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Infinitely Many Solutions for Fractional Hamiltonian Systems with Locally Defined Potentials

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Abstract: In this paper, we are concerned with the existence of infinitely many solutions for the following fractional Hamiltonian system

$$\begin{cases} {}_tD_\infty^\alpha({}_{-\infty}D_t^\alpha u)(t) + L(t)u(t) = \nabla W(t,u(t)),\ t\in\mathbb{R} \\ u\in H^\alpha(\mathbb{R}), \end{cases} \tag{1}$$

where ${}_{-\infty}D_t^\alpha$ and ${}_tD_\infty^\alpha$ are left and right Liouville-Weyl fractional derivatives of order $\frac{1}{2}<\alpha<1$ on the whole axis respectively, $L\in C(\mathbb{R},\mathbb{R}^{N^2})$ is a symmetric matrix valued function unnecessary coercive and $W(t,x)\in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$. The novelty of this paper is that, assuming that L is bounded from below and unnecessarily coercive at infinity, and W is only locally defined near the origin with respect to the second variable, we show that (1) possesses infinitely many solutions via a variant Symmetric Mountain Pass Theorem.

Keywords: Fractional Hamiltonian systems, infinitely many solutions, variational methods, locally defined potentials, symmetric mountain pass theorem.

1 Introduction

In the present paper, we are interested in the existence of infinitely many solutions for a class of fractional Hamiltonian systems of the following form

$$(\mathscr{FHS})\qquad \begin{cases} {}_tD_\infty^\alpha({}_{-\infty}D_t^\alpha u)(t) + L(t)u(t) = \nabla W(t,u(t)),\ t\in\mathbb{R} \\ u\in H^\alpha(\mathbb{R}), \end{cases}$$

where ${}_{-\infty}D_t^\alpha$ and ${}_tD_\infty^\alpha$ are left and right Liouville-Weyl fractional derivatives of order $\frac{1}{2}<\alpha<1$ on the whole axis respectively, $L\in C(\mathbb{R},\mathbb{R}^{N^2})$ is a symmetric matrix valued function unnecessary coercive and $W:\mathbb{R}\times\mathbb{R}^N\longrightarrow\mathbb{R}$ is a continuous function, differentiable in the second variable with continuous derivative $\frac{\partial W}{\partial x}(t,x)=\nabla W(t,x)$.

Recently, equations including left and right fractional derivatives have attracted extensive attentions because of its applications in mathematical modeling of processes in physics, mechanics, control theory, viscoelasticity, electrochemistry, bioengineering, economics and others. Therefore, the theory of fractional differential equations is an area intensively developed during the last decades. The existence and multiplicity of solutions for fractional differential equations have been established by the tools of nonlinear analysis, such as fixed point theory [1], topological degree theory [2], comparison methods [3], and so on.

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It should be noted that critical point theory and variational methods serve as effective tools in the study of integer-order differential equations. The underlying idea in this approach rests on finding critical points for a suitable energy functional defined on an appropriate function space. During the last three decades, the critical point theory has developed into a wonderful tool for investigated the existence criteria for the solutions of differential equations with variational structures, for example, see [4,5] and the references cited therein.
Motivated by the classical works in [4,5], for the first time, the authors [6] showed that critical point theory and variational methods are an effective approach to tackle the existence of solutions for the following fractional boundary value problem

$$\begin{cases} {}_tD_T^{\alpha}({}_0D_t^{\alpha}u)(t) = \nabla W(t,u(t)),\ t\in[0,T] \\ u(0)=u(T), \end{cases}$$

and obtained the existence of at least one nontrivial solution. Inspired by this work, Torres [7] consider the fractional Hamiltonian system $(\mathscr{FHS})$. Assuming that the functional L is positive definite symmetric matrix for all $t\in\mathbb{R}$ and coercive, and the potential $W(t,x)$ satisfies some suitable conditions, he showed that $(\mathscr{FHS})$ possesses at least one nontrivial solution using the Symmetric Mountain Pass Theorem. Since then, the existence and multiplicity of solutions for problem $(\mathscr{FHS})$ via critical point theory have been investigated in many papers, see [7]-[22] and the references cited therein. In all these papers, $W(t,x)$ was always required to satisfy some kinds of growth conditions at infinity with respect to x, such as superquadratic, subquadratic or asymptotically quadratic growth. Besides, the function L is required to satisfy one of the following conditions:
(1.1) There exists an $l\in C(\mathbb{R},\mathbb{R}_+)$ such that $l(t)\longrightarrow+\infty$ as $|t|\longrightarrow\infty$ and

$$L(t)x\cdot x\geq l(t)\,|x|^2,\ \forall(t,x)\in\mathbb{R}\times\mathbb{R}^N.$$

Here and in the following, "." denotes the usual inner product of $\mathbb{R}^N$ and $|.|$ is the induced norm.
(1.2) There are constants $0<\tau_1<\tau_2<+\infty$ such that

$$\tau_1\,|x|^2\leq L(t)x\cdot x\leq\tau_2\,|x|^2,\ \forall(t,x)\in\mathbb{R}\times\mathbb{R}^N.$$

(1.3) (i) There exists an $l\in C(\mathbb{R},\mathbb{R})$ such that

$$\inf_{t\in\mathbb{R}} l(t)>0 \ and\ L(t)x\cdot x\geq l(t)\,|x|^2,\ \forall(t,x)\in\mathbb{R}\times\mathbb{R}^N,$$

(ii) There exists a constant $r_0>0$ such that

$$\lim_{|s|\longrightarrow\infty} meas\Big(\{t\in]s-r_0,s+r_0[: L(t)<bI_N\}\Big)=0,\ \forall b>0,$$

where *meas* denotes the Lebesgue's measure on $\mathbb{R}$. The conditions (1.1), (1.2) and (1.3) guarantee the compactness of the Sobolev embedding.
In the present paper, we will study the existence of infinity many solutions for $(\mathscr{FHS})$ in the case where $W(t,x)$ is still only locally defined near the origin with respect to x and L satisfies some weaker conditions than $(1.1)-(1.3)$. More precisely, we make the following assumptions:
(L) There exists a constant $l_0>0$ such that

$$l(t)=\min_{|\xi|=1} L(t)\xi\cdot\xi\geq l_0,\ \forall t\in\mathbb{R}.$$

There exists a constant $\delta>0$ such that $W\in C(\mathbb{R}\times B_\delta(0),\mathbb{R})$ is continuously differentiable in the second variable with continuous derivative $\nabla W(t,x)=\frac{\partial W}{\partial x}(t,x)$, where $B_\delta(0)$ is the open ball in $\mathbb{R}^N$ centered at 0 with radius δ, and satisfies
(W_1) $W(t,x)$ is even in x and $W(t,0)=0,\forall t\in\mathbb{R}$;
(W_2) There exist constants $\nu\in]1,2[$, $\beta_1\in[1,2]$, $\beta_2\in[1,\frac{2}{2-\nu}]$ and nonnegative functions $a\in L^{\beta_1}(\mathbb{R},\mathbb{R}^+)$, $b\in L^{\beta_2}(\mathbb{R},\mathbb{R}^+)$ such that

$$|\nabla W(t,x)|\leq a(t)+b(t)\,|x|^{\nu-1},\ \forall(t,x)\in\mathbb{R}\times B_\delta(0);$$

(W_3)

$$\lim_{|x|\longrightarrow0}\frac{W(t,x)}{|x|^2}=+\infty,\ uniformly\ in\ t\in\mathbb{R}.$$

Our main result reads as follows.

Theorem 1. *Suppose that (L) and $(W_1)-(W_3)$ are satisfied. Then the fractional Hamiltonian system $(\mathscr{FHS})$ possesses a sequence of solutions (u_k) such that*

$$\max_{t\in\mathbb{R}}|u_k(t)|\longrightarrow0\ as\ k\longrightarrow\infty.$$

2 Preliminaries

In this Section, for the reader's convenience, first we will recall some facts about the fractional calculus on the whole real axis. On the other hand, we will give some preliminary lemmas for using in the sequel.

2.1 Liouville-Weyl Fractional Calculus

The Liouville-Weyl fractional integrals of order $0 < \alpha < 1$ on the whole axis $\mathbb{R}$ are defined as (see [23,24,25])

$$_{-\infty}I_t^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{t}(t-x)^{\alpha-1}u(x)dx, \tag{2}$$

and

$$_{t}I_{\infty}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_{t}^{\infty}(x-t)^{\alpha-1}u(x)dx. \tag{3}$$

The Liouville-Weyl fractional derivatives of order $0 < \alpha < 1$ on the whole axis $\mathbb{R}$ are defined as the left-inverse operators of the corresponding Liouville-Weyl fractional integrals (see [23,24,25])

$$_{-\infty}D_t^{\alpha}u(t) = \frac{d}{dt}(_{-\infty}I_t^{1-\alpha}u)(t), \tag{4}$$

and

$$_{t}D_{\infty}^{\alpha}u(t) = -\frac{d}{dt}(_{t}I_{\infty}^{1-\alpha}u)(t). \tag{5}$$

The definitions of (4) and (5) may be written in an alternative form as follows

$$_{-\infty}D_t^{\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)}\int_{0}^{\infty}\frac{u(t)-u(t-x)}{x^{\alpha+1}}dx, \tag{6}$$

and

$$_{t}D_{\infty}^{\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)}\int_{0}^{\infty}\frac{u(t)-u(t+x)}{x^{\alpha+1}}dx. \tag{7}$$

We establish the Fourier transform properties of the fractional integral and fractional differential operators. Recall that the Fourier transform $\widehat{u}$ of u is defined by

$$\widehat{u}(s) = \int_{-\infty}^{\infty}e^{-ist}u(t)dt.$$

Let u be defined on $\mathbb{R}$. Then the Fourier transform of the Liouville-Weyl integrals and differential operators satisfies (see [4,12])

$$\widehat{_{-\infty}I_t^{\alpha}u}(s) = (is)^{-\alpha}\widehat{u}(s), \tag{8}$$

$$\widehat{_{t}I_{\infty}^{\alpha}u}(s) = (-is)^{-\alpha}\widehat{u}(s), \tag{9}$$

$$\widehat{_{-\infty}D_t^{\alpha}u}(s) = (is)^{\alpha}\widehat{u}(s), \tag{10}$$

$$\widehat{_{t}D_{\infty}^{\alpha}u}(s) = (-is)^{\alpha}\widehat{u}(s). \tag{11}$$

Next, we present some properties for Liouville-Weyl fractional integrals and derivatives on the real axis, which were proved in [12].
Denote by $L^p(\mathbb{R})$ $(1 \le p < \infty)$, the Banach spaces of functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the norms

$$\|u\|_{L^p} = (\int_{\mathbb{R}}|u(t)|^p dt)^{\frac{1}{p}},$$

and $L^{\infty}(\mathbb{R})$ the Banach space of essentially bounded functions from $\mathbb{R}$ into $\mathbb{R}^N$ equipped with the norm

$$\|u\|_{\infty} = esssup\{|u(t)|\,/t \in \mathbb{R}\}.$$

Proposition 1. *1) Let $p,q \in [1,\infty]$, $\alpha > 0$. The operators ${}_{-\infty}I_t^{\alpha}$ and ${}_tI_{\infty}^{\alpha}$ are bounded from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$ if and only if*

$$0 < \alpha < 1,\ 1 < p < \frac{1}{\alpha},\ q = \frac{p}{1-\alpha p},$$

2) If $\alpha > 0$, for "sufficiently good" function u, the relations

$$({}_{-\infty}D_t^{\alpha}({}_{-\infty}I_t^{\alpha}u))(t) = u(t),\ ({}_tD_{\infty}^{\alpha}({}_tI_{\infty}^{\alpha}u))(t) = u(t) \tag{12}$$

are true. In particular, these relations hold for $u \in L^1(\mathbb{R})$,
3) Let $\alpha,\beta > 0$ and $p \geq 1$ be such that $\alpha+\beta = \frac{1}{p}$. If $u \in L^p(\mathbb{R})$, then

$$({}_{-\infty}I_t^{\beta}({}_{-\infty}I_t^{\alpha}u))(t) = {}_{-\infty}I_t^{\alpha+\beta}u(t),\ ({}_tI_{\infty}^{\beta}({}_tI_{\infty}^{\alpha}u))(t) = {}_tI_{\infty}^{\alpha+\beta}u(t), \tag{13}$$

4) If $\alpha > \beta > 0$, then

$$({}_{-\infty}D_t^{\beta}({}_{-\infty}I_t^{\alpha}u))(t) = {}_{-\infty}I_t^{\alpha-\beta}u(t),\ ({}_tD_{\infty}^{\beta}({}_tI_{\infty}^{\alpha}))(t) = {}_tI_{\infty}^{\alpha-\beta}u(t). \tag{14}$$

Proposition 2. *If $\alpha > 0$, then the relations*

$$\int_{\mathbb{R}} \varphi(t)\cdot({}_{-\infty}I_t^{\alpha}\psi)(t)dt = \int_{\mathbb{R}} ({}_tI_{\infty}^{\alpha}\varphi)(t)\cdot\psi(t)dt, \tag{15}$$

$$\int_{\mathbb{R}} u(t)\cdot({}_{-\infty}D_t^{\alpha}v)(t)dt = \int_{\mathbb{R}} ({}_tD_{\infty}^{\alpha}u)(t)\cdot v(t)dt, \tag{16}$$

are valid for "sufficiently good" functions φ,ψ,u,v. In particular, (15) holds for functions $\varphi \in L^p(\mathbb{R})$ and $\psi \in L^q(\mathbb{R})$, while (16) holds for $u \in {}_tI_{\infty}^{\alpha}(L^p(\mathbb{R}))$ and $v \in {}_{-\infty}I_t^{\alpha}(L^q(\mathbb{R}))$ provided that $p > 1$, $q > 1$ and $\frac{1}{p}+\frac{1}{q} = 1+\alpha$, where

$${}_tI_{\infty}^{\alpha}(L^p(\mathbb{R})) = \{u/\exists\varphi \in L^p(\mathbb{R}), u = {}_tI_{\infty}^{\alpha}\varphi\},$$

similarly, ${}_{-\infty}I_t^{\alpha}(L^q(\mathbb{R}))$ can be defined.

2.2 Fractional Derivative Spaces

In order to establish the variational structure which enables us to reduce the existence of solutions of $(\mathscr{FHS})$ to find critical points of the corresponding functional, it is necessary to construct the appropriate functional spaces.
For $\alpha > 0$, define the semi-norm

$$|u|_{I_{-\infty}^{\alpha}} = \|{}_{-\infty}D_t^{\alpha}u\|_{L^2}$$

and the norm

$$\|u\|_{I_{-\infty}^{\alpha}} = (\|u\|_{L^2} + |u|^2_{I_{-\infty}^{\alpha}})^{\frac{1}{2}},$$

and let

$$I_{-\infty}^{\alpha} = \overline{C_0^{\infty}(\mathbb{R})}^{\|\cdot\|_{I_{-\infty}^{\alpha}}}$$

where $C_0^{\infty}(\mathbb{R})$ denotes the space of infinitely differentiable functions from $\mathbb{R}$ into $\mathbb{R}^N$ with vanishing property at infinity.
Now, we can define the fractional Sobolev space $H^{\alpha}(\mathbb{R})$ in terms of the Fourier transform. Choose $0 < \alpha < 1$, define the semi-norm

$$|u|_{\alpha} = \left\||s|^{\alpha}\widehat{u}\right\|_{L^2}$$

and the norm

$$\|u\|_{\alpha} = (\|u\|_{L^2} + |u|^2_{\alpha})^{\frac{1}{2}},$$

and let

$$H^{\alpha}(\mathbb{R}) = \overline{C_0^{\infty}(\mathbb{R})}^{\|\cdot\|_{\alpha}}.$$

Moreover, we note that a function $u \in L^2(\mathbb{R})$ belongs to $I_{-\infty}^{\alpha}$ if and only if

$$|s|^{\alpha}\widehat{u} \in L^2(\mathbb{R}).$$

Especially, we have

$$|u|_{I^{\alpha}_{-\infty}} = \left\| |s|^{\alpha} \widehat{u} \right\|_{L^2}.$$

Therefore, $I^{\alpha}_{-\infty}$ and $H^{\alpha}(\mathbb{R})$ are equivalent with equivalent semi-norms and norms. Analogous to $I^{\alpha}_{-\infty}$, we introduce I^{α}_{∞}. Define the semi-norm

$$|u|_{I^{\alpha}_{\infty}} = \|{}_tD^{\alpha}_{\infty}u\|_{L^2}$$

and the norm

$$\|u\|_{I^{\alpha}_{\infty}} = (\|u\|_{L^2} + |u|^2_{I^{\alpha}_{\infty}})^{\frac{1}{2}},$$

and let

$$I^{\alpha}_{\infty} = \overline{C^{\infty}_0(\mathbb{R})}^{\|.\||_{I^{\alpha}_{\infty}}}$$

Then $I^{\alpha}_{-\infty}$ and I^{α}_{∞} are equivalent with equivalent semi-norms and norms. Let $C(\mathbb{R})$ denotes the space of continuous functions from $\mathbb{R}$ into $\mathbb{R}^N$. Then we obtain the following Sobolev lemma.

Lemma 1([7], Theorem 2.1). *If $\alpha > \frac{1}{2}$, then $H^{\alpha}(\mathbb{R}) \subset C(\mathbb{R})$, and there exists a constant $C = C_{\alpha}$ such that*

$$\|u\|_{L^{\infty}} = \sup_{t\in\mathbb{R}} |u(t)| \leq C_{\alpha} \|u\|_{\alpha}, \forall u \in H^{\alpha}(\mathbb{R}). \tag{17}$$

Remark 1. From Lemma 1, we know that if $u \in H^{\alpha}(\mathbb{R})$ with $\frac{1}{2} < \alpha < 1$, then $u \in L^p(\mathbb{R})$ for all $p \in [2,\infty]$, because

$$\int_{\mathbb{R}} |u(t)|^p \, dt \leq \|u\|^{p-2}_{\infty} \|u\|^2_{L^2}. \tag{18}$$

In what follows, we introduce the functional space in which we will construct the variational framework of $(\mathscr{F}\mathscr{H}\mathscr{S})$. Let

$$X^{\alpha} = \left\{ u \in H^{\alpha}(\mathbb{R}) / \int_{\mathbb{R}} [|{}_{-\infty}D^{\alpha}_t u(t)|^2 + L(t)u(t)\cdot u(t)]dt < \infty \right\}$$

then X^{α} is a reflexive and separable Hilbert space with the inner product

$$< u, v >_{X^{\alpha}} = \int_{\mathbb{R}} [{}_{-\infty}D^{\alpha}_t u(t) \cdot {}_{-\infty}D^{\alpha}_t v(t) + L(t)u(t)\cdot v(t)]dt$$

and the corresponding norm $\|u\|^2_{X^{\alpha}} = < u, u >_{X^{\alpha}}$. Evidently, X^{α} is continuously embedded into $H^{\alpha}(\mathbb{R})$. Hence X^{α} is continuously embedded in $L^p(\mathbb{R})$ for all $p \in [2,\infty]$ and compactly embedded in $L^p_{loc}(\mathbb{R})$ for all $p \in [2,\infty]$, where $L^p_{loc}(\mathbb{R})$ denotes the space of measurable functions u from $\mathbb{R}$ into $\mathbb{R}^N$ such that for all compact $K \subset \mathbb{R}$, $\int_K |u(t)|^p \, dt < \infty$. Consequently, for all $p \in [2,\infty]$, there exists a constant $\eta_p > O$ such that

$$\|u\|_{L^p} \leq \eta_p \|u\|_{X^{\alpha}}, \; \forall u \in X^{\alpha}. \tag{19}$$

To prove our main result via critical point theory, we shall use the following symmetric mountain pass theorem developed by Kajikiya [26]. We will first recall the notion of genus.
Let X be a Banach space and let A be a subset of X. A is said to be symmetric if $u \in A$ implies $-u \in A$. For a closed symmetric set A which does not contain the origin, we define the genus $\gamma(A)$ of A by the smallest integer k for which there exists an odd continuous mapping from $\mathbb{R}$ to $\mathbb{R}^k \setminus \{0\}$. If such a k does not exist, we define $\gamma(A) = +\infty$. Moreover, we set $\gamma(\phi) = 0$. Let

$$\Gamma_k = \{A \subset E / A \textit{ is a closed symmetric subset}, \; 0 \notin A, \; \gamma(A) \geq k\}.$$

The properties of genus used in the proof of our main result are summarized as follows.

Lemma 2([26], Proposition 7.5.). *Let A and B be closed symmetric subsets of E that do not contain the origin. Then the following hold.*
(i) If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.
(ii) The $N-$dimensional sphere S^N has a genus of $N+1$ by the Borsuk-Ulam theorem.

Lemma 3.[26] *Let X be an infinite-dimensional Banach space and $\Phi \in C^1(X,\mathbb{R})$ an even functional with $\Phi(0) = 0$. Suppose that Φ satisfies*
(1) Φ is bounded from below and satisfies the $(PS)-$condition;
(2) For each $k \in \mathbb{N}$, there exists $A_k \subset \Gamma_k$ such that

$$\sup_{u \in A_k} \Phi(u) < 0.$$

Then (a) or (b) below hold
(a) There exists a critical point sequence (u_k) such that $\Phi(u_k) < 0$ and $\lim_{k \longrightarrow \infty} u_k = 0$;
(b) There exist two critical point sequences (u_k) and (v_k) such that $\Phi(u_k) = 0$, $u_k \neq 0$ and $\lim_{k \longrightarrow \infty} u_k = 0$, $\Phi(v_k) < 0$, $\lim_{k \longrightarrow \infty} \Phi(v_k) = 0$ and (v_k) converges to a non-zero limit.

3 Proof of Theorem 1.1.

In order to prove our main result via critical point theory, we need to modify $W(t,x)$ for x outside a neighborhood of the origin to get $\tilde{W}(t,x)$ as follows. Choose a constant $r \in]0, \frac{\delta}{2}[$ and define a cut-off function $\chi \in C^1(\mathbb{R}^+,\mathbb{R}^+)$ such that $\chi(s) = 1$ for $0 \leq s \leq r$, $\chi(s) = 0$ for $s \geq 2r$ and $-\frac{2}{r} \leq \chi'(s) < 0$ for $r < s < 2r$. Let

$$\widetilde{W}(t,x) = \chi(|x|)W(t,x), \ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N. \tag{20}$$

Combining (W_1), (W_2) and the definition of χ, we obtain

$$\left|\widetilde{W}(t,x)\right| \leq a(t)\,|x| + b(t)\,|x|^{\nu}, \ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{21}$$

and

$$\left|\nabla\widetilde{W}(t,x)\right| \leq 5\left(a(t) + b(t)\,|x|^{\nu-1}\right), \ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N. \tag{22}$$

Now, we introduce the following modified system

$$(\widetilde{\mathscr{F}\mathscr{H}\mathscr{S}}) \qquad \begin{cases} {}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_t u)(t) + L(t)u(t) = \nabla\widetilde{W}(t,u(t)), \ t \in \mathbb{R} \\ u \in H^{\alpha}(\mathbb{R}), \end{cases}$$

and define the variational functional Φ associated with $(\widetilde{\mathscr{F}\mathscr{H}\mathscr{S}})$ by

$$\begin{aligned} \Phi(u) &= \frac{1}{2}\int_{\mathbb{R}} [|{}_{-\infty}D^{\alpha}_t u(t)|^2 + L(t)u(t)\cdot u(t)]dt - \int_{\mathbb{R}} \widetilde{W}(t,u(t))dt \\ &= \frac{1}{2}\|u\|^2 - \varphi(u) \end{aligned} \tag{23}$$

where $\varphi(u) = \int_{\mathbb{R}} \widetilde{W}(t,u(t))dt$.

Lemma 4. *Assume that (L), (W_1) and (W_2) are satisfied. Then $\varphi \in C^1(X^{\alpha},\mathbb{R})$ and $\varphi' : X^{\alpha} \longrightarrow (X^{\alpha})'$ is compact, and hence $\Phi \in C^1(X^{\alpha},\mathbb{R})$. Moreover*

$$\varphi'(u)v = \int_{\mathbb{R}} \nabla\widetilde{W}(t,u(t))\cdot v(t)dt \tag{24}$$

and

$$\Phi'(u)v = < u,v > - \int_{\mathbb{R}} \nabla\widetilde{W}(t,u(t))\cdot v(t)dt \tag{25}$$

for all $u,v \in X^{\alpha}$, and nontrivial critical points of Φ on X^{α} are homoclinic solutions of $(\widetilde{\mathscr{F}\mathscr{H}\mathscr{S}})$.

*Proof.*In the following, we will note

$$\overline{\beta}_1 = \frac{\beta_1}{\beta_1 - 1}, \ \overline{\beta}_2 = \frac{\nu\beta_2}{\beta_2 - 1}, (\overline{\beta}_1 = \infty, \overline{\beta}_2 = \infty, \ if \ \beta_1 = 1 \ or \ \beta_2 = 1). \tag{26}$$

It is easy to see that $\overline{\beta}_1, \overline{\beta}_2 \in [2,\infty]$. By (19), (21) and Hölder's inequality, we have for $u \in X^\alpha$

$$\begin{aligned} \int_{\mathbb{R}} \left|\widetilde{W}(t,u(t))\right| dt &\leq \int_{\mathbb{R}} a(t)\,|u(t)|\,dt + \int_{\mathbb{R}} b(t)\,|u(t)|^\nu\,dt \\ &\leq \|a\|_{L^{\beta_1}} \|u\|_{L^{\overline{\beta}_1}} + \|b\|_{L^{\beta_2}} \|u\|^\nu_{L^{\overline{\beta}_2}} \\ &\leq \eta_{\overline{\beta}_1} \|a\|_{L^{\beta_1}} \|u\| + \eta^\nu_{\overline{\beta}_2} \|b\|_{L^{\beta_2}} \|u\|^\nu < \infty, \end{aligned} \tag{27}$$

which implies that φ and Φ are both well defined. Now, we prove that $\varphi \in C^1(X^\alpha,\mathbb{R})$ and $\varphi' : X^\alpha \longrightarrow (X^\alpha)'$ is compact. By (22), for any $u,v \in X^\alpha$ and $s \in [0,1]$, there holds

$$\begin{aligned} \left|\nabla\widetilde{W}(t,u+sv)\cdot v\right| &\leq 5\left[a(t) + b(t)\,|u+sv|^{\nu-1}\right]|v| \\ &\leq 5\left[a(t) + b(t)\left(|u|^{\nu-1} + |v|^{\nu-1}\right)\right]|v| \\ &\leq 5\left[a(t) + b(t)\left(|u|^{\nu-1}|v| + |v|^\nu\right)\right]|v|. \end{aligned}$$

Hence, by the Mean Value Theorem and Lebesgue's Dominated Convergence Theorem, we get for all $u,v \in X^\alpha$

$$\begin{aligned} \lim_{s\longrightarrow 0} \frac{\varphi(u+sv) - \varphi(u)}{s} &= \lim_{s\longrightarrow 0} \int_{\mathbb{R}}\int_0^1 \nabla\widetilde{W}(t,u+rsv)\cdot v\,dr\,dt \\ &= \int_{\mathbb{R}} \nabla\widetilde{W}(t,u)\cdot v\,dt = \mathscr{L}(u)v. \end{aligned}$$

Moreover, it follows from (19), (22) and Hölder's inequality that

$$\begin{aligned} |\mathscr{L}(u)v| &\leq \int_{\mathbb{R}} \left|\nabla\widetilde{W}(t,u)\right| |v|\,dt \\ &\leq 5\left[\int_{\mathbb{R}} a(t)\,|v|\,dt + \int_{\mathbb{R}} a(t)\,|u|^{\nu-1}|v|\,dt\right. \\ &\leq 5\left[\|a\|_{L^{\beta_1}} \|v\|_{L^{\overline{\beta}_1}} + \|b\|_{L^{\beta_2}} \|u\|^{\nu-1}_{L^{\overline{\beta}_2}} \|v\|_{L^{\overline{\beta}_2}}\right] \\ &\leq 5\left[\eta_{\overline{\beta}_1}\|a\|_{L^{\beta_1}} + \eta^\nu_{\overline{\beta}_2}\|b\|_{L^{\beta_2}}\|u\|^{\nu-1}\right]\|v\|,\ \forall v \in X^\alpha, \end{aligned} \tag{28}$$

which means that $\mathscr{L}(u)$ is bounded. This means that φ is G$\widehat{a}$teaux-differentiable on X^α and its G$\widehat{a}$teaux-derivative at u is $\mathscr{L}(u)$. Let $u_n \rightharpoonup u$ in X^α as $n \longrightarrow \infty$, then (u_n) is bounded in X^α and

$$u_n \longrightarrow u \ in \ L^\infty_{loc}(\mathbb{R}) \ as \ n \longrightarrow \infty. \tag{29}$$

Therefore, there exists a constant $c_1 > 0$ such that

$$\|u_n\|^{\nu-1} + \|u\|^{\nu-1} \leq c_1,\ \forall n \in \mathbb{N}. \tag{30}$$

By (W_2), for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$\left(\int_{|t|\geq R_\varepsilon} (a(t))^{\beta_1} dt\right)^{\frac{1}{\beta_1}} \leq \frac{\varepsilon}{40\eta_{\overline{\beta}_1}} \tag{31}$$

$$\left(\int_{|t|\geq R_\varepsilon} (b(t))^{\beta_2} dt\right)^{\frac{1}{\beta_2}} \leq \frac{\varepsilon}{20c_1\eta^\nu_{\overline{\beta}_2}}. \tag{32}$$

Combining (22) with (30)-(32), the Hölder's inequality implies

$$\begin{aligned}
&\int_{|t|\geq R_\varepsilon}\left|\nabla\widetilde{W}(t,u_n)-\nabla\widetilde{W}(t,u)\right||v|\,dt\\
&\leq 5\int_{|t|\geq R_\varepsilon}\left[2a(t)+b(t)(|u_n|^{\nu-1}+|u|^{\nu-1})\right]|v|\,dt\\
&\leq 10\Big(\int_{|t|\geq R_\varepsilon}(a(t))^{\beta_1}dt\Big)^{\frac{1}{\beta_1}}\|v\|_{L^{\overline{\beta}_1}}\\
&+5\Big(\int_{|t|\geq R_\varepsilon}(b(t))^{\beta_2}dt\Big)^{\frac{1}{\beta_2}}\Big(\|u_n\|^{\nu-1}_{L^{\overline{\beta}_2}}+\|u\|^{\nu-1}_{L^{\overline{\beta}_2}}\Big)\|v\|_{L^{\overline{\beta}_2}}\\
&\leq 10\eta_{\overline{\beta}_1}\Big(\int_{|t|\geq R_\varepsilon}(a(t))^{\beta_1}dt\Big)^{\frac{1}{\beta_1}}\\
&+5\eta^{\nu}_{\overline{\beta}_2}\Big(\int_{|t|\geq R_\varepsilon}(b(t))^{\beta_2}dt\Big)^{\frac{1}{\beta_2}}\Big(\|u_n\|^{\nu-1}+\|u\|^{\nu-1}\Big)\\
&\leq\frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\frac{\varepsilon}{2},\ \forall n\in\mathbb{N},\ and\ \|v\|=1.
\end{aligned}\tag{33}$$

For the R_ε given above, by (19), (29) and the continuity of $\nabla\widetilde{W}$, there exists $n_0\in\mathbb{N}$ such that for all $n\geq n_0$ and $\|v\|=1$

$$\int_{-R_\varepsilon}^{R_\varepsilon}\left|\nabla\widetilde{W}(t,u_n)-\nabla\widetilde{W}(t,u)\right||v|\,dt\leq\eta_\infty\int_{-R_\varepsilon}^{R_\varepsilon}\left|\nabla\widetilde{W}(t,u_n)-\nabla\widetilde{W}(t,u)\right|dt<\frac{\varepsilon}{2}.\tag{34}$$

Combining (33) with (34), we get

$$\begin{aligned}
\|\mathscr{L}(u_n)-\mathscr{L}(u)\|_{(X^\alpha)'}&=\sup_{\|v\|=1}|(\mathscr{L}(u_n)-\mathscr{L}(u))v|\\
&=\sup_{\|v\|=1}\left|\int_{\mathbb{R}}(\nabla\widetilde{W}(t,u_n)-\nabla\widetilde{W}(t,u)).v\,dt\right|\\
&\leq\sup_{\|v\|=1}\int_{-R_\varepsilon}^{R_\varepsilon}\left|\nabla\widetilde{W}(t,u_n)-\nabla\widetilde{W}(t,u)\right||v|\,dt\\
&+\sup_{\|v\|=1}\int_{|t|\geq R_\varepsilon}\left|\nabla\widetilde{W}(t,u_n)-\nabla\widetilde{W}(t,u)\right||v|\,dt\\
&<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\ for\ all\ n\geq n_0.
\end{aligned}$$

This implies that $\mathscr{L}$ is continuous. Thus $\varphi\in C^1(X^\alpha,\mathbb{R})$ and (24) holds with $\varphi'=\mathscr{L}$. This together with the reflexivity of the Hilbert space X^α implies that φ' is compact. In addition, due to the form of Φ, we see that $\Phi\in C^1(X^\alpha,\mathbb{R})$ and (25) also holds. The proof of Lemma 3.1 is completed.

Lemma 5. *Assume that (L), (W_1) and (W_2) hold. Then Φ is bounded from below and satisfies the $(PS)-$condition.*

Proof. Firstly, we prove that Φ is bounded from below. By (27), it follows

$$\begin{aligned}
\Phi(u)&\geq\frac{1}{2}\|u\|^2-\int_{\mathbb{R}}\left|\nabla\widetilde{W}(t,u)\right|dt\\
&\geq\frac{1}{2}\|u\|^2-\eta_{\overline{\beta}_1}\|a\|_{L^{\beta_1}}\|u\|-\eta^{\nu}_{\overline{\beta}_2}\|b\|_{L^{\beta_2}}\|u\|^{\nu}.
\end{aligned}\tag{35}$$

Since $\nu<2$, it follows that Φ is bounded from below. Next, we show that Φ satisfies the $(PS)-$condition. Let (u_n) be a $(PS)-$sequence, that is

$$|\Phi(u_n)|\leq M,\ \forall n\in\mathbb{N},\ \Phi'(u_n)\longrightarrow 0\ as\ n\longrightarrow\infty,\tag{36}$$

for some constant $M>0$. By (35) and (36), it holds

$$M\geq\frac{1}{2}\|u_n\|^2-\eta_{\overline{\beta}_1}\|a\|_{L^{\beta_1}}\|u_n\|-\eta^{\nu}_{\overline{\beta}_2}\|b\|_{L^{\beta_2}}\|u_n\|^{\nu}$$

which implies that (u_n) is bounded in X^α since $\nu < 2$. Hence, up to a subsequence if necessary, we can assume that

$$u_n \rightharpoonup u \text{ in } X^\alpha \text{ as } n \longrightarrow \infty \tag{37}$$

for some $u \in X^\alpha$. By virtue of the Riez Representation Theorem, $\varphi : X^\alpha \longrightarrow (X^\alpha)'$ and $\Phi' : X^\alpha \longrightarrow (X^\alpha)'$ can be viewed as $\varphi : X^\alpha \longrightarrow X^\alpha$ and $\Phi' : X^\alpha \longrightarrow X^\alpha$ respectively. This together with (24) and (25) yields

$$u_n = \Phi'(u_n) + \varphi'(u_n),\ \forall n \in \mathbb{N}. \tag{38}$$

By Lemma 3.1, φ' is compact. Combining this with (36)-(38), the right side of (38) converges strongly in X^α and hence $u_n \longrightarrow u$ in X^α as $n \longrightarrow \infty$. Then Φ satisfies the $(PS)-$condition. The proof of Lemma 3.2 is completed.

Lemma 6. *Suppose that (L) and (W_3) hold. Then for each $k \in \mathbb{N}$, there exists an $A_k \subset X^\alpha$ with genus $\gamma(A_k) \geq k$ such that* $\sup_{u \in A_k} \Phi(u) < 0$.

Proof. Let (e_n) an orthonormal basis of X^α. Then for each $k \in \mathbb{N}$, let

$$X_k = \oplus_{m=1}^{k} span\{e_m\}.$$

Since X_k is finite dimensional, there exists a constant $\tau_k > 0$ such that

$$\|u\| \leq \tau_k \|u\|_{L^2},\ \forall u \in X_k. \tag{39}$$

By (W_3), there exists a constant $R_k > 0$ such that

$$\widetilde{W}(t,x) \geq \tau_k^2 |x|^2,\ \forall t \in \mathbb{R},\ |x| \leq R_k. \tag{40}$$

Let $u \in X^\alpha$ such that $\|u\| \leq \frac{R_k}{\eta_\infty}$. By (19), we know that $|u(t)| \leq R_k$ for all $t \in \mathbb{R}$, thus by (40), it holds

$$\widetilde{W}(t,u(t)) \geq \tau_k^2 |u(t)|^2,\ \forall t \in \mathbb{R}. \tag{41}$$

Therefore, by (39) and (41), for all $u \in X_k \setminus \{0\}$ with $0 < \|u\| = \frac{\min\{r,R_k\}}{\eta_\infty} = \rho_k$, we have

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} \nabla\widetilde{W}(t,u)dt \\
&\leq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} \tau_k^2 |u(t)|^2 dt \\
&\leq \frac{1}{2}\|u\|^2 - \|u\|^2 \\
&= -\frac{1}{2}\rho_k^2,
\end{aligned}$$

which implies

$$\{u \in X_k \setminus \{0\} / \|u\| = \rho_k\} \subset A_k = \left\{u \in X_k / \Phi(u) \leq -\frac{1}{2}\rho_k^2\right\}. \tag{42}$$

Thus, by Lemma 2.1, (42) implies

$$\gamma(A_k) \geq \gamma\Big(\{u \in X_k \setminus \{0\} / \|u\| = \rho_k\}\Big) \geq k$$

hence, by the definition of Γ_k, we have $A_k \subset \Gamma_k$. Moreover, the definition of A_k implies

$$\sup_{u \in A_k} \Phi(u) \leq -\frac{1}{2}\rho_k^2 < 0.$$

The proof of Lemma 3.3 is completed.

Consequently, Φ possesses a sequence of nontrivial critical points (u_k) satisfying $u_k \longrightarrow 0$ in X^α as $k \longrightarrow \infty$. By virtue of Lemma 3.1, (u_k) is a sequence of solutions of $(\widetilde{\mathcal{FHS}})$. By (19), it follows that $\max_{t\in\mathbb{R}} |u_k(t)| \longrightarrow 0$ as $k \longrightarrow \infty$. Therefore, there exists a positive constant $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, u_k is a solution of $(\mathcal{FHS})$. This ends the proof of Theorem 1.1.

4 Conclusion

In the present work, we established the existence of infinitely many solutions for a class of fractional Hamiltonian systems when the potential is only locally defined near the origin, via critical point theory and variational methods.

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