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Gull Alpha Power Moment Exponential Distribution: Statistical Properties, Estimation and Applications

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Abstract: This article introduces a new generalized distribution family called Gull alpha power moment exponential distribution. We discuss some of its statistical aspects, such as skewness, kurtosis, moments, incomplete moments, and entropy. The associated model parameters were estimated using maximum likelihood estimation (MLE). The performance of the ML approach to estimate the parameters was evaluated by a simulation study using Mote-Carlo simulations. The performance of the suggested model is demonstrated using two real data sets, and it is found to be more suited to both data sets than the other competitive models.

Keywords: Statistical Model; Numerical Results; Gull alpha power family; Moment exponential; Entropy; Estimation; Simulation; Applications.

1 Introduction

Probability distributions are most commonly used in statistical theory and practice as their fundamental concept. Data analysis is of paramount importance in many scientific disciplines. However, the data may simultaneously have two or more characteristics, such as skewness, kurtosis, and monotonic and non-monotonic failure rates. In contrast, certain family and most classical distributions cannot account for two or more data characteristics simultaneously because their cumulative density function has only one shape parameter. Numerous generalizations of probability distributions have been proposed in the literature to provide distributions with additional shape parameters, such as the variable transformation, the exponentiation method, the quantile method, the combination of two or more distributions/models, and so on. Consequently, many novel distributions have been extended and refined over the last decade. For example, the gamma-G in [1], Weibull odd Burr III −G in [9], Kumaraswamy-G in [2], type-I half logistic Burr X −G in [3], T-X family in [4], beta odd log-logistic generalized in [5], logistic-X family in [6], Burr X generator in [7], odd-Burr generalized family in [8], odd Fréchet −G in [10], exponentiated Kumaraswamy −G in [11], generalized inverted Kumaraswamy−G in [12], generalized truncated Fréchet−G in [13], Type II exponentiated half logistic−G in [14], new extended cosine−G in [15], Marshall-Olkin odd Burr III−G in [16], odd generalized N-H−G in [17], new truncated Muth−G in [18], sine-exponentiated Weibull−G in [19], odd inverse power generalized Weibull−G in [20], Type II half-logistic odd Fréchet−G in [21], ratio exponentiated general−G in [22], alpha power transformed Weibull−G in [23], compounded Bell−G in [24], sine Burr−G in [25], exponentiated M−G in [26], generalized odd Burr III−G in [27], truncated burr X−G in [28], Topp-Leone odd Fréchet−G in [29], Truncated Cauchy power Weibull−G in [30], exponentiated power generalized Weibull power series−G in [31], odd Perks−G in [32], and Kumarswamy truncated Lomax distribution in [33].

In 2020, [34] proposed the Gull alpha power (GAP) family of distribution. This distribution family has a single shape of the parameters of its cumulative function (CDF). Recently, researchers have extended the GAP family of distributions by adding two or more data features and making the family more flexible. Some of these extensions are the GAP Chen-G in [35], the GAP Ampadu-G by [36], the exponentiated generalized GAP exponential (EGGAPE) model in [41], the exponentiated generalized GAP Rayleigh (EGGAPR) model in [40], the exponentiated GAP exponential (EGAPE) distribution by [42], and an extended Kumaraswamy- GAP exponential (K-GAPE) model in [43].

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According to [34], the GAP family of distributions with CDF and probability density function (PDF), respectively, are given as follows

$$F(z;\lambda,\omega)=\lambda^{1-G(z;\omega)}G(z;\omega),\quad z\in\mathbb{R},\quad \lambda>0,\quad \lambda\neq 1, \tag{1}$$

and

$$f(z;\lambda,\omega)=\lambda^{1-G(z;\omega)}g(z;\omega)\left[1-\log(\lambda)G(z;\omega)\right],\quad z\in\mathbb{R},\quad \lambda>0,\quad \lambda\neq 1. \tag{2}$$

where$g(z;\omega)$and$G(z;\omega)$are the PDF and the CDF of parent distribution.
Moment distributions play an important role in probability theory. In [44] the moment exponential distribution (ME) was investigated, which is more flexible than the exponential distribution. The ME has another name which is the length biased exponential (LBE) distribution. The CDF and PDF of the ME distribution are

$$G(z;\ \psi)=1-\left(1+\frac{z}{\psi}\right)e^{-\frac{z}{\psi}},\quad z,\psi>0. \tag{3}$$

and

$$g(z;\ \psi)=\frac{z}{\psi^2}e^{-\frac{z}{\psi}},\quad z,\psi>0. \tag{4}$$

Many authors studied the ME or LBE such as; [37] studied the Bayesian and non-Bayesian estimation of dynamic cumulative residual Tsallis entropy for the ME distribution, [38] introduced the Kavya-Manoharan inverse LBE distribution and [39] discussed the extended Marshall-Olkin LBE distribution.
This article aims to introduce a new generalized family of distributions, the Gull alpha power ME (GAP-ME) distribution. We determine some main properties and investigate the meaning and flexibility of the new distribution. This article can be organized as follows. Section 2 studies and develops the GAP-ME distribution. In Section 3, we discuss some mathematical properties of the new distribution. Section 4 presents maximum likelihood estimation, while Section 5 investigates the performance of the GAP-ME model using a simulation study. In Section 6, we apply the GAP-ME model to two real data sets, followed by some concluding remarks in Section 7.

2 The GAP-ME Distribution

The CDF of random variable Z can be obtained by inserting (3) in (1) as below

$$F(z;\lambda,\psi)=\lambda^{\left(1+\frac{z}{\psi}\right)e^{-\frac{z}{\psi}}}\left[1-\left(1+\frac{z}{\psi}\right)e^{-\frac{z}{\psi}}\right],\quad z,\psi,\lambda>0,\quad \lambda\neq 1. \tag{5}$$

The accompanying PDF is

$$f(z;\lambda,\psi)=\lambda^{\left(1+\frac{z}{\psi}\right)e^{-\frac{z}{\psi}}}\frac{z}{\psi^2}e^{-\frac{z}{\psi}}\left[1-\log(\lambda)\left(1-\left(1+\frac{z}{\psi}\right)e^{-\frac{z}{\psi}}\right)\right],\quad z,\psi,\lambda>0,\quad \lambda\neq 1, \tag{6}$$

where, ψ and λ are two scale parameters. The survival function of Z is provided as

$$R(z;\lambda,\psi)=1-\left[\lambda^{\left(1+\frac{z}{\psi}\right)e^{-\frac{z}{\psi}}}\left[1-\left(1+\frac{z}{\psi}\right)e^{-\frac{z}{\psi}}\right]\right]$$

. The hazard rate function (HRF), reversed HRF and cumulative HRF of Z are provided as:

$$h(z;\lambda,\psi)=\frac{\lambda^{\left(1+\frac{z}{\psi}\right)e^{-\frac{z}{\psi}}}\frac{z}{\psi^2}e^{-\frac{z}{\psi}}\left[1-\log(\lambda)(1-\left(1+\frac{z}{\psi}\right)e^{-\frac{z}{\psi}})\right]}{\left[1-\lambda^{\left(1+\frac{z}{\psi}\right)e^{-\frac{z}{\psi}}}\right]\left[\left(1+\frac{z}{\psi}\right)e^{-\frac{z}{\psi}}\right]}$$

$$\tau(z;\lambda,\psi)=\frac{\frac{z}{\psi^2}e^{-\frac{z}{\psi}}\left[1-\log(\lambda)(1-\left(1+\frac{z}{\psi}\right)e^{-\frac{z}{\psi}})\right]}{\left[1-\left(1+\frac{z}{\psi}\right)e^{-\frac{z}{\psi}}\right]}$$

$$H(z;\lambda,\psi)=-\log(1-\left[\lambda^{\left(1+\frac{z}{\psi}\right)e^{-\frac{z}{\psi}}}\left[1-\left(1+\frac{z}{\psi}\right)e^{-\frac{z}{\psi}}\right]\right])$$

. Figures 1 and 2 show the PDF and the HRF for the GAP-ME distribution, with different parameter values shown.

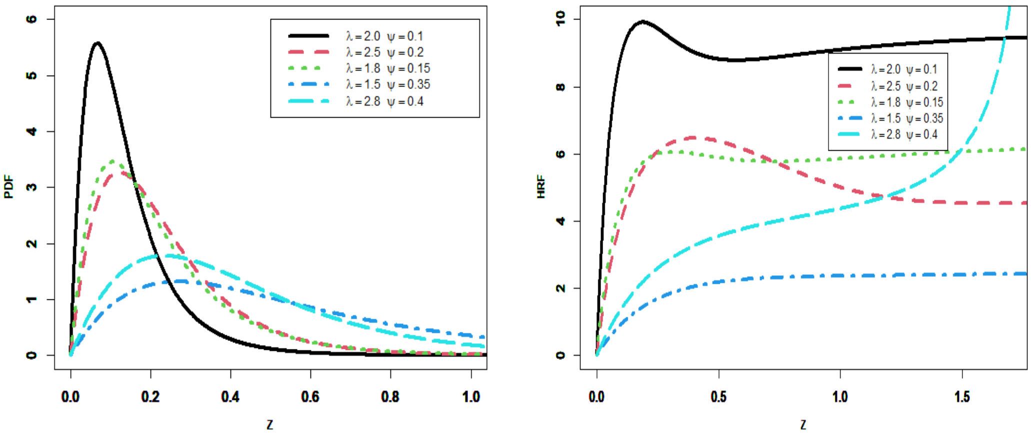

Fig. 1 The PDF and HRF plots for the GAP-ME distribution

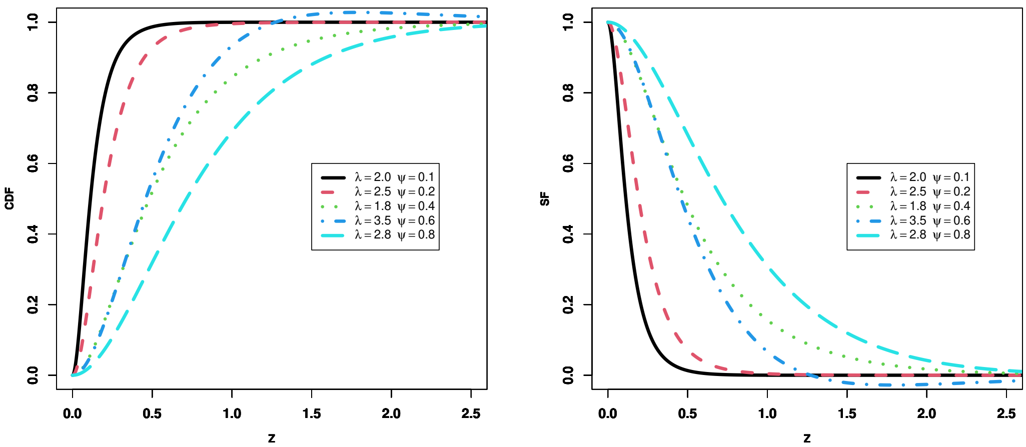

Fig. 2 The CDF and SF plots for the GAP-ME distribution

3 Characterizations of the GAP-ME Distribution

In this section, we examine several important statistical characters of the GAP-ME model.

3.1 Moments

In probability theory and statistics, moments of a distribution is essential for determining its features and creating informed decisions of the model. They provide insightful information on the central tendencies, shape, and spread of the data. We derive he r_{th} moments of GAP-ME distribution. If Z has the PDF (6), then $\mu_r^{'}$ is obtained as follows

$$\mu_r^{'} = \int_0^{\infty} z^r f(z;\lambda,\psi)\, dz \tag{7}$$

by employing Eq. (6) in Eq. (7) we have

$$\mu_r' = \frac{1}{\psi^2}\int_0^\infty z^{r+1}\lambda^{\left(1+\frac{z}{\psi}\right)e^{-\frac{z}{\psi}}} e^{-\frac{z}{\psi}}\left[1-\log(\lambda)\left(1-\left(1+\frac{z}{\psi}\right)e^{-\frac{z}{\psi}}\right)\right]dz. \tag{8}$$

Since, the power series can be written as

$$\lambda^v = \sum_{i=0}^\infty \frac{(\log(\lambda))^i}{i!}v^i. \tag{9}$$

By inserting (9) in (8), we get

$$\mu_r' = \frac{1}{\psi^2}\sum_{i=0}^\infty \frac{(\log(\lambda))^i}{i!}\int_0^\infty z^{r+1}\left(1+\frac{z}{\psi}\right)^i e^{-\frac{(i+1)z}{\psi}}\left[1-\log(\lambda)\left(1-\left(1+\frac{z}{\psi}\right)e^{-\frac{z}{\psi}}\right)\right]dz.$$

We can re-write the above equation as below

$$\begin{aligned}\mu_r' = {} & \frac{(1-\log(\lambda))}{\psi^2}\sum_{i=0}^\infty \frac{(\log(\lambda))^i}{i!}\int_0^\infty z^{r+1}\left(1+\frac{z}{\psi}\right)^i e^{-\frac{(i+1)z}{\psi}}dz \\ & -\frac{1}{\psi^2}\sum_{i=0}^\infty \frac{(\log(\lambda))^{i+1}}{i!}\int_0^\infty z^{r+1}\left(1+\frac{z}{\psi}\right)^{i+1} e^{-\frac{(i+2)z}{\psi}}dz.\end{aligned} \tag{10}$$

By applying the binomial expansion $(1+z)^i = \sum_{j=0}^{i}\binom{i}{j}z^j$ to (10), then

$$\begin{aligned}\mu_r' = {} & (1-\log(\lambda))\sum_{i=0}^\infty\sum_{j=0}^{i}\binom{i}{j}\frac{(\log(\lambda))^i}{\psi^{2+j}i!}\int_0^\infty z^{r+j+1}e^{-\frac{(i+1)z}{\psi}}dz \\ & -\sum_{i=0}^\infty\sum_{j=0}^{i+1}\binom{i+1}{j}\frac{(\log(\lambda))^{i+1}}{\psi^{2+j}i!}\int_0^\infty z^{r+j+1}e^{-\frac{(i+2)z}{\psi}}dz.\end{aligned}$$

Then,

$$\begin{aligned}\mu_r' = {} & (1-\log(\lambda))\sum_{i=0}^\infty\sum_{j=0}^{i}\binom{i}{j}\frac{(\log(\lambda))^i\Gamma(r+j+2)\psi^r}{i!(i+1)^{r+j+2}} \\ & -\sum_{i=0}^\infty\sum_{j=0}^{i+1}\binom{i+1}{j}\frac{(\log(\lambda))^{i+1}\Gamma(r+j+2)\psi^r}{i!(i+2)^{r+j+2}}.\end{aligned} \tag{11}$$

where $\Gamma(.)$ is the gamma function. Setting r =1, 2, 3, and 4 in Eq. (11) the first four ordinary moments of Z are obtained.
Table 1 illustrates the numerical results of μ_1', μ_2', μ_3', μ_4', the variance (σ^2), the skewness (SK), the kurtosis (KR) and coefficient of variation (COV) for the GAP-ME distribution.

3.2 *Incomplete moments*

The incomplete moment is an important metric that can be used to determine conditional moments, the mean waiting time, and measures of income inequality. The s_{th} upper incomplete moment of the GAP-ME distribution can be provided by

$$\omega_s(t) = \int_0^t z^s f(z;\lambda,\psi)\,dz \tag{12}$$

Plugging Eq. (6) in Eq. (12), we have

$$\omega_s(t) = \int_{-\infty}^t z^s\lambda^{\left(1+\frac{z}{\psi}\right)e^{-\frac{z}{\psi}}}\frac{z}{\psi^2}e^{-\frac{z}{\psi}}\left[1-\log(\lambda)\left(1-\left(1+\frac{z}{\psi}\right)e^{-\frac{z}{\psi}}\right)\right]dz.$$

Table 1 Numerical results of $\mu_1^{'}, \mu_2^{'}, \mu_3^{'}, \mu_4^{'}$, SK, KR, and COV for the GAP-ME model

ψ	λ	$\mu_1^{'}$	$\mu_2^{'}$	$\mu_3^{'}$	$\mu_4^{'}$	σ^2	SK	KR	COV
1.5	1.8	0.151	0.109	0.084	0.069	0.086	1.671	4.230	1.932
	2.3	0.188	0.134	0.104	0.084	0.099	1.336	3.191	1.675
	2.8	0.223	0.159	0.123	0.100	0.109	1.073	2.542	1.481
	3.3	0.257	0.183	0.141	0.114	0.117	0.852	2.118	1.327
	3.8	0.291	0.206	0.159	0.129	0.122	0.660	1.838	1.197
	4.3	0.324	0.229	0.176	0.143	0.124	0.488	1.660	1.086
	4.8	0.357	0.252	0.194	0.157	0.125	0.331	1.560	0.988
	5.3	0.390	0.275	0.211	0.170	0.123	0.184	1.526	0.899
	5.8	0.422	0.297	0.227	0.184	0.119	0.047	1.549	0.818
	6.3	0.453	0.319	0.244	0.197	0.113	0.008	1.626	0.742
3	1.8	0.051	0.037	0.030	0.024	0.035	3.708	15.617	3.668
	2.3	0.064	0.047	0.037	0.031	0.043	3.203	11.978	3.226
	2.8	0.078	0.057	0.045	0.037	0.051	2.830	9.627	2.904
	3.3	0.091	0.067	0.053	0.043	0.058	2.537	7.985	2.655
	3.8	0.104	0.076	0.060	0.050	0.066	2.296	6.775	2.454
	4.3	0.117	0.086	0.068	0.056	0.072	2.094	5.848	2.288
	4.8	0.131	0.096	0.075	0.062	0.079	1.919	5.117	2.146
	5.3	0.144	0.105	0.083	0.068	0.084	1.765	4.528	2.023
	5.8	0.157	0.115	0.090	0.074	0.090	1.627	4.043	1.915
	6.3	0.170	0.124	0.097	0.080	0.095	1.503	3.640	1.819
5	1.8	0.020	0.015	0.012	0.010	0.015	6.207	41.450	5.933
	2.3	0.026	0.019	0.015	0.013	0.019	5.444	32.166	5.236
	2.8	0.032	0.023	0.019	0.015	0.022	4.888	26.181	4.734
	3.3	0.037	0.028	0.022	0.018	0.026	4.460	22.001	4.348
	3.8	0.043	0.032	0.025	0.021	0.030	4.115	18.916	4.040
	4.3	0.048	0.036	0.028	0.023	0.033	3.829	16.547	3.785
	4.8	0.054	0.040	0.032	0.026	0.037	3.587	14.669	3.571
	5.3	0.059	0.044	0.035	0.029	0.040	3.377	13.145	3.387
	5.8	0.065	0.048	0.038	0.032	0.044	3.193	11.884	3.227
	6.3	0.070	0.052	0.041	0.034	0.047	3.029	10.823	3.085
6.5	1.8	0.013	0.009	0.007	0.006	0.009	8.038	68.637	7.620
	2.3	0.016	0.012	0.009	0.008	0.012	7.074	53.435	6.732
	2.8	0.020	0.014	0.012	0.010	0.014	6.376	43.645	6.092
	3.3	0.023	0.017	0.014	0.011	0.017	5.840	36.812	5.602
	3.8	0.026	0.020	0.016	0.013	0.019	5.410	31.771	5.211
	4.3	0.030	0.022	0.018	0.015	0.021	5.056	27.900	4.889
	4.8	0.033	0.025	0.020	0.016	0.024	4.756	24.833	4.618
	5.3	0.037	0.027	0.022	0.018	0.026	4.498	22.343	4.386
	5.8	0.040	0.030	0.024	0.020	0.028	4.273	20.281	4.184
	6.3	0.044	0.032	0.026	0.021	0.030	4.073	18.547	4.006

After some simplification, then

$$\omega_s(t) = (1-\log(\lambda))\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}\frac{(\log(\lambda))^i\gamma\left(r+j+2,\frac{(i+1)t}{\psi}\right)\psi^r}{i!(i+1)^{r+j+2}} - \sum_{i=0}^{\infty}\sum_{j=0}^{i+1}\binom{i+1}{j}\frac{(\log(\lambda))^{i+1}\gamma\left(r+j+2,\frac{(i+2)t}{\psi}\right)\psi^r}{i!(i+2)^{r+j+2}}.$$

3.3 Entropy

Rényi entropy (RE) [45] is a statistical measure of the distribution's essential shape, and it is defined using ($\nu > 0, \nu \neq 1$):

$$I_R(\nu) = \frac{1}{1-\nu}\log[\int_0^{\infty} f(z;\lambda,\psi)^{\nu}dz]\,. \tag{13}$$

Where $\nu > 0$, $\nu \neq 1$. Based on PDF (6), $f(z;\lambda,\psi)^{\nu}$ can be formed as follows:

$$f(z;\lambda,\psi) = \lambda^{\left(1+\frac{(z)^{\nu}}{\psi}\right)e^{-\frac{\nu z}{\psi}}} \frac{(z)^{\nu}}{\psi^2} e^{-\frac{\nu z}{\psi}} [1-\log(\lambda)(1-\left(1+\frac{z}{\psi}\right)e^{-\frac{z}{\psi}})]^{\nu} \tag{14}$$

Putting Eq. (13) in Eq. (14), we have

$$I_R(\nu) = \frac{1}{1-\nu}\log[\int_0^{\infty} \lambda^{\left(1+\frac{(z)^{\nu}}{\psi}\right)e^{-\frac{\nu z}{\psi}}} \frac{(z)^{\nu}}{\psi^2} e^{-\frac{\nu z}{\psi}}[1-\log(\lambda)(1-\left(1+\frac{z}{\psi}\right)e^{-\frac{z}{\psi}})]^{\nu}\, dz] \tag{15}$$

Table 2 shows some numerical findings of the RE for the GAP-ME model.

Table 2 Some numerical findings of the RE for the GAP-ME model.

ψ	λ	δ=1.2	δ=1.5	δ=2	δ=2.5	δ=3	δ=3.5	δ=4
	1.8	8.321	4.079	2.647	2.160	1.911	1.758	1.654
1.5	2.3	7.005	3.424	2.214	1.801	1.589	1.459	1.37
	2.8	5.948	2.898	1.866	1.512	1.33	1.218	1.142
	3.3	5.065	2.458	1.574	1.271	1.114	1.017	0.951
	3.8	4.306	2.081	1.324	1.063	0.928	0.844	0.787
	4.3	3.641	1.749	1.105	0.881	0.765	0.693	0.643
	4.8	3.049	1.455	0.909	0.719	0.62	0.558	0.515
	5.3	2.515	1.189	0.733	0.573	0.489	0.436	0.399
	5.8	2.030	0.947	0.573	0.440	0.37	0.325	0.293
	6.3	1.584	0.725	0.426	0.318	0.26	0.223	0.197
3	1.8	15.070	7.425	4.853	3.981	3.537	3.265	3.081
	2.3	13.648	6.716	4.381	3.588	3.184	2.937	2.769
	2.8	12.507	6.146	4.002	3.273	2.901	2.673	2.518
	3.3	11.554	5.670	3.686	3.010	2.665	2.453	2.308
	3.8	10.735	5.262	3.414	2.784	2.462	2.264	2.128
	4.3	10.018	4.904	3.176	2.586	2.284	2.098	1.971
	4.8	9.380	4.585	2.964	2.410	2.126	1.95	1.83
	5.3	8.805	4.298	2.773	2.251	1.983	1.817	1.704
	5.8	8.282	4.037	2.600	2.107	1.853	1.696	1.589
	6.3	7.802	3.798	2.440	1.975	1.734	1.585	1.483
5	1.8	20.593	10.175	6.675	5.492	4.892	4.526	4.278
	2.3	19.141	9.449	6.192	5.090	4.53	4.189	3.957
	2.8	17.976	8.868	5.805	4.767	4.24	3.918	3.7
	3.3	17.003	8.381	5.481	4.498	3.998	3.692	3.484
	3.8	16.168	7.964	5.203	4.266	3.789	3.498	3.3
	4.3	15.436	7.598	4.959	4.064	3.607	3.328	3.138
	4.8	14.785	7.273	4.743	3.883	3.445	3.176	2.994
	5.3	14.198	6.980	4.547	3.721	3.298	3.04	2.864
	5.8	13.664	6.713	4.370	3.573	3.165	2.916	2.746
	6.3	13.174	6.468	4.207	3.437	3.043	2.802	2.637
6.5	1.8	23.536	11.642	7.650	6.302	5.619	5.203	4.922
	2.3	22.077	10.913	7.164	5.898	5.255	4.864	4.599
	2.8	20.906	10.328	6.774	5.573	4.963	4.591	4.34
	3.3	19.928	9.839	6.449	5.302	4.719	4.364	4.123
	3.8	19.088	9.419	6.169	5.069	4.509	4.168	3.937
	4.3	18.352	9.052	5.924	4.865	4.326	3.997	3.774
	4.8	17.698	8.725	5.706	4.683	4.162	3.844	3.629
	5.3	17.108	8.430	5.510	4.520	4.015	3.707	3.498
	5.8	16.571	8.162	5.331	4.371	3.881	3.582	3.379
	6.3	16.079	7.916	5.167	4.234	3.759	3.468	3.27

4 Method of ML Estimation

In this section, we used the ML estimates (MLEs) method to estimate the unknown parameters of the GAP-ME distribution. Let $z_1, \ldots, z_n$ be the random sample of size n, from the GAP-ME distribution Eq. (6). The log-likelihood function of the GAP-ME distribution is given by

$$\log L = \sum_{i=0}^{n}(1+\frac{z_i}{\psi})\, e^{-\frac{z_i}{\psi}} \log(\lambda) + \sum_{i=0}^{n}\log(z_i) \; - \log(\psi^2) \; - \sum_{i=0}^{n}\frac{z_i}{\psi} + \sum_{i=0}^{n}\left[1-\log(\lambda)\left(1-\left(1+\frac{z_i}{\psi}\right)e^{-\frac{z_i}{\psi}}\right)\right]. \tag{16}$$

By differentiating Eq. (16) with regarded to λ and ψ as next

$$\frac{\partial L}{\partial \lambda} = \sum_{i=0}^{n}\left(\frac{e^{-\frac{z_i}{\psi}}(1+\frac{z_i}{\psi})}{\lambda} + \frac{1-e^{-\frac{z_i}{\psi}}(1+\frac{z_i}{\psi})}{\lambda}\right), \tag{17}$$

and

$$\frac{\partial L}{\partial \psi} = -\frac{2}{\psi} + \sum_{i=0}^{n}\frac{z_i}{\psi^2} + \sum_{i=0}^{n}\left(-\log(\lambda)\left(\frac{z_i\, e^{-\frac{z_i}{\psi}}}{\psi^2} - \frac{z_i\, e^{-\frac{z_i}{\psi}}\left(1+\frac{z_i}{\psi}\right)}{\psi^2}\right)\right) + \sum_{i=0}^{n}\left(-\frac{z_i\, e^{-\frac{z_i}{\psi}}\log(\lambda)}{\psi^2} + \frac{z_i\, e^{-\frac{z_i}{\psi}}(1+\frac{z_i}{\psi})\log(\lambda)}{\psi^2}\right). \tag{18}$$

To get the MLEs of the parameters λ and ψ, set equations (17) and (18) to zero and solve these nonlinear systems of equations concurrently.

5 Monte Carlo Simulations

In order to study the performance of the ML approach for estimating parameters, simulation study was conducted using Mote Carlo simulations. The calculations in this section are carried out using R program. The simulation process is structured as follows:

1. The random samples from the GAP-ME distribution were generated by using the inverse of equation (5).
2. Mote Carlo simulations were preformed 1000 times with $n = 30, 50, 70, 100$ and 200.
3. Tables 3, 4, and 5 list the selected values for the parameters.
4. Mean, Bias, MSE and AIL are computed.

Tables from 3 to 5 display the simulation results of the GAP-ME distribution with various values of λ, ψ and n. Based on the results in the tables, the estimated MSE and AIL decrease, when n increase. It can be concluded that the simulation performs well enough to estimate the parameters λ and ψ.

Table 3 Numerical outcomes for the GAP-ME model at λ = 0.1 and ψ =0.3

n	parameter	mean	Bias	MSE	AIL
30	λ	0.1208	0.0208	0.0174	0.3760
	ψ	0.2972	0.0028	0.0025	0.1954
50	λ	0.1166	0.0166	0.0103	0.3133
	ψ	0.2988	0.0013	0.0012	0.1377
70	λ	0.1102	0.0102	0.0064	0.2655
	ψ	0.2989	0.0011	0.0009	0.1165
100	λ	0.1063	0.0063	0.0040	0.2303
	ψ	0.2992	0.0008	0.0006	0.0950
200	λ	0.1047	0.0047	0.0020	0.1754
	ψ	0.2996	0.0004	0.0003	0.0662

Table 4 Numerical outcomes for the GAP-ME model at λ = 0.07 and ψ =0.1

n	parameter	mean	Bias	MSE	AIL
30	λ	0.0905	0.0205	0.0131	0.3110
	ψ	0.0990	0.0010	0.0003	0.0628
50	λ	0.0832	0.0132	0.0059	0.2316
	ψ	0.0990	0.0010	0.0001	0.0444
70	λ	0.0767	0.0067	0.0035	0.1924
	ψ	0.0992	0.0008	0.00003	0.0378
100	λ	0.0779	0.0079	0.0027	0.1792
	ψ	0.0998	0.0002	0.00006	0.0316
200	λ	0.0719	0.0019	0.0013	0.1385
	ψ	0.0999	0.0001	0.00003	0.0222

Table 5 Numerical outcomes for the GAP-ME model at λ = 0.08 and ψ =0.2

n	parameter	mean	Bias	MSE	AIL
30	λ	0.1006	0.0206	0.0112	0.3039
	ψ	0.1987	0.0013	0.0009	0.1162
50	λ	0.0929	0.0129	0.0076	0.2621
	ψ	0.1988	0.0012	0.0006	0.0944
70	λ	0.0919	0.0119	0.0048	0.2255
	ψ	0.1994	0.0006	0.0004	0.2744
100	λ	0.0828	0.0028	0.0028	0.1868
	ψ	0.1983	0.0017	0.0003	0.0626
200	λ	0.0842	0.0042	0.0014	0.1474
	ψ	0.1998	0.0002	0.0001	0.0439

6 Modelling to Real Data

To demonstrate the effectiveness of the GAP-ME distribution in a data-fitting situation, two data sets generated from the real world are used in this section. The MLEs of the parameters of this distribution and other competing distributions are presented, and the goodness-of-fit statistics for this distribution and other competing distributions are compared. To compare the corresponding models, we consider seven well-referenced metrics of goodness-of-fit. These metrics include the Akaike information criterion (AK-IC), the Bayesian-IC (B-IC), the Kolmogorov–Smirnov test (Ko-Sm), the Anderson-Darling test (A-D), the Cramer-von Mises test (C-VM) and the p-value (PV).

The First Dataset

According to the information presented in [46], we investigate the number of months that it takes for patients undergoing renal dialysis to get infected. The dates at the time of infection are: 5.5, 6.5, 6.5, 7.5, 3.5, 7.5, 12.5, 3.5, 2.5, 2.5, 12.5, 13.5, 3.5, 11.5, 7.5, 14.5, 14.5, 4.5, 7.5, 8.5, 9.5, 10.5, 21.5, 25.5, 27.5, 21.5, 22.5, and 22.5. Now, we divide these data by thirty to execute a normalization procedure, producing values ranging from 0 to 1. The collected data are updated: 0.116667, 0.25000, 0.28333, 0.450000, 0.08333, 0.25000, 0.35000, 0.38333 0.483333, 0.416667, 0.416667, 0.750000,0.483333, 0.116667, 0.850000, 0.316667, 0.116667, 0.15000, 0.18333, 0.216667, 0.916667, 0.216667, 0.25000,0.25000, 0.08333, 0.716667, 0.716667, and 0.750000.

We assess the goodness-of-fit of the GAP-ME model to analyze this dataset. The fits of the GAP-ME distribution is compared with power xlindley (PXL) model in [47], inverse power Lindley model (IPL) in [48], Kumaraswamy (Kw) model in [49], and beta (B) model [50].

Table 6 shows the MLEs and standard errors (SEs) of the model parameters. Table 7 illustrates the MLEs of the estimated parameters and the goodness of fit of the GAP-ME model compared to the other competing models. As shown in Tables 6-7, the GAP-ME model has the lowest values and the largest PV for all goodness-of-fit criteria, indicating that it fits the data set better than the other models. Figure 3 illustrates the histogram with fitted PDF, fitted CDF and PP plot of the GAP-ME model.

The Second Dataset

The data set below (n = 40) comes from [51] and represents the time to failure (103h) of a turbocharger of one engine type. The data are

Table 6 MLEs and SEs (in the parentheses) for the first dataset.

Model	Estimates	
GAP-ME (λ,ψ)	0.81232 (0.64044)	0.17562 (0.04924)
PXL $(\eta,\ \beta)$	1.637 (0.2409)	4.2239 (0.9428)
IPL $(\eta,\ \beta)$	1.1641 (0.1421)	0.3153 (0.0827)
Kw $(\eta,\ \beta)$	1.265 (0.2544)	2.0797 (0.5714)
B $(\eta,\ \beta)$	1.3567 (0.3332)	2.1058 (0.5496)

Table 7 Goodness-of-fit measures for the first dataset.

Models	AK-IC	B-IC	KO-SM	A-D	C-VM	PV
GAP-ME	-3.79049	-1.12609	0.11229	0.38855	0.04881	0.87193
PXL	-3.5549	-0.8905	0.12081	0.465	0.0627	0.80843
IPL	0.2734	2.9378	0.13099	0.7071	0.1012	0.72263
Kw	-3.325	-0.6606	0.13772	0.7049	0.1136	0.66296
B	-3.5552	-0.8908	0.14118	0.6859	0.1101	0.63213

8.5 3.0 4.6 5.3 6.07.3 7.7 8.0 8.4 2.0 3.9 5.0 5.6 1.6 3.5 4.8 5.4 6.0 6.5 7.0 6.1 6.5 7.1 7.3 7.8 8.1 8.4 2.6 4.5 5.1 5.8 6.3 6.7 7.3 7.7 7.9 8.3 8.7 8.8 9.0.

The fits of the GAP-ME distribution is compared with beta Fréchet (BFr) model in [52], exponentiated Fréchet (EFr) model in [53], Marshall-Olkin log logistic (MO-LLoG) model in [54], generalized Gompertz (G-Gom) model in [55] and Marshall-Olkin extended inverse Weibull (MO-IW) model in [56].

Tables 8-9 show the MLEs of the parameters of each fitted model with their standard errors (SEs) and the goodness of fit statistics are presented. Tables 8-9 show that the GAP-ME model achieves the lowest value in all goodness-of-fit metrics and the largest PV compared to the other competing models. In addition, the histogram with the fitted PDF, the fitted CDF and the PP diagram of the GAP-ME model are shown in Figure 4.

Table 8 MLEs and SEs (in the parentheses) for the second dataset.

Model	Estimates			
GAP-ME (λ,ψ)	0.00391 (0.00541)	1.58401 (0.13927)		
BFr (a, b, λ, α)	71.1502 (2.1131)	259.1749 (111.2123)	74.9748 (38.1707)	0.1685 (0.0323)
EFr (a, b, λ)	1.6108 (0.1674)	1.9440 (0.2033)	7.9296 (3.3140)	
MO-LLoG (α,β,γ)	4.6878 (0.3149)	4.8416 (0.6536)	3.9474 (0.0772)	
G-Gom (α,β,λ)	9.5146 (5.2499)	0.4498 (0.2038)	2.9039×10^{-6} (0.035)	
MO-IW (α,λ,θ)	48362 (8.4147×10^{-6})	4.8391 (0.6512)	0.9275 (0.2303)	

7 Conclusion

In this paper, a new two-parameter lifetime distribution called Gull alpha power Moment Exponential Distribution is proposed. Several statistical and computational features of the GAP-ME model are calculated, including skewness,

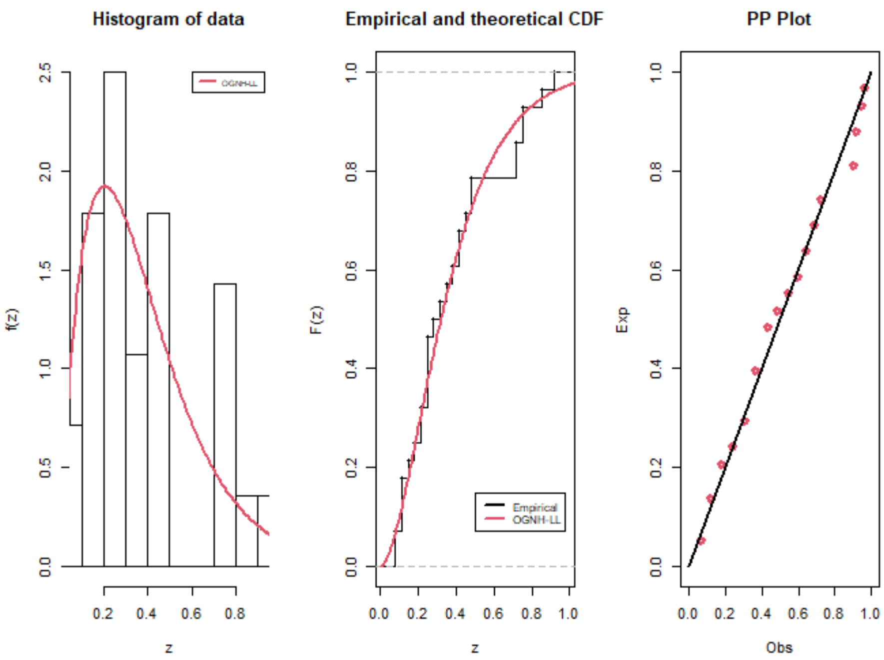

Fig. 3 Plots of the estimated PDF, CDF, and PP plot of the GAP-ME model for the first dataset.

Table 9 Goodness-of-fit measures for the second dataset.

Models	AK-IC	B-IC	KO-SM	A-D	C-VM	PV
GAP-ME	179.2559	182.6337	0.13866	1.29934	0.19466	0.42534
BFr	190.5317	197.2872	0.1523	1.9815	0.3161	0.3109
EFr	209.1836	214.2502	0.2438	3.4798	0.6067	0.0172
MO-LLoG	183.4	188.5	0.1437	1.4072	0.2142	0.3807
G-Gom	186.3	191.4	0.1542	1.7601	0.2757	0.2976
MO-IW	183.4	188.5	0.1438	1.4131	0.2153	0.3800

kurtosis, moments, incomplete moments and entropy. The maximum likelihood method is used to estimate the model parameters. To investigate the performance of the ML approach to estimate the parameters, a simulation study using Mote-Carlo simulations was conducted. An application to two real data sets was performed to investigate the significance and flexibility of the presented model. We proved that both data sets were well fitted by the GAP-ME model.

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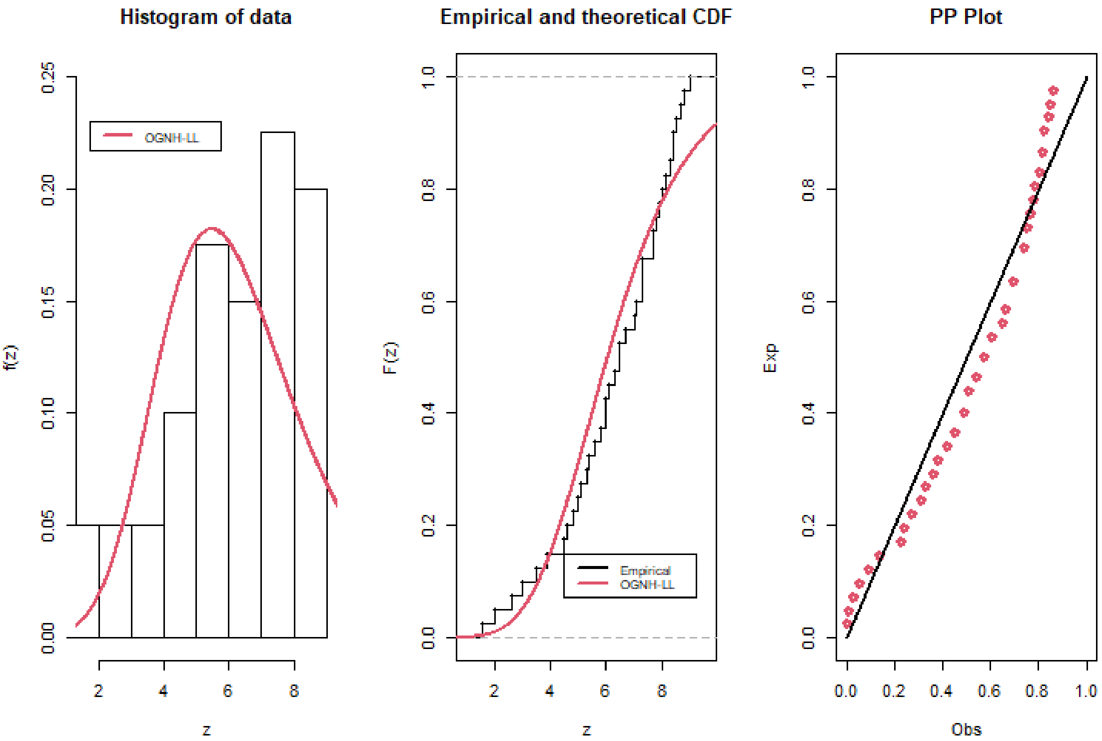

Fig. 4 Plots of the estimated PDF, CDF, and PP plot of the GAP-ME model for the second dataset.

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