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On Lacunary Statistically Convergent Triple Sequences in Probabilistic Normed Space

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Abstract: In this paper we study the concept of lacunary statistical convergent triple sequences in probabilistic normed spaces and prove some basic properties.

Keywords: Double Statistical convergence, Triple sequence, t norm, Probabilictic normed space

1 Introduction and Background

An interesting and important generalization of the notion of metric space was introduced by Menger [1] under the name of statistical metric, which is now called probabilistic metric space. The notion of a probabilistic metric space corresponds to the situations when we do not know exactly the distance between two points, we know only probabilities of possible values of this distance. The theory of probabilistic metric space was developed by numerous authors, as it can be realized upon consulting the list of references in [2], as well as those in [3,4]. Probabilistic normed spaces (briefly, PN-spaces) are linear spaces in which the norm of each vector is an appropriate probability distribution function rather than a number. Such spaces were introduced by Serstnev in 1963 [5]. In [6], Alsina et al. gave a new definition of PN-spaces which includes Serstnev's a special case and leads naturally to the identification of the principle class of PN-spaces, the Menger spaces. An important family of probabilistic metric spaces are probabilistic normed spaces. The theory of probabilistic normed spaces is important as a generalization of deterministic results of linear normed space.

It seems therefore reasonable to think if the concept of statistical convergence can be extended to probabilistic normed spaces and in that case enquire how the basic properties are affected. But basic properties do not hold on probabilistic normed spaces. The problem is that the triangle function in such spaces.

In this paper we study the concept of lacunary statistical convergent triple sequences on probabilistic normed spaces. Since the study of convergence in PN-spaces is fundamental to probabilistic functional analysis, we feel that the concept of lacunary statistical convergent triple sequences in a PN-space would provide a more general framework for the subject.

2 Preliminaries

Now we recall some notations and definitions used in this paper.

For the following concepts, we refer to Menger [1] and Schweizer-Sklar [4,5,6].

Definition 2.1.[1] A function $f:\mathbb{R}\rightarrow\mathbb{R}_o^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t\in\mathbb{R}} f(t)=0$ and $\sup_{t\in\mathbb{R}} f(t)=1$. We will denote the set of all distribution functions by D.

Definition 2.2.[1] A triangular norm, briefly t-norm, is a binary operation on $[0,1]$ which is continuous, commutative, associative, non-decreasing and has 1 as neutral element, that is, it is the continuous mapping $\circledast:[0,1]\times[0,1]\rightarrow[0,1]$ such that for all $a,b,c\in[0,1]$:

(1) $a\circledast 1=a$,
(2) $a\circledast b=b\circledast a$,
(3) $c\circledast d\geq a\circledast b$ if $c\geq a$ and $d\geq b$,
(4) $(a\circledast b)\circledast c=a\circledast(b\circledast c)$.

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Example 2.1. The $\circledast$ operations $a \circledast b = \max\{a+b-1,0\}$, $a \circledast b = a.b$ and $a \circledast b = \min\{a,b\}$ on $[0,1]$ are t-norms.

Definition 2.3.[3,4] A triple $(X,N,\circledast)$ is called a probabilistic normed space or shortly PN-space if X is a real vector space, N is a mapping from X into D (for $x \in X$, the distribution function $N(x)$ is denoted by N_x and $N_x(t)$ is the value of N_x at $t \in \mathbb{R}$) and $\circledast$ is a t-norm satisfying the following conditions:

(PN-1) $N_x(0) = 0$,

(PN-2) $N_x(t) = 1$ for all $t > 0$ if and only if $x = 0$,

(PN-3) $N_{\alpha x}(t) = N_x\left(\frac{t}{|\alpha|}\right)$ for all $\alpha \in R \backslash \{0\}$,

(PN-4) $N_{x+y}(s+t) \geq N_x(s) \circledast N_x(t)$ for all $x, y \in X$ and $s,t \in \mathbb{R}_o^+$.

Example 2.2. Suppose that $(X, \|.\|)$ is a normed space $\mu \in D$ with $\mu(0) = 0$ and $\mu \neq h$, where

$$h(t) = \begin{cases} 0, t \leq 0 \\ 1, t > 0 \end{cases}.$$

Define

$$N_x(t) = \begin{cases} h(t), x = 0 \\ \mu\left(\frac{t}{\|x\|}\right), x \neq 0 \end{cases},$$

where $x \in X, t \in \mathbb{R}$. Then $(X,N,\circledast)$ is a PN-space. For example if we define the functions μ and ν on $\mathbb{R}$ by

$$\mu(x) = \begin{cases} 0, x \leq 0 \\ \frac{x}{1+x}, x > 0 \end{cases}, \quad \nu(x) = \begin{cases} 0, x \leq 0 \\ e^{\frac{-1}{x}}, x > 0 \end{cases}$$

then we obtain the following well-known $\circledast$ norms:

$$N_x(t) = \begin{cases} h(t), x = 0 \\ \frac{t}{t+\|x\|}, x \neq 0 \end{cases}, M_x(t) = \begin{cases} h(t), x = 0 \\ e^{\left(\frac{-\|x\|}{t}\right)}, x \neq 0 \end{cases}.$$

We recall the concept of convergence double sequences in a probabilistic normed space.

Definition 2.4.[7] Let $(X,N,\circledast)$ is a PN-space. Then a double sequence $x = (x_{k,l})$ is said to be convergent to $L \in X$ with respect to the probabilistic norm N if, for every $\varepsilon > 0$ and $\theta \in (0,1)$, there exists a positive integer k_o such that $N_{x_{k,l}-L}(\varepsilon) > 1-\theta$ whenever $k,l \geq k_o$. It is denoted by $N_2 - \lim x = L$ or $x_{k,l} \overset{N_2}{\to} L$ as $k,l \to \infty$.

3 Triple Lacunary Convergence in PN-spaces

The idea of statistical convergence was first introduced by Steinhaus in 1951 [8] and then studied by various authors, e.g. Fast [9], Salat [10], Fridy [11], Esi [12], Esi and Ozdemir [13], Tripathy [14,15], Savaş [16], Savaş and Das [17], Tripathy and Mahanta [18] Mohiuddine et.al [19] and many others and in normed space by Kolk [20]. Karakus [21] has studied the concept of statistical convergence in probabilistic normed spaces for single sequences. Karakuş and Demirci [7] have studied this concept for double sequences. After then Mohiuddine and Savaş [22,23] and Savaş and Mohiuddine [24] have studied lacunary statistically convergent sequences and $\overline{\lambda}-$statistically convergent sequences in probabilistic normed space, respectively. Recently Dutta et.al. [25] have studied statistically convergent triple sequence spaces.

Definition 3.1. Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two dimensional set of positive integers and let $K(n,m)$ be the numbers of (k,l) in K such that $k \leq n, l \leq m$. Then the two dimensional of natural density can be defined as follows: The lower asymptotic density of a set $K \subseteq \mathbb{N} \times \mathbb{N}$ is defined as $\underline{\delta_2}(K) = \lim_{n,m} \inf \frac{K(n,m)}{nm}$.In case the sequence $\left(\frac{K(n,m)}{nm}\right)$ has a limit in Pringsheim's sense, then we say that K has a double natural density and is defined as $\delta_2(K) = \lim_{n,m} \frac{K(n,m)}{nm}$.

Definition 3.2.[26] A real double sequence $x = (x_{k,l})$ is to be statistically convergent to L, provided that for each $\varepsilon > 0$

$$P - \lim_{m,n} \frac{1}{mn} \left| \left\{ (k,l) : \ k \leq m \text{ and } l \leq n, \left| x_{k,l} - L \right| \geq \varepsilon \right\} \right| = 0,$$

where the vertical bars denote the cardinality of the enclosed set. In this case we write $S^L - \lim x = L$ or $x_{k,l} \to L\left(S^L\right)$.

Definition 3.3. [27] The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary sequence if there exist two increasing of integers such that

$$k_o = 0, \ h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty \text{ and } l_o = 0,$$

$$\bar{h}_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty.$$

Notations: $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \bar{h}_s$, $\theta_{r,s}$ is determined by

$$I_{r,s} = \{(k,l) : \ k_{r-1} < k \leq k_r \text{ and } l_{s-1} < l \leq l_s\},$$

$$q_r = \frac{k_r}{k_{r-1}}, \ \bar{q}_s = \frac{l_s}{l_{s-1}} \text{ and } q_{r,s} = q_r \bar{q}_s.$$

Definition 3.4.[28] A subset K of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is said to have natural density $\delta_3(K)$ if

$$\delta_3(K) = P - \lim_{p,q,r \to \infty} \frac{\left|K_{p,q,r}\right|}{pqr} \text{ exists}$$

where the vertical bars denote the number of (k,l,m) in K such that $k \leq p, l \leq q, m \leq r$.Then, a real triple sequence x is said to be statistically convergent to L if for each $\varepsilon > 0$

$$\delta_3\left(\left\{(k,l,m) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \ \left|x_{k,l,m} - L\right| \geq \varepsilon\right\}\right) = 0.$$

In this case, one writes $st_3 - \lim_{k,l,m} x_{k,l,m} = L$.

Definition 3.1.[29] Let $(X,N,\circledast)$ be a PN-space. Then, a triple sequence $x = (x_{jkl})$ is said to be convergent to $L \in X$ with respect to the probabilistic norm N provided that, for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists a positive integer k_o such that $N_{x_{jkl}-L}(\varepsilon) > 1-\lambda$ whenever $j,k,l \geq k_o$. It is denoted by $N_3 - \lim x = L$ or or $x_{jkl} \xrightarrow{N} L$ as $j,k,l \to \infty$.

Definition 3.5. [29] Let $(X,N,\circledast)$ be a PN-space. A triple sequence $x = (x_{jkl})$ is statistically convergent to $L \in X$ with respect to the probabilistic norm N provided that, for every $\varepsilon > 0$ and $\lambda \in (0,1)$

$$K = \left\{(j,k,l) : \ j \leq n, k \leq m \text{ and } l \leq s, \ N_{x_{jkl}-L}(\varepsilon) \leq 1-\lambda\right\}$$

has triple natural density zero, that is if $K(n,m,s)$ become the numbers of (j,k,l) in K

$$\lim_{n,m,s} \frac{K(n,m,s)}{nms} = 0.$$

In this case, one writes $st_{N_3} - \lim_{j,k,l} x_{j,k,l} = L$, where L is said to be $st_{N_3}-$limit. Also one denotes the set of all statistically convergent triple sequences with respect to the probabilistic norm N by st_{N_3}.

Definition 3.6. The triple sequence $\theta_{r,s,t} = \{(j_r,k_s,l_t)\}$ is called triple lacunary sequence if there exist three increasing sequences of integers such that

$$j_o = 0, \ h_r = j_r - j_{r-1} \to \infty \ as \ r \to \infty,$$

$$k_o = 0, \ h_s = k_s - k_{s-1} \to \infty \ as \ s \to \infty$$

and

$$l_o = 0, \ h_t = l_t - l_{t-1} \to \infty \ as \ t \to \infty.$$

Let $k_{r,s,t} = j_r k_s l_t$, $h_{r,s,t} = h_r h_s h_t$ and $\theta_{r,s,t}$ is determined by

$$I_{r,s,t} = \{(j,k,l) : j_{r-1} < j \leq j_r \ , k_{s-1} < k \leq k_s \text{ and } l_{t-1} < l \leq l_t\},$$

$$q_r = \frac{j_r}{j_{r-1}}, q_s = \frac{k_s}{k_{s-1}}, q_t = \frac{l_t}{l_{t-1}} \text{ and } q_{r,s,t} = q_r q_s q_t.$$

Let $K \subset \mathbb{N}\times\mathbb{N}\times\mathbb{N}$. The number

$$\delta_3^{\theta}(K) = \lim_{r,s,t} \frac{1}{h_{r,s,t}} \left|\{(j,k,l) \in I_{r,s,t} : \ (j,k,l) \in K\}\right|$$

is said to be the $\theta_{r,s,t} - density$ of K, provided the limit exits.

Definition 3.7. Let $(X,N,\circledast)$ be a PN-space and $\theta_{r,s,t}$ be a triple lacunary sequence. We say that a triple sequence $x = (x_{jkl})$ is said to be $S_{\theta_{r,s,t}} - convergent$ to L in probabilistic normed space X $\left(\text{for short, } S^{(PN)}_{\theta_{r,s,t}} - convergent\right)$ if for every $\varepsilon > 0$ and $\gamma \in (0,1)$,

$$\delta_3^{\theta}\left(\left\{(j,k,l) \in I_{r,s,t} : \ N_{x_{k,l}-L}(\varepsilon) \leq 1-\gamma\right\}\right) = 0$$

or equivalently

$$\delta_3^{\theta}\left(\left\{(j,k,l) \in I_{r,s,t} : \ N_{x_{k,l}-L}(\varepsilon) > 1-\gamma\right\}\right) = 1.$$

In this case we write $x_{jkl} \xrightarrow{N} L\left(S_{\theta_{r,s,t}}\right)$ or $S^{(PN)}_{\theta_{r,s,t}} - \lim x_{jkl} = L$ and denote the set of all $S_{\theta_{r,s,t}} - convergent$ triple sequences in probabilistic normed space by $\left(S_{\theta_{r,s,t}}\right)_N$.

By using (3.1) and well-known density properties, one can easily establish the following.

Lemma 3.1. Let $(X,N,\circledast)$ is a PN-space. Then, for every $\varepsilon > 0$ and $\gamma \in (0,1)$, the following statements are equivalent:

(i)
$\lim_{r,s,t} \frac{1}{h_{r,s,t}} \left|\left\{(j,k,l) \in I_{r,s,t} : \ N_{x_{jkl}-L}(\varepsilon) \leq 1-\gamma\right\}\right| = 0,$

(ii)
$\delta_3^{\theta}\left(\left\{(j,k,l) \in I_{r,s,t} : N_{x_{jkl}-L}(\varepsilon) \leq 1-\gamma\right\}\right) = 0,$

(iii)
$\delta_3^{\theta}\left(\left\{(j,k,l) \in I_{r,s} : N_{x_{jkl}-L}(\varepsilon) > 1-\gamma\right\}\right) = 1,$

(iv)
$\lim_{r,s,t} \frac{1}{h_{r,s,t}} \left|\left\{(j,k,l) \in I_{r,s,t} : \ N_{x_{jkl}-L}(\varepsilon) > 1-\gamma\right\}\right| = 1.$

Theorem 3.2. Let $(X,N,\circledast)$ is a PN-space and $\theta_{r,s,t}$ be a triple lacunary sequence. If a sequence $x = (x_{jkl})$ triple lacunary statistical convergent in probabilistic normed space X, then $S^{(PN)}_{\theta_{r,s,t}}$-limit is unique.

Proof. Omitted.

Theorem 3.3. Let $(X,N,\circledast)$ is a PN-space and $\theta_{r,s,t}$ be a triple lacunary sequence. If $N_3 - \lim x_{jkl} = L$ then $S^{(PN)}_{\theta_{r,s,t}} - \lim x_{jkl} = L$. But converse need not be true.

Proof. Let $N_3 - \lim x_{jkl} = L$. Then for every $\gamma \in (0,1)$ and $\varepsilon > 0$, there exists a number $(j_o,k_o,l_o) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ such that $N_{x_{jkl}-L}(\varepsilon) > 1-\gamma$ for all $j \geq j_o, k \geq k_o$ and $l \geq l_o$. Hence the set $\left\{(j,k,l) \in I_{r,s,t} : \ N_{x_{jkl}-L_1}(\varepsilon) \leq 1-\gamma\right\}$ has natural density zero and hence $\delta_3^{\theta}\left(\left\{(j,k,l) \in I_{r,s,t} : \ N_{x_{jkl}-L_1}(\varepsilon) \leq 1-\gamma\right\}\right) = 0$, that is, $S^{(PN)}_{\theta_{r,s,t}} - \lim x_{jkl} = L$.

For converse part, we construct the following example:

Example 3.1. Let $(\mathbb{R},|.|)$ denote the space of real numbers with the usual norm. Let $a \circledast b = ab$ and $N_x(\varepsilon) = \frac{\varepsilon}{\varepsilon+|x|}$, where $x \in X$ and $\varepsilon > 0$. In this case, we observe that $(R,N,\circledast)$ is a PN-space. Define a sequence $x = (x_{jkl})$ by

$$x_{jkl} = \begin{cases} jkl \ , \text{for } j_r - \left[\left|\sqrt{h_r}\right|\right] + 1 \leq j \leq j_r, \\ \qquad k_s - \left[\left|\sqrt{h_s}\right|\right] + 1 \leq k \leq k_s \\ \qquad \text{and } l_t - \left[\left|\sqrt{h_t}\right|\right] + 1 \leq l \leq l_t \\ 0 \quad \text{, otherwise} \end{cases}.$$

For $\varepsilon > 0$ and $\gamma \in (0,1)$, let $K(\gamma,\varepsilon) = \left\{(j,k,l) \in I_{r,s,t} : \ N_{x_{jkl}}(\varepsilon) \leq 1-\gamma\right\}$. Then

$$K(\gamma,\varepsilon) = \left\{(j,k,l) \in I_{r,s,t} : \ \frac{\varepsilon}{\varepsilon+|x_{jkl}|} \leq 1-\gamma\right\}$$

$$= \left\{(j,k,l) \in I_{r,s,t}: \ |x_{jkl}| \geq \frac{\gamma\varepsilon}{1-\gamma} > 0 \right\}$$
$$= \{(j,k,l) \in I_{r,s,t}: \ |x_{jkl}| = jkl\}$$

$$= \Big\{(j,k,l) \in I_{r,s,t} : j_r - \left[\left|\sqrt{h_r}\right|\right] + 1 \leq j \leq j_r, \\ k_s - \left[\left|\sqrt{h_s}\right|\right] + 1 \leq k \leq k_s \\ \text{and } l_t - \left[\left|\sqrt{h_t}\right|\right] + 1 \leq l \leq l_t \Big\}$$

and so, we get

$$\lim_{r,s,t} \frac{1}{h_r h_s h_t} \Big| \Big\{(j,k,l) \in I_{r,s,t} : j_r - \left[\left|\sqrt{h_r}\right|\right] + 1 \leq j \leq j_r, \\ k_s - \left[\left|\sqrt{h_s}\right|\right] + 1 \leq k \leq k_s \\ \text{and } l_t - \left[\left|\sqrt{h_t}\right|\right] + 1 \leq l \leq l_t \Big\} \Big|$$
$$\leq \lim_{r,s,t} \frac{\sqrt{h_r}\sqrt{h_s}\sqrt{h_t}}{h_r h_s h_t} = 0.$$

This implies that $S_{\theta_{r,s,t}}^{(PN)} - \lim x_{jkl} = 0$. On the other hand $N_3 - \lim x \neq 0$, since

$$N_{x_{jkl}}(\varepsilon) = \frac{\varepsilon}{\varepsilon + |x_{jkl}|} =$$

$$\begin{cases} \frac{\varepsilon}{\varepsilon+|x_{jkl}|}, & \text{for } j_r - \left[\left|\sqrt{h_r}\right|\right] + 1 \leq j \leq j_r, \\ & k_s - \left[\left|\sqrt{h_s}\right|\right] + 1 \leq k \leq k_s \\ & \text{and } l_t - \left[\left|\sqrt{h_t}\right|\right] + 1 \leq l \leq l_t \\ 1 & \text{, otherwise} \end{cases}$$

$$\leq 1.$$

This completes the proof.

Theorem 3.4. Let $(X, N, \circledast)$ is a PN-space and $\theta_{r,s,t}$ be a triple lacunary sequence. Then, $S_{\theta_{r,s,t}}^{(PN)} - \lim x_{jkl} = L$ iff there exists a subset

$$K = \{(j_n, k_n, l_n) : \ j_1 < j_2 < j_3 < ...; k_1 < k_2 < k_3 < ...; \\ l_1 < l_2 < l_3 < ...\} \subset \mathbb{N}\times\mathbb{N}\times\mathbb{N}$$

such that $\delta_3^{\theta}(K) = 1$ and $N_3 - \lim_{n\to\infty} x_{j_n k_n l_n} = L$.

Proof. We first assume that $S_{\theta_{r,s,t}}^{(PN)} - \lim x_{jkl} = L$. Now, for any $\varepsilon > 0$ and $r \in \mathbb{N}$, let

$$K(r,\varepsilon) = \left\{(j,k,l) \in I_{r,s,t}: \ N_{x_{jkl}-L}(\varepsilon) \leq 1 - \frac{1}{r}\right\},$$
$$M(r,\varepsilon) = \left\{(j,k,l) \in I_{r,s,t}: \ N_{x_{jkl}-L}(\varepsilon) > 1 - \frac{1}{r}\right\}.$$

Then $\delta_3^{\theta}(\{K(r,\varepsilon)\}) = 0$ and

$i)$ $M(1,\varepsilon) \supset M(2,\varepsilon) \supset ... \supset M(i,\varepsilon) \supset M(i+1,\varepsilon) \supset ...$

$ii)$ $\delta_3^{\theta}(\{M(r,\varepsilon)\}) = 1, \ r = 1,2,3,....$

Now we have to show that for $(j,k,l) \in M(r,\varepsilon)$, $x = (x_{jkl})$ is $N_3 -$ *convergent* to L. Suppose that $x = (x_{jkl})$ is not $N_3 -$ *convergent* to L. Therefore there is $\lambda > 0$ such that

$$\left\{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}: \ N_{x_{jkl}-L}(\varepsilon) \leq 1-\lambda\right\}$$

for finitely many terms. Let

$$M(\lambda,\varepsilon) = \left\{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}: \ N_{x_{jkl}-L}(\varepsilon) \geq 1-\lambda\right\}, \\ \lambda > \frac{1}{r} \ (r = 1,2,3,...).$$

Then

$iii)$ $\delta_3^{\theta}(\{M(\lambda,\varepsilon)\}) = 0$ and by (i), $M(r,\varepsilon) \subset M(\lambda,\varepsilon)$.

Hence $\delta_3^{\theta}(\{M(r,\varepsilon)\}) = 0$ which contradict (ii). Therefore $x = (x_{jkl})$ is $N_3 -$ *convergent* to L.

Conversely, suppose that there exists a subset

$$K = \{(j_n, k_n, l_n) : \ j_1 < j_2 < j_3 < ...; k_1 < k_2 < k_3 < ...; \\ l_1 < l_2 < l_3 < ...\} \subset \mathbb{N}\times\mathbb{N}\times\mathbb{N}$$

such that $\delta_3^{\theta}(K) = 1$ and $N_3 - \lim_{n\to\infty} x_{j_n k_n l_n} = L$. Then, there exists $k_o \in \mathbb{N}$ such that for every $\lambda \in (0,1)$ and for any $\varepsilon > 0$

$$N_{x_{jkl}-L}(\varepsilon) > 1-\lambda, \forall j,k,l \geq k_o.$$

Now

$$M(\lambda,\varepsilon) = \left\{(j,k,l) \in I_{r,s,t}: \ N_{x_{jkl}-L}(\varepsilon) \leq 1-\lambda\right\} \subset$$

$$\mathbb{N}\times\mathbb{N}\times\mathbb{N}/\{(j_{k_o+1}, k_{k_o+1}, l_{k_o+1}), (j_{k_o+2}, k_{k_o+2}, l_{k_o+2}), ...\}.$$

Therefore, $\delta_3^{\theta}(\{M(\lambda,\varepsilon)\}) \leq 1-1 = 0$. Hence, we conclude that $S_{\theta_{r,s,t}}^{(PN)} - \lim x_{jkl} = L$.

Lemma 3.5. Let $(X, N, \circledast)$ be a PN-space.

(i) If $S_{\theta_{r,s,t}}^{(PN)} - \lim x_{jkl} = L_1$ and $S_{\theta_{r,s,t}}^{(PN)} - \lim y_{jkl} = L_2$, then $S_{\theta_{r,s,t}}^{(PN)} - \lim (x_{jkl} + y_{jkl}) = L_1 + L_2$.

(ii) If $S_{\theta_{r,s,t}}^{(PN)} - \lim x_{jkl} = L$. and $\alpha \in \mathbb{R}$, then $S_{\theta_{r,s,t}}^{(PN)} - \lim \alpha x_{jkl} = \alpha L$.

(iii) If $S_{\theta_{r,s,t}}^{(PN)} - \lim x_{jkl} = L_1$ and $S_{\theta_{r,s,t}}^{(PN)} - \lim y_{jkl} = L_2$, then $S_{\theta_{r,s,t}}^{(PN)} - \lim (x_{jkl} - y_{jkl}) = L_1 - L_2$.

Proof (i). Let $S_{\theta_{r,s,t}}^{(PN)} - \lim x_{jkl} = L_1$ and $S_{\theta_{r,s,t}}^{(PN)} - \lim y_{jkl} = L_2$, $\varepsilon > 0$ and $\lambda \in (0,1)$. Choose $\gamma \in (0,1)$ such that $(1-\gamma) \circledast (1-\gamma) \geq 1-\lambda$. Then, we define the following sets:

$$K_1(\gamma,\varepsilon) = \left\{(j,k,l) \in I_{r,s,t}: \ N_{x_{jkl}-L_1}(\varepsilon) \leq 1-\gamma\right\},$$
$$K_2(\gamma,\varepsilon) = \left\{(j,k,l) \in I_{r,s,t}: \ N_{x_{jkl}-L_2}(\varepsilon) \leq 1-\gamma\right\}.$$

Since $S_{\theta_{r,s,t}}^{(PN)}-\lim x_{jkl}=L_1$, we have $\delta_3^{\theta}\left(\left\{K_1\left(\gamma,\varepsilon\right)\right\}\right)=0$ for all $\varepsilon>0$ and since $S_{\theta_{r,s,t}}^{(PN)}-\lim y_{jkl}=L_2$, we get $\delta_3^{\theta}\left(\left\{K_2\left(\gamma,\varepsilon\right)\right\}\right)=0$ for all $\varepsilon>0$. Now let $K\left(\gamma,\varepsilon\right)=K_1\left(\gamma,\varepsilon\right)\cap K_2\left(\gamma,\varepsilon\right)$. Then observe that $\delta_3^{\theta}\left(\left\{K\left(\gamma,\varepsilon\right)\right\}\right)=0$ for all $\varepsilon>0$ which implies $\delta_3\theta\left(\left\{I_{r,s,t}/K\left(\gamma,\varepsilon\right)\right\}\right)=1$. If $\left(j,k,l\right)\in I_{r,s,t}/K\left(\gamma,\varepsilon\right)$, then we have

$$N_{\left(x_{jkl}-L_1\right)+\left(y_{jkl}-L_2\right)}\left(\varepsilon\right)\geq N_{x_{jkl}-L_1}\left(\frac{\varepsilon}{2}\right)\circledast N_{y_{jkl}-L_2}\left(\frac{\varepsilon}{2}\right)$$
$$>\left(1-\gamma\right)\circledast\left(1-\gamma\right)\geq 1-\lambda.$$

This shows that

$$\delta_3^{\theta}\left(\left\{\left(j,k,l\right)\in I_{r,s,t}:\ N_{\left(x_{jkl}-L_1\right)+\left(y_{jkl}-L_2\right)}\left(\varepsilon\right)\leq 1-\lambda\right\}\right)=0$$

so, $S_{\theta_{r,s,t}}^{(PN)}-\lim\left(x_{jkl}+y_{jkl}\right)=L_1+L_2$.

(ii) Let $S_{\theta_{r,s,t}}^{(PN)}-\lim x_{jkl}=L,$, $\varepsilon>0$ and $\lambda\in\left(0,1\right)$. First of all, we consider the case of $\alpha=0$. In this case

$$N_{0x_{jkl}-0L}\left(\varepsilon\right)=N_0\left(\varepsilon\right)=1>1-\lambda.$$

So we obtain $N_3-\lim 0x_{jkl}=0$. Then from Theorem 3.3, we have $S_{\theta_{r,s,t}}^{(PN)}-\lim 0x_{jkl}=0$. Now we consider the case $\alpha\neq 0$.Since $S_{\theta_{r,s,t}}^{(PN)}-\lim x_{jkl}=L$, if we define the set

$$K\left(\lambda,\varepsilon\right)=\left\{\left(j,k,l\right)\in I_{r,s,t}:\ N_{x_{jkl}-L}\left(\varepsilon\right)\leq 1-\lambda\right\}$$

then, we can say that

$$\delta_3^{\theta}\left(\left\{K\left(\lambda,\varepsilon\right)\right\}\right)=0 \text{ for all } \varepsilon>0.$$

In this case

$$\delta_3^{\theta}\left(\left\{I_{r,s,t}/K\left(\lambda,\varepsilon\right)\right\}\right)=1.$$

If $\left(j,k,l\right)\in I_{r,s,t}/K\left(\lambda,\varepsilon\right)$, then

$$N_{\alpha x_{jkl}-\alpha L}\left(\varepsilon\right)=N_{x_{jkl}-L}\left(\frac{\varepsilon}{|\alpha|}\right)\circledast N_0\left(\frac{\varepsilon}{|\alpha|}-\varepsilon\right)$$
$$=N_{x_{jkl}-L}\left(\varepsilon\right)\circledast 1=N_{x_{jkl}-L}\left(\varepsilon\right)>1-\lambda$$

for $\alpha\in\mathbb{R}\ \left(\alpha\neq 0\right)$. This shows that

$$\delta_3^{\theta}\left(\left\{\left(j,k,l\right)\in I_{r,s,t}:\ N_{\alpha x_{jkl}-\alpha L}\left(\varepsilon\right)\leq 1-\lambda\right\}\right)=0$$

so, $S_{\theta_{r,s,t}}^{(PN)}-\lim\alpha x_{jkl}=\alpha L$.

(iii) The proof is clear from (i) and (ii).

Definition 3.8. Let $\left(X,N,\circledast\right)$ be a PN-space and $\theta_{r,s,t}$ be a triple lacunary sequence. Then a triple sequence $x=\left(x_{jkl}\right)\in X$ is said to be $S_{\theta_{r,s,t}}^{(PN)}-$ Cauchy in PN-space X if for every $\varepsilon>0$ and $\lambda\in\left(0,1\right)$, there exist $M=M\left(\varepsilon\right),T=T\left(\varepsilon\right),\ P=P\left(\varepsilon\right)\in\mathbb{N}$ such that for all $j,p\geq M,k,q\geq T$ and $l,u\geq P$; the set

$$\left\{\left(j,k,l\right)\in I_{r,s,t}:N_{x_{jkl}-x_{pqu}}\left(\varepsilon\right)\leq 1-\lambda\right\}$$

has triple natural density zero.

Now using a similar technique in the proof of Theorem 3.4, one can get the following result at once.

Theorem 3.6. Let $\left(X,N,\circledast\right)$ be a PN-space and $\theta_{r,s,t}$ be a triple lacunary sequence. Then, a triple sequence $x=\left(x_{jkl}\right)\in X$ is $S_{\theta_{r,s,t}}^{(PN)}-$ convergent if and only if it is $S_{\theta_{r,s,t}}^{(PN)}-$ Cauchy in PN-space X.

Proof. Omitted.

4 Conclusion

In this paper we obtained some results on statistical convergence for triple sequences in probabilistic normed space. As every ordinary norm induces a probabilistic norm, the results obtained here are more general then the corresponding of normed spaces.

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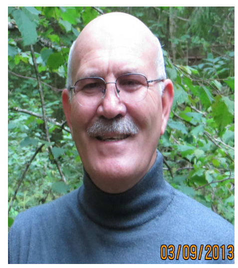

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