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A Recurrence Relation of Hypergeometric Series Through Record Statistics and a Characterization

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Abstract: In this paper, a new general recurrence relation of hypergeometric series is derived using distribution function of upper record statistics. The result can be extended easily to k-records statistics. A characterization is given on the basis of this recurrence relation.

Keywords: Upper record values, Hypergeometric series, Recurrence relation, Incomplete Gamma function.

1 Introduction

There are several situations where only those observations are considered which have values larger (smaller) than a specific observation available in the data, hence these observations are retained and the rest of the observations are discarded. This specific observation is generally referred to as a record.

A record value may be defined as an observation in a sequence below or above which all the observations are smaller or larger. Consider a record in a sequence of data; if all the observations have values above the value of this record observation, then the said observation is an "upper record". For example, in the Olympic records of hammer throwing, the maximum distance, to which an athlete throws the hammer, will be considered as an upper record. Now only those distances will form the upper record sequence whose values are larger than this maximum distance. On the other hand, if all the observations have values below the value of the record observation, then this record observation is called a "lower record". For example, in the progression of Olympic records for 100 meters women's free style swimming, the minimum time to cover the distance will be considered as a lower record and only those times will form the lower record sequence whose values are below this minimum time.

There are such situations where record value data naturally occurs. [13] observes that in some hydrological and material testing situation only record values are stored. He also presents similar form of data set which arises in real-time machine monitoring.

Consider a sequence of random variables $\{X_1, X_2, X_3, \ldots\}$ from a distribution function $F(x)$. Let $Y_n = \max(\min)\{X_1, X_2, X_3, \ldots, X_n, \text{for}\, n \geq 1\}$, then Y_j is an upper (lower) record value of $\{X_n : n \geq 1\}$ if $Y_j > Y_{j-1}$ for $j > 1$ or $Y_j < Y_{j-1}$ for $j > 1$. It is evident from the above definition of the record values that Y_1 is an upper as well as a lower record value.

The concept of record values and record statistics is first introduced by [7] and later developed by [8] in connection with gambling problems. [2], [3] and [10] give a comprehensive and in-depth study of record values and record statistics. Many authors derive recurrence relations of moments of record statistics for different distributions. [9] develop some useful relationships of record statistics for Inverse Rayleigh Distribution with Gamma Distribution and recurrence relationships of moments. [14] proposes recurrence relations between the single moments of record values from the modified Weibull distribution as well as between the double moments. [5] establish relations for single and product moments of record values from Gumbel distribution. [4] also develop the recurrence relations for single and product moments of record statistics for Lomax distribution. Various authors derive the recurrence relations for k-th record values for different distributions; see details [11], [12], [6].

The hypergeometric confluent functions are extensively used in wide range of practical problems such

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as networks, wave analysis, finance etc. In this article, we deal with a recurrence relation of hypergeometric series using record statistics from the absolute continuous distribution and a characterization based on this recurrence relation.

2 Recurrence Relation of Hypergeometric Series Using Record Statistics

Let $F(x)$ and $f(x)$ be the cumulative distribution and probability density functions, respectively, of the sequence of identical and independently distributed random variables $X_1, X_2, \ldots$. Denote the upper record statistics by $X_U(n), n \geq 1$ from the sequence $\{X_n; n \geq 1\}$. We use the notations $R(x) = -\ln(1 - F(x))$ and $H(x) = -\ln F(x)$. Now, denote the distribution function and the probability density function of $X_U(n)$ by $F_n(x)$ and $f_n(x)$ respectively, then it is known that

$$F_n(x) = P(X_U(n) \geq x)$$
$$= \int_{-\infty}^{R(x)} \frac{y^{n-1} e^{-y}}{(n-1)!} dy, \quad -\infty < y < +\infty, \tag{1}$$
$$f_n(x) = \frac{R^{n-1}(x)}{(n-1)!} f(x), \quad -\infty < y < +\infty. \tag{2}$$

(1), on integration by parts, gives

$$1 - F_n(x) = e^{-R(x)} \sum_{j=0}^{n-1} \frac{R^j(x)}{j!}. \tag{3}$$

From the relation $R(x) = -\ln(1 - F(x))$, we get $F(x) = 1 - e^{-R(x)}$, and consequently, $f(x) = R'(x)\, e^{-R(x)}$. Subtituting this in (2), we get

$$f_n(x) = \frac{R^{n-1}(x) R'(x) e^{-R(x)}}{(n-1)!}, \quad -\infty < x < \infty. \tag{4}$$

This is another form of the pdf for the upper record values. For brevity, in what follows, we will write R for $R(x)$.

Theorem 1. *Let $\{X_n, n > 1\}$ be a sequence of independent and identically distributed non-negative continuous random variables with a common cumulative distribution function $F(x)$ and probability density function $f(x)$. Then the following recurrence relation holds*

$$R\, {}_1F_1(2; n+2; R) = (n+1)[n - (n-R)\; {}_1F_1(1; n+1; R)],$$

where

$${}_1F_1(a; b; x) = 1 + \frac{a}{b}x + \frac{a(a+1)}{b(b+1)} \frac{x^2}{2!} + \frac{a(a+1)(a+2)}{b(b+1)(b+1)} \frac{x^3}{3!} + \ldots$$

is the hypergeometric series.

Proof. From (3), we have

$$1 - F_n(x) = e^{-R} \sum_{j=0}^{n-1} \frac{R^j}{j!}$$
$$= e^{-R}\left[1 + R + \frac{R^2}{2!} + \ldots + \frac{R^{n-1}}{(n-1)!}\right]$$
$$= e^{-R}[e^R - \frac{R^n}{n!}\{1 + \frac{R}{(n+1)} + \frac{R^2}{(n+1)(n+2)} + \ldots\}]$$
$$= e^{-R}\left[e^R - \frac{R^n}{n!}\, {}_1F_1(1; n+1; R)\right]$$
$$= 1 - \frac{R^n e^{-R}}{n!}\, {}_1F_1(1; n+1; R). \tag{5}$$

Differentiating both sides of (5) with respect to x, we get

$$-f_n(x) = -\frac{n R' R^{n-1} e^{-R}}{n!}\, {}_1F_1(1; n+1; R) + \frac{R' R^n e^{-R}}{n!}\, {}_1F_1(1; n+1; R) - \frac{R^n e^{-R}}{n!}\, {}_1F_1'(1; n+1; R). \tag{6}$$

Using (4) in (6), we obtain

$$-\frac{R^{n-1} R' e^{-R}}{(n-1)!} = -\frac{R' R^{n-1} e^{-R}}{n!} [n\, {}_1F_1(1; n+1; R) - R\, {}_1F_1(1; n+1; R) + \frac{R\, {}_1F_1'(1; n+1; R)}{R'}]. \tag{7}$$

Simplifying (7), we get

$${}_1F_1'(1; n+1; R) = \frac{[n - (n-R)\, {}_1F_1(1; n+1; R)]\, R'}{R}. \tag{8}$$

But we also know that

$${}_1F_1(1; n+1; R) = 1 + \frac{R}{(n+1)} + \frac{R^2}{(n+1)(n+2)} + \ldots$$

Differentiating both sides of above equation with respect to x, we get

$${}_1F_1'(1; n+1; R) = \frac{R'}{(n+1)} + \frac{2RR'}{(n+1)(n+2)} + \frac{3R^2 R'}{(n+1)(n+2)(n+3)} + \ldots$$
$$= \frac{R'}{(n+1)}[1 + \frac{2R}{(n+2)} + \frac{3R^2}{(n+2)(n+3)} + \ldots]$$
$$= \frac{R'}{(n+1)}\, {}_1F_1(2; n+2; R). \tag{9}$$

Equating (8) and (9), we get

$$R\,{}_1F_1\,(2;n+2;R) = (n+1)\,[n-(n-R)\,{}_1F_1\,(1;n+1;R)]. \quad (10)$$

(10) can also be rewritten as

$${}_1F_1\,(2;n+2;R) = [(n+1)\,{}_1F_1\,(1;n+1;R) - n\,{}_1F_1\,(1;n+2;R)]. \quad (11)$$

Remarks

1. Following the above arguments, we can easily show that the result of Theorem 1 is also true if we use the lower record statistic with R replaced by $H(x)$.
2. The so-called $k-$records for $(k \geq 1)$ (see, for example [10]) are the natural extension of records. The result for k-th record is given by

$$k\,R\,{}_1F_1\,(2;n+2;kR) = (n+1)\,[n-(n-kR)\,{}_1F_1\,(1;n+1;kR)].$$

3 Characterization

Theorem 2. *Suppose the recurrence relation in Theorem 1 holds and let R=x, then the recurrence relation in Theorem 1 is a special case of the recurrence relation for the Incomplete Gamma function, i.e.*

$$\gamma(n,x) = n^{-1}x^n\, e^{-x}\,{}_1F_1\,(1;n+1;x), \quad x \geq 0, n \neq 0,-1,-2,\ldots$$

Proof. The Incomplete Gamma function $\gamma(n,x)$ is defined by

$$\gamma(n,x) = \int_0^x t^{n-1}\, e^{-t}\, dt. \quad (12)$$

Then (12) can also be expressed in infinite series form as

$$\gamma(n,x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\frac{x^{n+k}}{n+k} = n^{-1}x^n\, e^{-x}\,{}_1F_1\,(1;n+1;x).$$

This gives (10) as

$${}_1F_1\,(2;n+2;x) = n\,(n+1)\,x^{-n-1}\,e^x\,[x\,\gamma(n,x) - \gamma(n+1,x)]. \quad (13)$$

We know the recurrence relation

$$\gamma(n+1,x) = n\,\gamma(n,x) - x^{-n}\,e^{-x}, \quad (14)$$

and substituting this in (13), we get

$$x\,{}_1F_1\,(2;n+2;x) = n\,(n+1)\left[x^{-n}e^x\,(x-n)\;\gamma(n,x)+1\,\right]. \quad (15)$$

Also we have from (10)

$$x\,{}_1F_1\,(2;n+2;x) = (n+1)\,[n-(n-x)\,{}_1F_1\,(1;n+1;x)]. \quad (16)$$

Equating (15) and (16), we get the desired result

$$\gamma(n,x) = n^{-1}x^n\, e^{-x}\,{}_1F_1\,(1;n+1;x), \quad x \geq 0,\; n \neq 0,-1,-2,\ldots \quad (17)$$

4 Conclusion

Researchers are taking keen interest in record values data as it occurs in many real life situations. In statistical modeling, e.g. in reliability analysis, life time studies, testing the strength of materials etc., the realizations of many experiments come up in records set of observations, therefore the use of record statistics is necessary. In this article we derived a new form of the probability distribution for the upper values and subsequently we derived a new recurrence relation in hypergeometric confluent function terms which is useful in characterizing some functions such as gamma function. This recurrence relation can be still more useful if we assume that $R(x) = x$ and apply some additional conditions such that it follows some given distribution. Another application of the recurrence relation could be to find similar results for the entropy of the records, or likewise for some other functions.

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