}E_{\alpha,\alpha}(\lambda_jt^{\alpha}) \\ & & & E_{\alpha,\alpha}(\lambda_jt^{\alpha}) \end{bmatrix}
\end{aligned} \tag{24}$$

The term $K_s^i, 1 \le i \le n_j - 1$ denotes the binomial coefficients. It is noticeable that if $|\arg(\lambda_j(F))| > \frac{\alpha\pi}{2}$, $1 \le j \le M$ and $t \to \infty$, with further calculations, we have

$$|E_{\alpha,\alpha}(\lambda_jt^{\alpha})| \to 0, \quad \left|\frac{1}{i!}\left(\frac{\partial}{\partial\lambda_j}\right)^i E_{\alpha,\alpha}(\lambda_jt^{\alpha})\right| \to 0, \text{ for } 0 \le i \le n_j,\ 1 \le j \le M.$$

In fact, it is obvious that

$$E_{\alpha,\alpha}(\lambda_jt^{\alpha}) = -\sum_{s=2}^{k}\frac{(\lambda_jt^{\alpha})^{-s}}{\Gamma(\alpha-\alpha s)} + \mathscr{O}(|\lambda_jt^{\alpha}|^{-1-k}),$$

which means that $|E_{\alpha,\alpha}(\lambda_jt^{\alpha})| \to 0$ as $t \to \infty$, and

$$\begin{aligned}
\frac{1}{i!}\left(\frac{\partial}{\partial\lambda_j}\right)^i E_{\alpha,\alpha}(\lambda_jt^{\alpha}) &= \frac{1}{i!}\left(\frac{\partial}{\partial\lambda_j}\right)^i\left\{-\sum_{s=2}^{k}\frac{(\lambda_jt^{\alpha})^{-s}}{\Gamma(\alpha-\alpha s)} + \mathscr{O}(|\lambda_jt^{\alpha}|^{-1-k})\right\} \\
&= -\sum_{s=2}^{k}\frac{(-1)^i(s+i-1)\ldots(s+1)s\lambda_j^{-i-s}t^{\alpha s}}{i!\Gamma(\alpha-\alpha s)} + \mathscr{O}(|\lambda_j|^{-1-k-i}|t^{\alpha}|^{-1-k}) \\
&= -\sum_{s=2}^{k}\frac{(-1)^i(s+i-1)!\lambda_j^{-i-m}t^{\alpha s}}{i!(s-1)!\Gamma(\alpha-\alpha s)} + \mathscr{O}(|\lambda_j|^{-1-k-i}|t^{\alpha}|^{-1-k}),
\end{aligned}$$

which results to $\left|\frac{1}{i!}\left(\frac{\partial}{\partial\lambda_j}\right)^i E_{\alpha,\alpha}(\lambda_jt^{\alpha})\right| \to 0,\ 1 \le i \le n_j - 1$ as $t \to \infty$. It follows that $\|u(t)\| = \|u_0t^{\alpha-1}E_{\alpha,\alpha}(Ft^{\alpha})\| \to 0$ as $t \to \infty$, for any non-zero initial value u_0. This completes the proof.

Theorem 33 Let $0 < \alpha < 2$. For internal stability, we consider the fractional autonomous system at $a = 0$, that is

$$^{RL}\mathscr{D}_{0,t}^{\alpha}u(t) = Fu(t), \tag{25}$$

with $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_n]^{\mathrm{T}}$ which has $n-$ dimensional representation

$$\begin{aligned}
{}^{RL}\mathscr{D}_{0,t}^{\alpha_1}u_1(t) &= f_{11}u_1(t) + f_{12}u_2(t) + \cdots + f_{1n}u_n(t), \\
{}^{RL}\mathscr{D}_{0,t}^{\alpha_2}u_2(t) &= f_{21}u_1(t) + f_{22}u_2(t) + \cdots + f_{2n}u_n(t), \\
\vdots & \qquad \cdots \\
{}^{RL}\mathscr{D}_{0,t}^{\alpha_n}u_n(t) &= f_{n1}u_1(t) + f_{n2}u_2(t) + \cdots + f_{nn}u_n(t),
\end{aligned} \tag{26}$$

where α_i's are fractional numbers between 0 and 2. Suppose φ is the least common multiple of the denominators x_i's of α_i's, where $\alpha_i = y_i/x_i$, $x_i \in \mathbb{Z}^+$ for $i = 1,2,\ldots,n$ and we set $\beta = 1/\kappa$. We define

$$\det\begin{pmatrix} \lambda^{\kappa\alpha_1} - f_{11} & -f_{12} & \cdots & -f_{1n} \\ -f21 & \lambda^{\kappa\alpha_2} - f_{22} & \cdots & -f_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ -f_{n1} & -f_{n2} & \cdots & \lambda^{\kappa\alpha_n} - f_{nn} \end{pmatrix} = 0. \tag{27}$$

If all α_i's are fractional number, the characteristic polynomial equation (27) can be transformed to integer order polynomial. Then the zero solution of n-dimensional fractional differential system (26) is globally asymptotically stable provided all roots λ_i's of the characteristic equation (27) satisfy

$$|\arg(\lambda_i)| > \beta\frac{\pi}{2}, \ \forall i.$$

By putting $\lambda = \hbar^{\beta}$ in equation (27), the characteristic equation can be written in the form $\det(\hbar^{\beta}\mathbf{I} - F) = 0$.

*Proof.*See Deng et al. [34] for a similar proof.

Theorem 34 Assume all the eigenvalues of matrix F satisfy (25), that is,

$$|\arg(\lambda(F))| \geq \frac{\alpha\pi}{2},$$

and the critical eigenvalues found satisfying $|\arg(\lambda(F))| = \frac{\alpha\pi}{2}$ have the same geometric and algebraic multiplicities, then we can say that the zero solution of fractional differential system (19) is not asymptotically stable but rather stable.

*Proof.*Let us assume there is a critical eigenvalue λ_ρ which satisfies $|\arg(\lambda_\rho)| = \frac{\alpha\pi}{2}$ with both geometric and algebraic multiplicity equal to unity. Then, from the above, we present the solution of (19) as

$$\begin{aligned} u(t) &= u_0 t^{\alpha-1} E_{\alpha,\alpha}(Ft^{\alpha}) = \\ &\quad u_0 t^{\alpha-1} A\,\mathrm{diag}[E_{\alpha,\alpha}(\mathbf{J}_1 t^{\alpha}),\ldots,E_{\alpha,\alpha}(\mathbf{J}_{\rho-1}t^{\alpha}),E_{\alpha,\alpha}(\lambda_\rho t^{\alpha}),E_{\alpha,\alpha}(\mathbf{J}_{\rho+1}t^{\alpha}),\ldots,E_{\alpha,\alpha}(\mathbf{J}_\rho t^{\alpha})]A^{-1} \end{aligned} \tag{28}$$

where $\mathbf{J}_q'$s are Jordan block matrices with order q, $|\arg(\lambda_q(F))| > \frac{\alpha\pi}{2}$, and $\sum_{q=1}^{\rho-1} n_q + \sum_{q=\rho+1}^{s} n_q + 1 = n, q = 1,\ldots,\rho-1,\rho+1,\ldots,\omega$.

We obtain from above

$$E_{\alpha,\alpha}(\lambda_\rho t^{\alpha}) = \frac{1}{\alpha}(\lambda_\rho t^{\alpha})^{(1-\alpha)/\alpha}\exp((\lambda_\rho t^{\alpha})^{1/\alpha}) - \sum_{q=2}^{s}\frac{(\lambda_\rho t^{\alpha})^{-q}}{\Gamma(\alpha-\alpha q)} + \mathscr{O}(|(\lambda_\rho t^{\alpha})|^{-1-s}). \tag{29}$$

Assume

$$\lambda_\rho = s\left(\cos\frac{\alpha\pi}{2} + i\sin\frac{\alpha\pi}{2}\right),$$

recall that $i^2 = -1$, and ω denotes the modulus of λ_ρ. Then,

$$\begin{aligned} E_{\alpha,\alpha}(\lambda_\rho t^{\alpha}) &= \frac{1}{\alpha}\left[\omega t^{\alpha}\left(\cos\frac{\alpha\pi}{2} + i\sin\frac{\alpha\pi}{2}\right)\right]^{(1-\alpha)/\alpha}\exp\left(\left[\omega t^{\alpha}\left(\cos\frac{\alpha\pi}{2} + i\sin\frac{\alpha\pi}{2}\right)\right]^{(1-\alpha)/\alpha}\right) \\ &\quad - \sum_{q=2}^{s}\frac{\left[\omega t^{\alpha}\left(\cos\frac{\alpha\pi}{2} + i\sin\frac{\alpha\pi}{2}\right)\right]^{-q}}{\Gamma(\alpha-\alpha q)} + \mathscr{O}\left[\left|\omega t^{\alpha}\left(\cos\frac{\alpha\pi}{2} + i\sin\frac{\alpha\pi}{2}\right)\right|^{-1-s}\right] \\ &= \frac{1}{\alpha}\left(\omega^{(1-\alpha)/\alpha}t^{1-\alpha}\left(\sin\frac{\alpha\pi}{2} + i\cos\frac{\alpha\pi}{2}\right)\right)\exp\left(i\omega^{1/\alpha}t\right) \\ &\quad - \sum_{q=2}^{s}\frac{\left[\omega^{-q}t^{-\alpha q}\left(\cos\frac{\alpha q\pi}{2} - i\sin\frac{\alpha q\pi}{2}\right)\right]^{-q}}{\Gamma(\alpha-\alpha q)} + \mathscr{O}\left((\omega t^{\alpha})^{-1-s}\right), \end{aligned} \tag{30}$$

similarly, we have

$$\begin{aligned} t^{\alpha-1}E_{\alpha,\alpha}(\lambda_\rho t^{\alpha}) &= \frac{1}{\alpha}\left(\omega^{(1-\alpha)/\alpha}\left(\sin\frac{\alpha\pi}{2} + i\cos\frac{\alpha\pi}{2}\right)\right)\exp\left(i\omega^{1/\alpha}t\right) \\ &\quad - t^{\alpha-1}\sum_{q=2}^{s}\frac{\left[\omega^{-q}t^{-\alpha q}\left(\cos\frac{\alpha q\pi}{2} - i\sin\frac{\alpha q\pi}{2}\right)\right]^{-q}}{\Gamma(\alpha-\alpha q)} + \mathscr{O}\left(\omega t^{-s\alpha-1}\right). \end{aligned} \tag{31}$$

From the above, it is obvious that the absolute value of the first term is $\frac{1}{\alpha}\omega^{(1-\alpha)/\alpha}$, but the remainder terms tend to zero as $t \to +\infty$. We can see from the result presented in Theorem 32 that $E_{\alpha,\alpha}(\mathbf{J}_q t^{\alpha}) \to 0$ as $t \to +\infty$, for $q = 1, \ldots, \rho-1, \rho+1, \ldots, \omega$, which show that the solution of fractional differential system (19) is stable but not asymptotically stable. This completes the proof.

4 Numerical Experiments

In this section, we present some numerical experiments of the chaotic time-fractional differential equations arising from the fields of mathematical biology, physics and finance in which the time classical derivatives are replaced with the Riemann-Liouville fractional derivatives of order $\alpha \in (0,1)$. Numerical approximation is done with the forward difference scheme which we execute with the Matlab R2012a package.

4.1 Lotka-Volterra System

Let us consider the time-fractional order Lotka-Volterra System [10]

$$
\begin{aligned}
{}^{RL}\mathscr{D}^{\alpha}_{0,t}u(t) &= au(t) + \varepsilon u^2(t) - \beta u(t)v(t) - \tau w(t)u^2(t)\\
{}^{RL}\mathscr{D}^{\alpha}_{0,t}v(t) &= -cv(t) + du(t)v(t)\\
{}^{RL}\mathscr{D}^{\alpha}_{0,t}w(t) &= -\rho w(t) + \tau w(t)u^2(t)
\end{aligned}
\tag{32}
$$

where $u(t) > 0, v(t) > 0, w(t) > 0$ are the two-predator and one-prey densities. The parameters $a, \beta, c, d, \varepsilon, \rho, \tau$ are

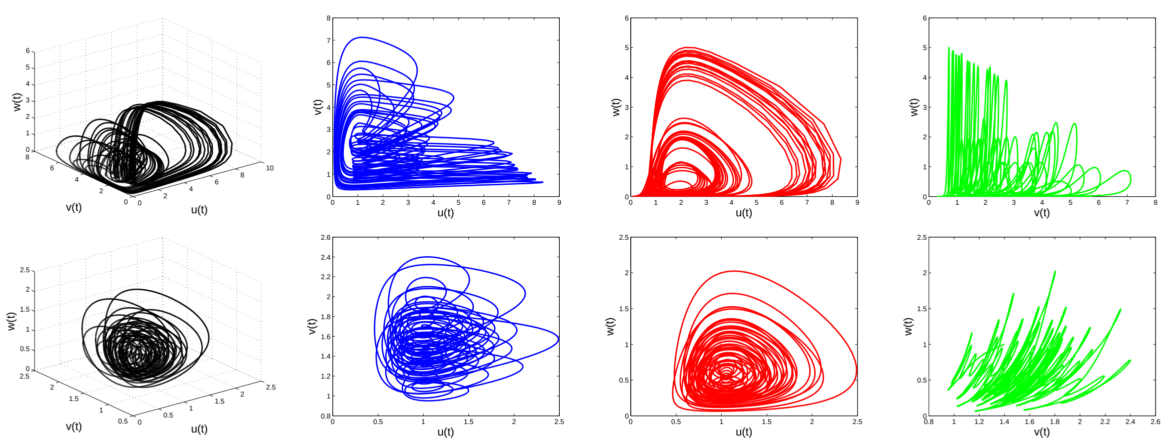

Fig. 1: Simulation result for fractional Lotka-Volterra system (32). The upper and lower rows correspond to species evolution at $\alpha = 0.73$ and $\alpha = 0.95$ respectively. Other parameters are given in (33). Simulation runs for t=100.

assumed positive. Whenever $\rho = 0, \tau = 0$, we recall the fractional order one-predator and one-prey model discussed in [35]. The Lotka-Volterra system (32) displays chaotic attractors in Figure 1 at different values of α (as given in the caption) for the parameter values

$$
a = 1,\ \beta = 1,\ c = 1,\ d = 1,\ \varepsilon = 2,\ \rho = 3,\ \tau = 2.67,\ u(0) = 1,\ v(0) = 1.4,\ w(0) = 1. \tag{33}
$$

4.2 Hyperchaotic System

Here, we consider the time-fractional order problem which describe a novel hyperchaotic system given by the four-dimensional dynamics [36]

$$\begin{aligned} {}^{RL}\mathscr{D}^{\alpha}_{0,t}u(t) &= \gamma(v(t)-u(t))+w(t)+z(t) \\ {}^{RL}\mathscr{D}^{\alpha}_{0,t}v(t) &= \beta u(t)-u(t)w(t)+z(t) \\ {}^{RL}\mathscr{D}^{\alpha}_{0,t}w(t) &= u(t)v(t)-\tau w(t) \\ {}^{RL}\mathscr{D}^{\alpha}_{0,t}z(t) &= -\rho(u(t)+v(t)) \end{aligned} \tag{34}$$

where $u(t), v(t), w(t)$ and $z(t)$ denotes the states, and $\gamma, \beta, \tau, \rho$ are positive (constants) parameters. The 4D system (34) as

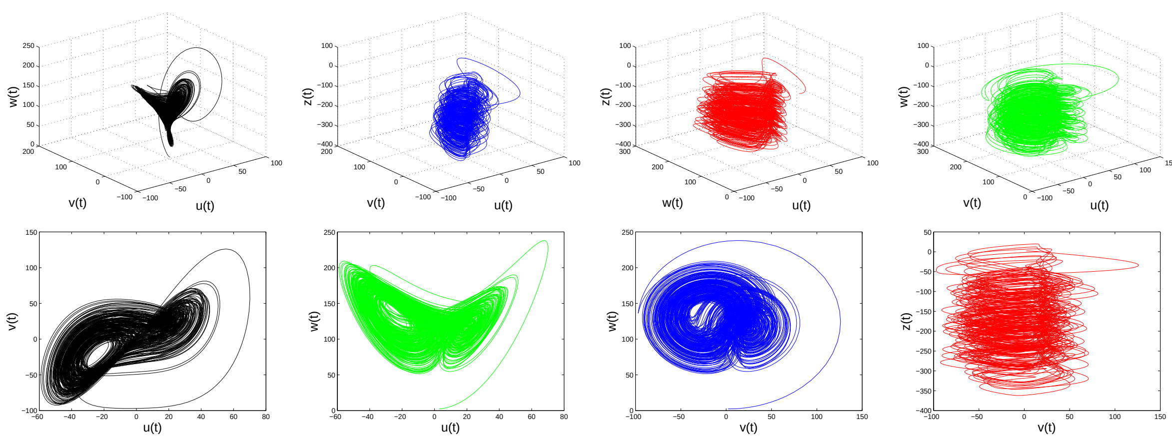

Fig. 2: Numerical results at $\alpha = 0.91$ showing the 3D (upper-row) and 2D (lower-row) projections of the novel hyperchaotic system (34) with initial conditions $u(0) = 2.6, v(0) = 2.1, w(0) = 2.5$ and $z(0) = 2.3$. Other parameters are given in (35). Simulation runs for t=100.

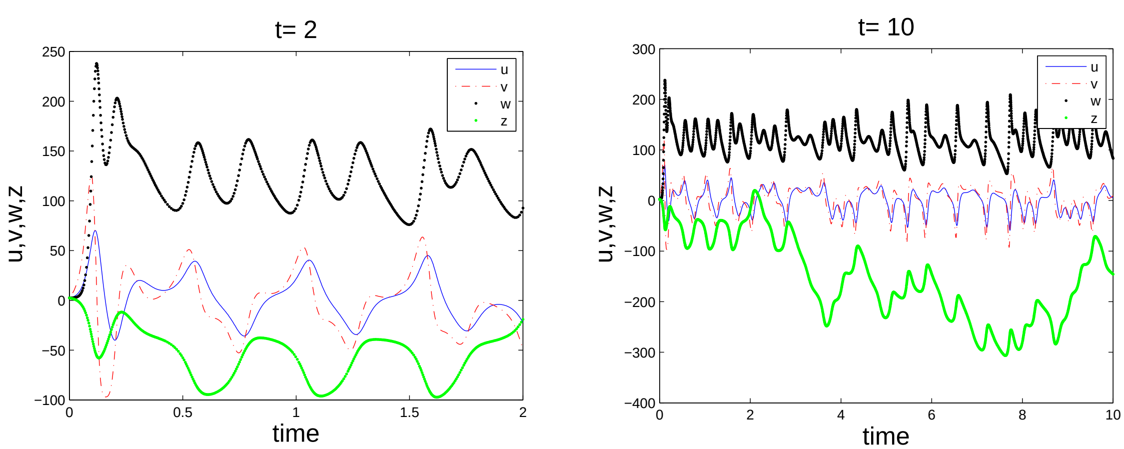

Fig. 3: Time series result of the novel hyperchaotic system (34) at $\alpha = 0.87$ with different instances of simulation time $t = 2, 10$. Other parameters are given in Figure 2.

presented in Figures 2 and 3, displays a strange hyperchaotic attractors for the parameter values

$$\gamma = 18,\ \beta = 125,\ \tau = 4,\ \rho = 6. \tag{35}$$

4.3 *Halvorsen Circulant Chaotic System*

Lastly, we consider the Halvorsen circulant chaotic system, described by the 3D dynamics [37]

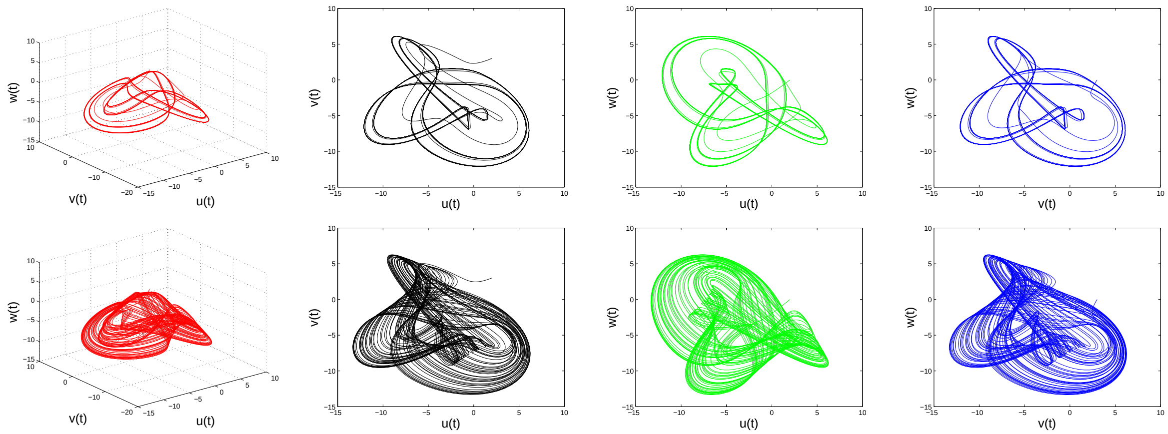

Fig. 4: Strange chaotic attractors of the Halvorsen system (36) in $\mathbb{R}^2$ and $\mathbb{R}^3$ obtained at $\alpha = 0.93$ with upper-row ($\gamma = 1.68, \beta = 3.82$) and lower-row ($\gamma = 1.27, \beta = 4$). Simulation runs for time $t = 100$.

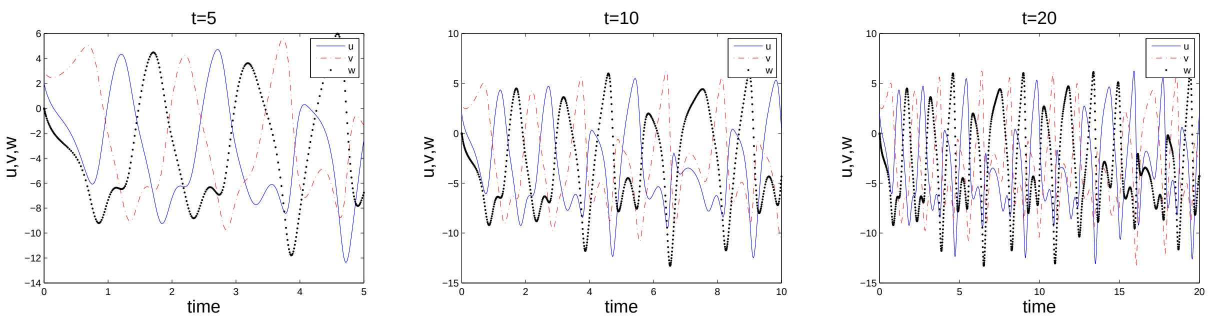

Fig. 5: Time series result of the Halvorsen circulant chaotic system (36) showing chaotic oscillations at $\alpha = 0.85$ for some instances of simulation time $t = 5, 10, 20$ with $\gamma = 1.27$ and $\beta = 4$.

$$
\begin{aligned}
{}^{RL}\mathscr{D}_{0,t}^{\alpha} u(t) &= -\gamma u(t) - \beta(v(t) + w(t)) - v^2(t) \\
{}^{RL}\mathscr{D}_{0,t}^{\alpha} v(t) &= -\gamma v(t) - \beta(u(t) + w(t)) - w^2(t) \\
{}^{RL}\mathscr{D}_{0,t}^{\alpha} w(t) &= -\gamma w(t) - \beta(u(t) + v(t)) - u^2(t)
\end{aligned}
\tag{36}
$$

which is symmetric with respect to cyclic interchanges of components u, v and w. Halvorsen have shown that the circulant chaotic system described in (36) is chaotic when $\gamma = 1.27$ and $\beta = 4$. In the numerical experiment, we simulate with the initial conditions $u(0) = 0.2$, $v(0) = 0.6$ and $w(0) = 0.2$. Figure 4 illustrates the strange attractor of system (36) in $\mathbb{R}^3$. Various 2D projections on $(u,v),(u,w)$ and (v,w) coordinate are shown. It should be mentioned that, apart from the patterns obtained for the examples considered here, other chaotic and more-complex structures are obtainable, depending on the choices of initial conditions and other parameter values.

5 Conclusion

We have given a mathematical analysis with numerical treatment for general time-fractional differential equations. In the model, we the standard time derivative is replaced with the Riemann-Liouville fractional derivative of order α, defined in the interval $(0,1]$. We observed that strange attractors can only occur if the right-hand side of the differential equation is coupled. Some numerical examples which have wide application in physics, biology and engineering are illustrated for different values of α to cover the pitfall and naturally arise question. In the future, the mathematical analysis and numerical treatment reported in this paper will be extended to time-space fractional reaction-diffusion equations.

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References

[1] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Netherlands, 2006.
[2] J. A. T. Machado, V. Kiryakova and F. Mainardi, Recent history of fractional calculus, *Commun. Nonlin. Sci.* **16**, 1140–1153 (2011).
[3] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, 1999.
[4] F. A. Rihan, Numerical modeling of fractional-order biological systems, *Abstr. Appl. Anal.* **2013**, Article ID 816803, 11 pages (2013).
[5] F. A. Rihan, S. Lakshmanan, A. H. Hashish, R. Rakkiyappan and E. Ahmed, Fractional order delayed predator-prey systems with Holling type-II functional response, *Nonlinear Dynam.* **80**, 777–789 (2015).
[6] A. Atangana, *Derivative with a new parameter: theory, methods and applications*, Academic Press, New York, 2016.
[7] A. Atangana, *Fractional operators with constant and variable order with application to geo-hydrology*, Academic Press, New York, 2017.
[8] J. F. Gómez-Aguilar, M. G. López-López, V. M. Alvarado-Martínez, J. Reyes-Reyes and M. Adam-Medina, Modeling diffusive transport with a fractional derivative without singular kernel, *Physica A.* **447**, 467–481 (2016).
[9] J. F. Gómez-Aguilar and A. Atangana, New insight in fractional differentiation: power, exponential decay and Mittag-Leffler laws and applications, *Eur. Phys. J. Plus* **132**:13, (2017), DOI 10.1140/epjp/i2017-11293-3.
[10] I. Petrás, *Fractional-order nonlinear systems: modeling, analysis and simulation*, Springer, Berlin 2011.
[11] R. Garrappa, Numerical evaluation of two and three parameter Mittag-Leffler functions, *SIAM J. Numer. Anal.* **53**, 1350–1369 (2015).
[12] R. Magin, M. Ortigueira, I. Podlubny and J. J. Trujillo, On the fractional signals and systems, *Signal Proc.* **91**, 350–371 (2011).
[13] A. H. Bhrawy, T. M. Taha and J. A. T. Machado, A review of operational matrices and spectral techniques for fractional calculus, *Nonlinear Dynam.* **81**, 1023–1052 (2015).
[14] A. H. Bhrawy and M. A. Abdelkawy, A fully spectral collocation approximation for multi-dimensional fractional Schrodinger equations, *J. Comput. Phys.* **294**, 462–483 (2015).
[15] A. Bueno-Orovio, D. Kay and K. Burrage, Fourier spectral methods for fractional-in-space reaction-diffusion equations, *BIT Numer. Math.* **54**, 937–954 (2014).
[16] K. M. Owolabi, Mathematical analysis and numerical simulation of patterns in fractional and classical reaction-diffusion systems, *Chaos Solit. Fract.* **93**, 89–98 (2016).
[17] K. M. Owolabi and A. Atangana, Numerical solution of fractional-in-space nonlinear Schrödinger equation with the Riesz fractional derivative, *Eur. Phys. J. Plus* **131**, 335 (2016) DOI 10.1140/epjp/i2016-16335-8
[18] K. M. Owolabi, Robust and adaptive techniques for numerical simulation of nonlinear partial differential equations of fractional order, *Commun. Nonlin. Sci.* **44**, 304–317 (2017).
[19] K. M. Owolabi and A. Atangana, Numerical approximation of nonlinear fractional parabolic differential equations with Caputo-Fabrizio derivative in Riemann-Liouville sense, *Chaos Solit. Fract.* **99**, 171–179 (2017).
[20] K. M. Owolabi and A. Atangana, Numerical simulation of oninteger order system in subdiffussive, diffusive, and superdiffusive scenarios, *J. Comput. Nonlin. Dynam.* **12**, 031010-1 (2017) 7-pages, DOI: 10.1115/1.4035195.
[21] K. M. Owolabi, Mathematical modelling and analysis of two-component system with Caputo fractional derivative order, *Chaos Solit. Fract.* **103**, 544–554 (2017).
[22] C. Li and F. Zeng, *Numerical methods for fractional calculus*, CRC Press, Taylor and Francis Group, London, 2015.
[23] B. M. Vinagre, Y. Q. Chen and I. Petrás, Two direct Tustin discretization methods for fractional-order differentiator/integrator, *J. Franklin Inst.* **340**, 349–362 (2003).

[24] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, John Wiley & Sons, New York, USA, 1993.
[25] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integrals and derivatives. Theory and applications*, Gordon and Breach, Yverdon, 1993.
[26] S. Abbas, M. Benchohra and G. N. N'Guérékata, *Topics in fractional differential equations*, Springer, New York, 2012.
[27] A. Atangana and D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model, *Thermal Sci.* **20**, 763–769 (2016).
[28] D. Baleanu, R. Caponetto and J. A. T. Machado, Challenges in fractional dynamics and control theory, *J. Vibr. Contr.* **22**, 2151–2152 (2016).
[29] J. A. T. Machado, Entropy analysis of integer and fractional dynamical systems, *Nonlinear Dynam.* **62**, 371–378 (2010).
[30] M. Caputo and M. Fabrizio, Applications of new time and spatial fractional derivatives with exponential kernels, *Progr. Fract. Differ. Appl.* **2**, 1–11 (2016).
[31] M. Caputo and M. Fabrizio, A new definition of fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.* **1**, 73-85 (2015).
[32] A. Atangana and J. F. Gómez-Aguilar, Numerical approximation of Riemann-Liouville definition of fractional derivative: From Riemann-Liouville to Atangana-Baleanu, *Numer. Meth. Part. Differ. Equ.*, to appear.
[33] D. Matignon, Generalized fractional differential and difference equations: stability properties and modelling issues, *Proc. of the Math. Theory of Networks and Systems Symposium*, Padova, Italy, 1998.
[34] W. Deng, C. Li and J. Lu, Stability analysis of linear fractional differential system with multiple time delays, *Nonlinear Dynam.* **48**, 409–416 (2007).
[35] E. Ahmed, A. M. A. El-Sayed and A. A. El-Saka, Equilibrium points, stability and numerical solutions of fractional-order predator-prey and rabies models, *J. Math. Anal. Appl.*, **325**, 542–553 (2007).
[36] A. T. Azar and S. Vaidyanathan, *Advances in chaos theory and intelligent control*, Springer, Switzerland, 2016.
[37] J. C. Sprott, *Elegant chaos: algebraically simple chaotic flows*, World Scientific, Singapore, 2010.